# MATHEMATICAL QUESTIONS

## Question 1

**Derive the impedance matrix of the op-amp circuit shown in Fig. 1 and show that the impedance matrix looks like that of a gyrator. Is this circuit an exact equivalent implementation of a gyrator?**

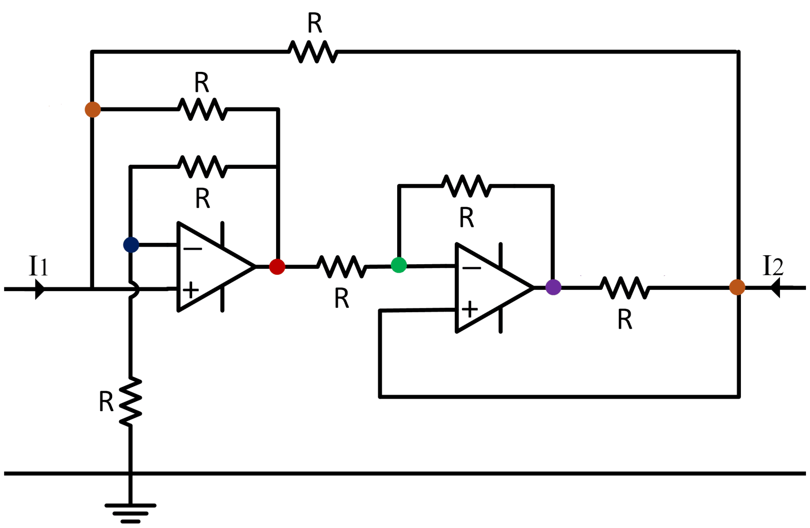


Figure 1: An Op-Amp circuit.

Let $V_1$ and $V_2$ be the input voltages of the left and right op-amps, respectively. Clearly, the left op-amp is a non-inverting amplifier and its output voltage is $V_1(1+R/R) = 2V_1$. Wrting a KCL for the inverting leg of the right op-amp,

$$\frac{V_2 - 2V_1}{R} + \frac{V_2 - V_{o2}}{R} = 0 \Rightarrow V_{o2} = 2(V_2 - V_1)$$

Now,

$$I_1 = \frac{V_1 - 2V_1}{R} + \frac{V_1 - V_2}{R} = -\frac{V_2}{R} \Rightarrow V_2 = -RI_1$$

$$I_2 = \frac{V_2 - V_1}{R} + \frac{V_2 - V_{o2}}{R} = \frac{V_1}{R} \Rightarrow V_1 = RI_2$$

So,

$$\boldsymbol{Z} = \begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix}$$

The impedance matrix is equivalent two that of a gyrator with the parameter $\alpha = R$, so the circuit may be considered as an implementation of a gyrator. However, the real gyrator is a

Question 1 continued on next page...

passive circuit but this op-amp circuit is active. Further, the ports in a gyrator are isolated unlike this op-amp circuit whose ports are not isolated.

## Question 2

**The impedance matrix $Z$ of a two-port is given. Prove the equations that relate the admittance, hybrid H, and ABCD transmittance matrices of the two-port to the elements of its impedance matrix $Z$.**

For the admittance matrix,

$$\boldsymbol{V} = \boldsymbol{ZI} \Rightarrow \boldsymbol{I} = \boldsymbol{Z}^{-1}\boldsymbol{V} = \boldsymbol{YV} \Rightarrow \boldsymbol{Y} = \boldsymbol{Z}^{-1} = \begin{bmatrix} \frac{Z_{22}}{\Delta Z} & -\frac{Z_{12}}{\Delta Z} \\ -\frac{Z_{21}}{\Delta Z} & \frac{Z_{11}}{\Delta Z} \end{bmatrix}$$

Now, note that

$$\begin{cases} V_1 = z_{11}I_1 + z_{12}I_2 \\ V_2 = z_{21}I_1 + z_{22}I_2 \end{cases}$$

Using the equations above,

$$h_{11} = \frac{V_1}{I_1}|_{V_2=0} = \frac{z_{11}I_1 + z_{12}I_2}{I_1} = z_{11} + z_{12}\frac{I_2}{I_1} = z_{11} + z_{12}\frac{-z_{21}}{z_{22}} = \frac{\Delta \boldsymbol{Z}}{z_{22}}$$

$$h_{21} = \frac{I_2}{I_1}|_{V_2=0} = -\frac{z_{21}}{z_{22}}$$

$$h_{22} = \frac{I_2}{V_2}|_{I_1=0} = \frac{I_2}{z_{22}I_2} = \frac{1}{z_{22}}$$

$$h_{12} = \frac{V_1}{V_2}|_{I_1=0} = \frac{z_{12}I_2}{z_{22}I_2} = \frac{z_{12}}{z_{22}}$$

$$\boldsymbol{H} = \begin{bmatrix} \frac{\Delta \boldsymbol{Z}}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{z_{12}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$$

Similarly, for the transmittance ABCD matrix,

$$A = \frac{V_1}{V_2}\Big|_{I_2=0} = \frac{z_{11}I_1}{z_{21}I_1} = \frac{z_{11}}{z_{21}}$$

$$B = \frac{V_1}{-I_2}\Big|_{V_2=0} = \frac{z_{11}I_1 + z_{12}I_2}{-I_2} = -z_{11}\frac{I_1}{I_2} - z_{12} = -z_{11}\frac{-z_{22}}{z_{21}} - z_{12} = \frac{\Delta \boldsymbol{Z}}{z_{21}}$$

$$C = \frac{I_1}{V_2}\Big|_{I_2=0} = \frac{I_1}{z_{21}I_1} = \frac{1}{z_{21}}$$

$$D = \frac{I_1}{-I_2}\Big|_{V_2=0} = \frac{I_1}{\frac{z_{21}}{z_{22}}I_1} = \frac{z_{22}}{z_{21}}$$

$$\boldsymbol{T} = \begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta \boldsymbol{Z}}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$$

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## Question 3

**Obtain both the impedance and admittance parameters for the two-port networks of Fig. 2.**

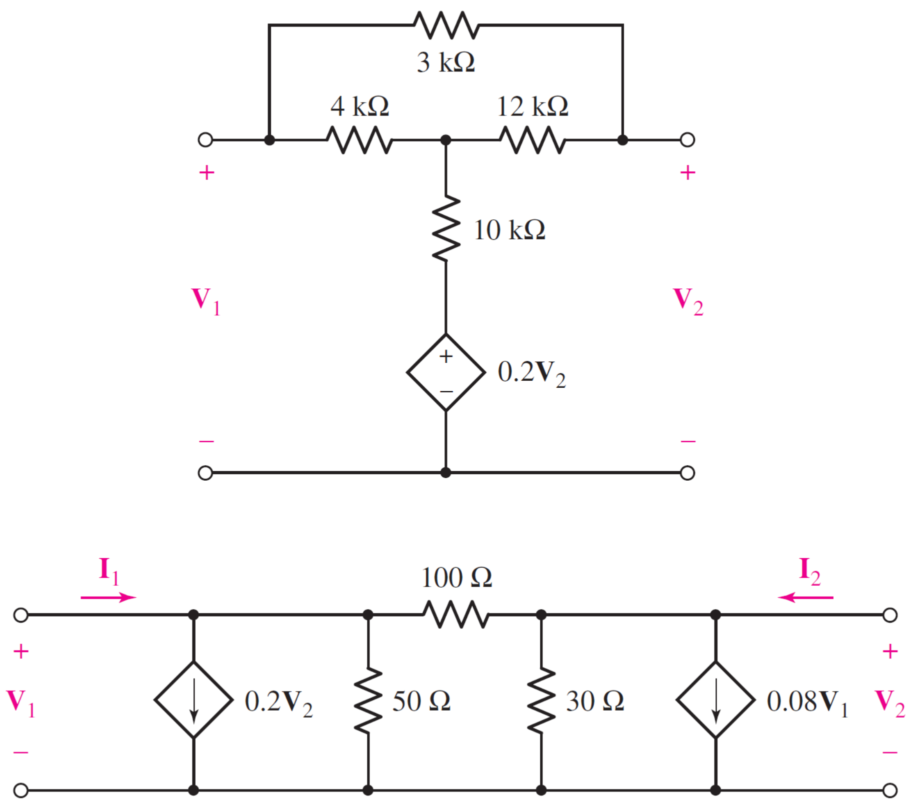


Figure 2: Two two-ports for which the impedance and admittance specifications are required.

The impedance and admittance parameters for the up two-port network is calculated as follows.
KVL @ 3:

$$3(I_1 - I) + 12(I_1 + I_2 - I) - 4I = 0 \Rightarrow I = \frac{15}{19}I_1 + \frac{12}{19}I_2$$

KVL @ 2:

$$12(I_1 + I_2 - I) + 10(I_1 + I_2) + 0.2V_2 = V_2$$
$$\Rightarrow V_2 = \frac{297.5}{19}I_1 + \frac{342.5}{19}I_2 = 15.66I_1 + 18.03I_2$$

KVL @ 1:

$$4I + 10(I_1 + I_2) + 0.2V_2 = V_1$$
$$\Rightarrow V_1 = \frac{309.5}{19}I_1 + \frac{306.5}{19}I_2 = 16.29I_1 + 16.13I_2$$

Question 3 continued on next page. . .

Hence,

$$\boldsymbol{Z} = \begin{bmatrix} 16.29 & 16.13 \\ 15.66 & 18.03 \end{bmatrix}$$

$$\boldsymbol{Y} = \boldsymbol{Z}^{-1} = \begin{bmatrix} 0.44 & -0.39 \\ -0.38 & 0.40 \end{bmatrix}$$

The impedance and admittance parameters for the down two-port network is calculated as follows.
KVL:

$$100(I_1 - 0.2V_2 - 0.02V_1) + V_2 - V_1 = 0 \Rightarrow I_1 = 0.03V_1 + 0.19V_2$$

KCL @ A:

$$\begin{aligned} I_2 - 0.08V_1 &= \frac{V_2}{30} - (I_1 - 0.2V_2 - \frac{V_1}{50}) \\ I_1 + I_2 &= 0.1V_1 + 0.23V_2 \\ I_2 &= 0.1V_1 + 0.23V_2 - I_1 \\ I_2 &= 0.1V_1 + 0.23V_2 - (0.03V_1 + 0.19V_2) \\ I_2 &= 0.07V_1 + 0.04V_2 \end{aligned}$$

Hence,

$$\boldsymbol{Y} = \begin{bmatrix} 0.03 & 0.19 \\ 0.07 & 0.04 \end{bmatrix}$$

$$\boldsymbol{Z} = \boldsymbol{Y}^{-1} = \begin{bmatrix} -3.31 & 15.7 \\ 5.79 & -2.48 \end{bmatrix}$$

## Question 4

**The purpose of this problem is to justify a method for checking whether a given two-port is a reciprocal two-port at frequency $\omega_0$. Consider the two situations shown in Fig. 3. All measurements are sinusoidal steady-state measurements made at frequency $\omega_0$; consequently $V_1$, $V_2$, $I_1$, $I_2$, $\hat{V}_1$, $\hat{V}_2$, $\hat{I}_1$, and $\hat{I}_2$ are the phasors representing the sinusoidal waveforms. The impedances $\hat{Z}_1$ and $Z_2$ and the internal impedance of the generator are arbitrary, except that $\frac{I_1}{I_2} \neq \frac{\hat{I}_1}{\hat{I}_2}$. Show that the two-port is reciprocal at frequency $\omega_0$ if and only if $V_1\hat{I}_1 + V_2\hat{I}_2 = \hat{V}_1 I_1 + \hat{V}_2 I_2$.**



Question 4 continued on next page...

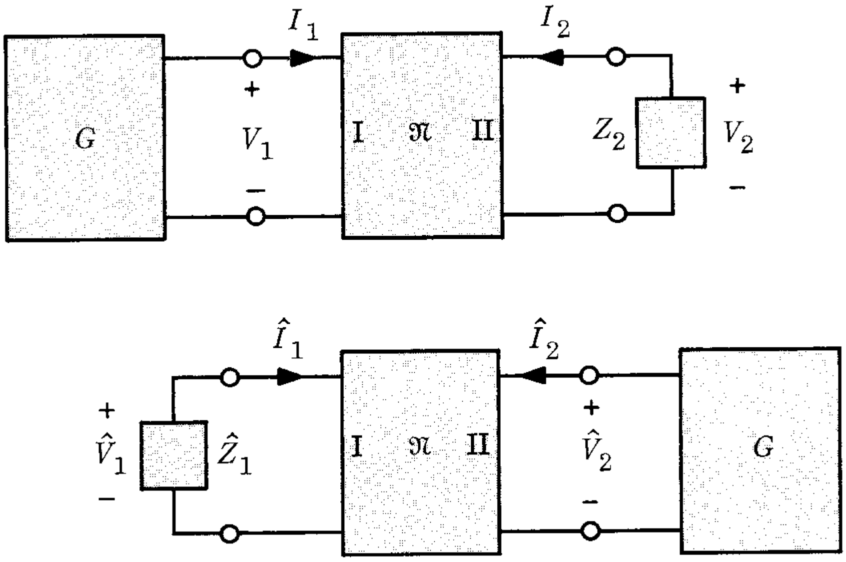


Figure 3: Test setup for checking the reciprocity of a two-port at frequency $\omega_0$.

We know from the impedance description that

$$V_1(j\omega) = z_{11}(j\omega)I_1(j\omega) + z_{12}(j\omega)I_2(j\omega)$$

$$V_2(j\omega) = z_{21}(j\omega)I_1(j\omega) + z_{22}(j\omega)I_2(j\omega)$$
$$\hat{V}_1(j\omega) = z_{11}(j\omega)\hat{I}_1(j\omega) + z_{12}(j\omega)\hat{I}_2(j\omega)$$
$$\hat{V}_2(j\omega) = z_{21}(j\omega)\hat{I}_1(j\omega) + z_{22}(j\omega)\hat{I}_2(j\omega)$$

We have,

$$V_1(j\omega)\hat{I}_1(j\omega) + V_2(j\omega)\hat{I}_2(j\omega) = \hat{V}_1(j\omega)I_1(j\omega) + \hat{V}_2(j\omega)I_2(j\omega)$$

, which simplifies to

$$[z_{11}(j\omega)I_1(j\omega) + z_{12}(j\omega)I_2(j\omega)]\hat{I}_1(j\omega) + [z_{21}(j\omega)I_1(j\omega) + z_{22}(j\omega)I_2(j\omega)]\hat{I}_2(j\omega)$$

$$= [z_{11}(j\omega)\hat{I}_1(j\omega) + z_{12}(j\omega)\hat{I}_2(j\omega)]I_1(j\omega) + [z_{21}(j\omega)\hat{I}_1(j\omega) + z_{22}(j\omega)\hat{I}_2(j\omega)]I_2(j\omega)$$

and then,

$$z_{12}(j\omega)I_2(j\omega)\hat{I}_1(j\omega) + z_{21}(j\omega)I_1(j\omega)\hat{I}_2(j\omega) = z_{12}(j\omega)\hat{I}_2(j\omega)I_1(j\omega) + z_{21}(j\omega)\hat{I}_1(j\omega)I_2(j\omega)$$

The equality holds if

$$z_{12}(j\omega) = z_{21}(j\omega)$$

, which is the condition required for the reciprocity of the two-port.

## Question 5

**Find the admittance matrix $Y$ for the two-port shown in Fig. 4.**

*Question 5 continued on next page...*

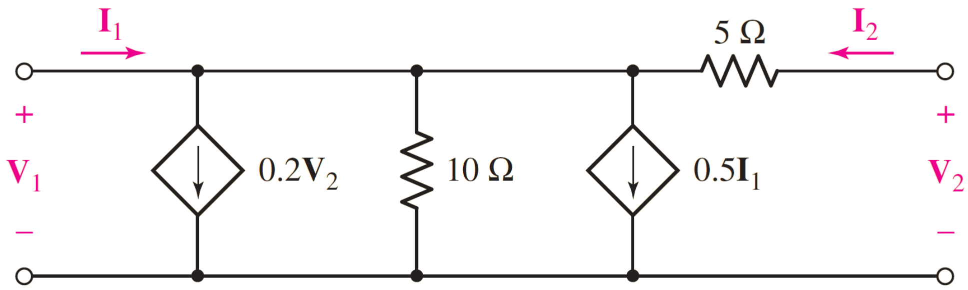


Figure 4: An LTI circuit with dependent sources.

Let $V_2 = 0$. We have

$$I_1 - \frac{V_1}{10\,\Omega} - \frac{V_1}{5\,\Omega} - 0.5I_1 - 0.2V_2 = 0 \Rightarrow \frac{3}{10}V_1 = 0.5I_1$$

$$\Rightarrow y_{11} = \frac{I_1}{V_1}\bigg|_{V_2=0} = \frac{6}{10}$$

$$I_2 R_{5\Omega} = -V_1 \Rightarrow 5I_2 = -V_1 \Rightarrow \frac{I_2}{V_1} = -\frac{1}{5}$$

$$\Rightarrow y_{21} = \frac{I_2}{V_1}\bigg|_{V_2=0} = -\frac{1}{5}$$

Now, let $V_1 = 0$. We have

$$I_2 R_{5\Omega} = V_2 \Rightarrow 5I_2 = V_2 \Rightarrow \frac{I_2}{V_2} = \frac{1}{5}$$

$$\Rightarrow y_{22} = \frac{I_2}{V_2}\bigg|_{V_1=0} = \frac{1}{5}$$

$$-I_2 + 0.5I_1 - I_1 + 0.2V_2 = 0$$

$$I_2 = 0.2V_2 \Rightarrow -0.2V_2 + 0.5I_1 - I_1 + 0.2V_2 = 0 \Rightarrow 0.5I_1 = 0$$

$$\Rightarrow y_{12} = \frac{I_1}{V_2}\bigg|_{V_1=0} = 0$$

The admittance matrix is

$$\boldsymbol{Y} = \begin{bmatrix} 0.6 & 0 \\ -0.2 & 0.2 \end{bmatrix}$$

## Question 6

**Find the impedance matrix $Z$ for the small-signal model of the bipolar current mirror shown in Fig. 5.**

*Question 6 continued on next page...*

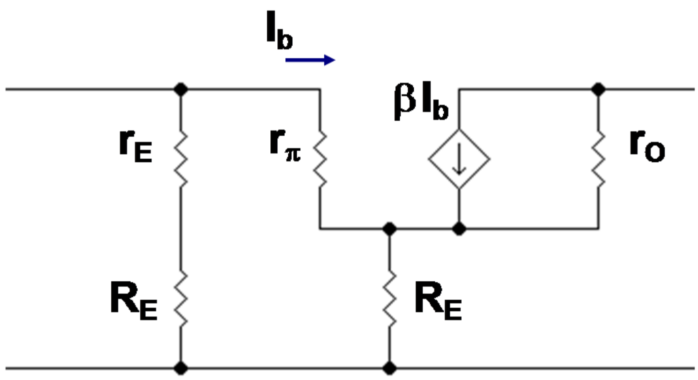


Figure 5: Small-signal model of the bipolar current mirror.

Let $I_2 = 0$. Now,

$$V_1 = I_1[(R_E + r_E) \parallel (R_E + r_\pi)]$$

$$\Rightarrow z_{11} = \frac{V_1}{I_1} = (R_E + r_E) \parallel (R_E + r_\pi) = \frac{(R_E + r_E)(R_E + r_\pi)}{2R_E + r_\pi + r_E}$$

$$V_2 = I_b R_E - \beta I_b r_o \ , \quad I_b = \frac{R_E + r_E}{2R_E + r_\pi + r_E} I_1$$

$$\Rightarrow V_2 = I_b(R_E - \beta r_o) = \frac{(R_E + r_E)(R_E - \beta r_o)}{2R_E + r_\pi + r_E} I_1$$

$$\Rightarrow z_{21} = \frac{V_2}{I_1} = \frac{(R_E + r_E)(R_E - \beta r_o)}{2R_E + r_\pi + r_E}$$

For $I_1 = 0$,

$$V_2 = (I_2 - \beta I_b) r_o + [R_E \parallel (R_E + r_\pi + r_E)] I_2 \ , \quad I_b = -\frac{R_E}{2R_E + r_\pi + r_E} I_2$$

$$\Rightarrow V_2 = I_2[(1 + \frac{\beta R_E}{2R_E + r_\pi + r_E}) r_o + (R_E \parallel (R_E + r_\pi + r_E))]$$

$$\Rightarrow z_{22} = \frac{V_2}{I_2} = (1 + \frac{\beta R_E}{2R_E + r_\pi + r_E}) r_o + \frac{R_E(R_E + r_\pi + r_E)}{2R_E + r_\pi + r_E}$$

$$V_1 = -I_b(R_E + r_E) = \frac{R_E(R_E + r_E)}{2R_E + r_\pi + r_E} I_2$$

$$\Rightarrow z_{12} = \frac{V_1}{I_2} = \frac{R_E(R_E + r_E)}{2R_E + r_\pi + r_E}$$

The impedance matrix:

$$\boldsymbol{Z} = \frac{1}{2R_E + r_\pi + r_E} \begin{bmatrix} (R_E + r_E)(R_E + r_\pi) & (R_E + r_E)(R_E - \beta r_o) \\ R_E(R_E + r_E) & (\beta + 1) R_E r_o + (R_E + r_o)(R_E + r_\pi + r_E) \end{bmatrix}$$

# SOFTWARE QUESTIONS

## Question 7

**Use AC analysis of PSpice to obtain the frequency curves of the impedance and admittance parameters for the double-tuned circuit shown in Fig. 6. You should provide a Bode diagram for each parameter.**

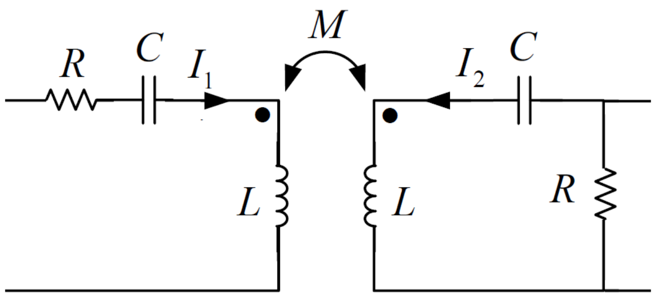


Figure 6: Double-tuned circuit.

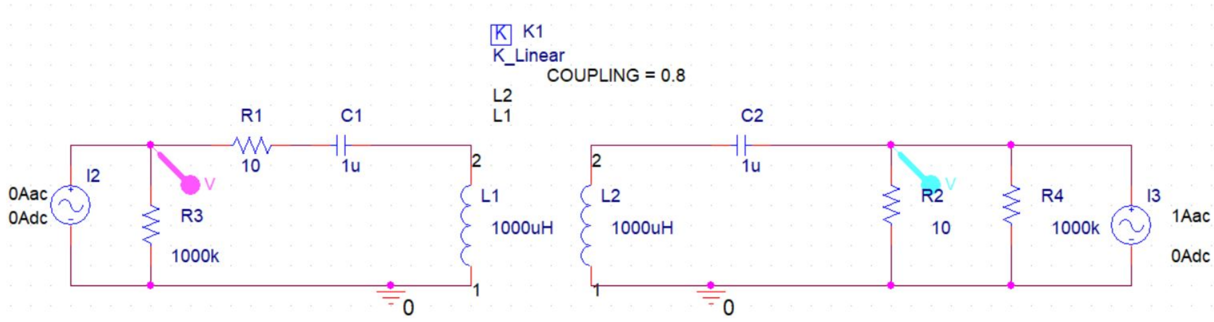


Figure 7: Simulation setup for impedance parameters.



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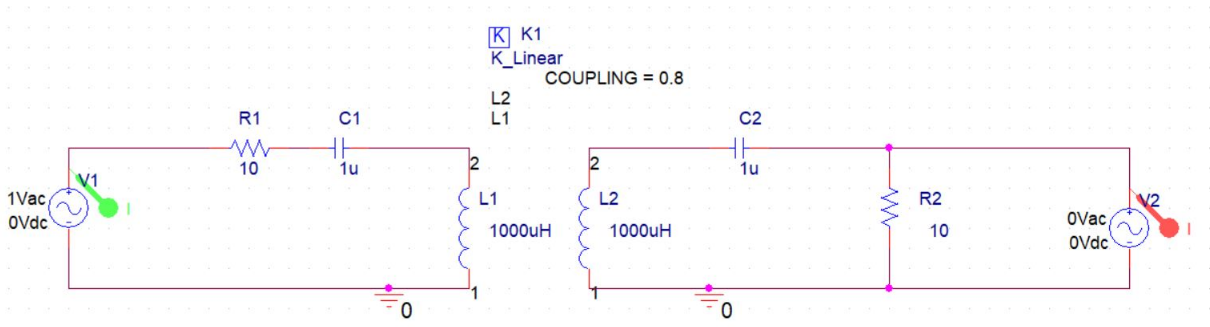


Figure 8: Simulation setup for admittance parameters.

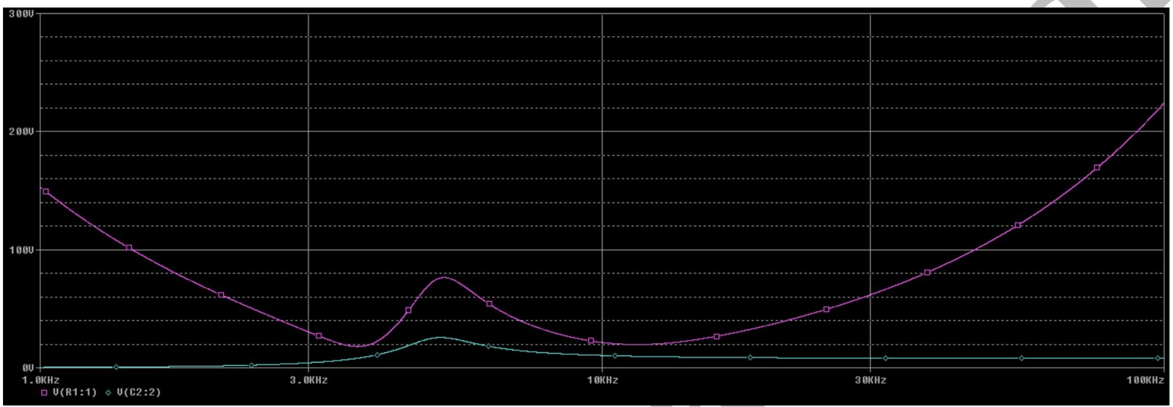


Figure 9: Magnitude response of the parameters $z_{11}$ and $z_{21}$.

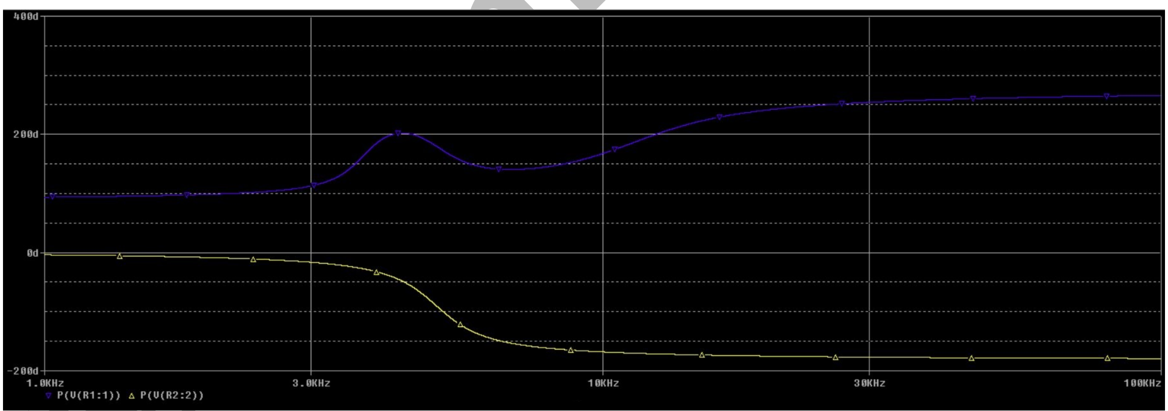


Figure 10: Phase response of the parameters $z_{11}$ and $z_{21}$.

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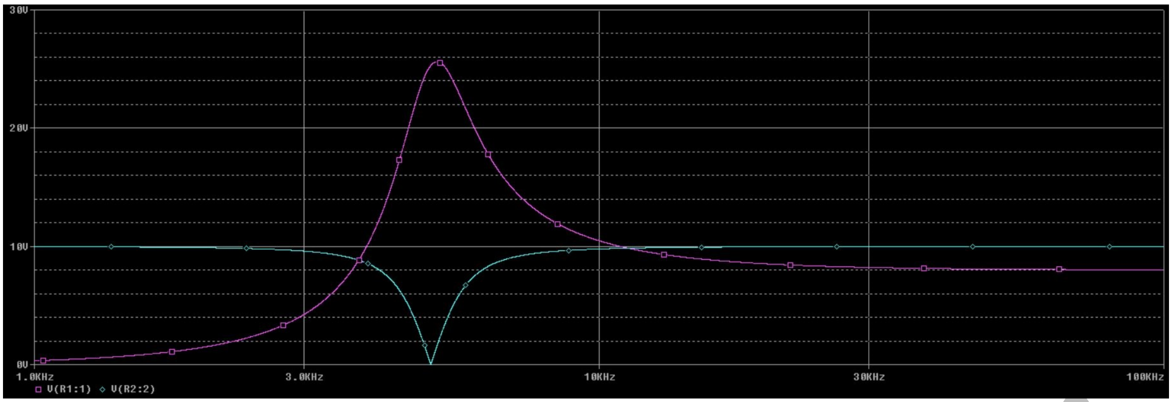


Figure 11: Magnitude response of the parameters $z_{12}$ and $z_{22}$.

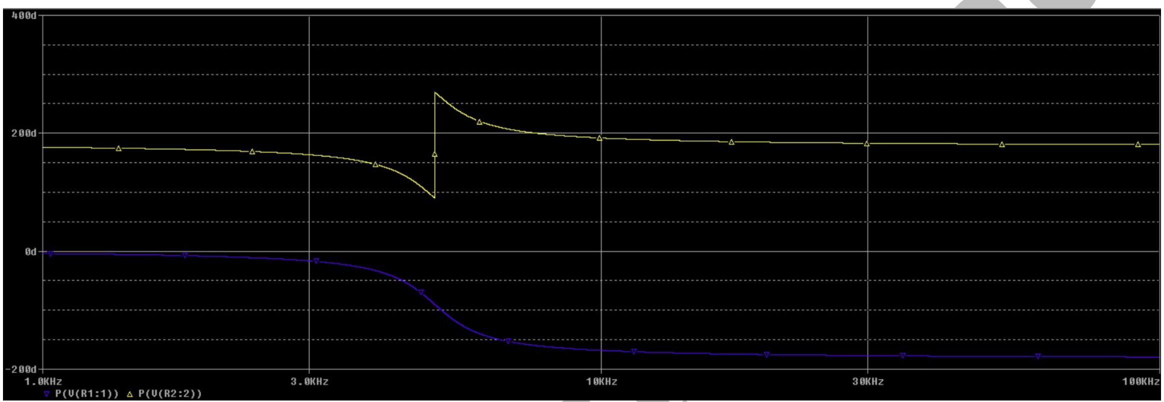


Figure 12: Phase response of the parameters $z_{12}$ and $z_{22}$.

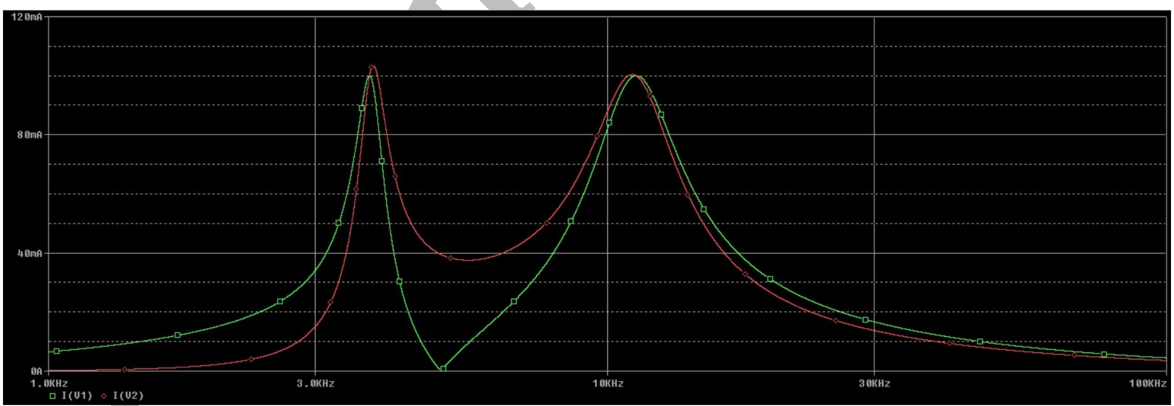


Figure 13: Magnitude response of the parameters $y_{11}$ and $y_{21}$.

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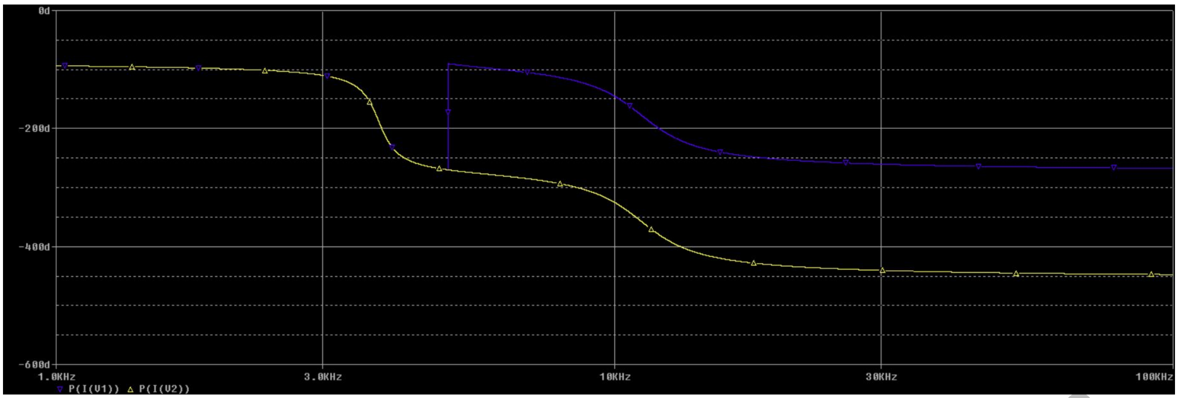


Figure 14: Phase response of the parameters $y_{11}$ and $y_{21}$.

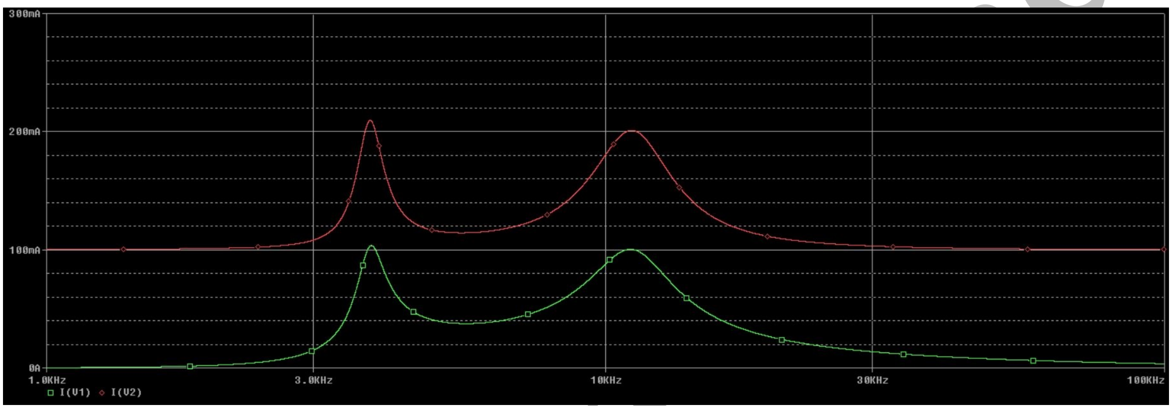


Figure 15: Magnitude response of the parameters $y_{12}$ and $y_{22}$.

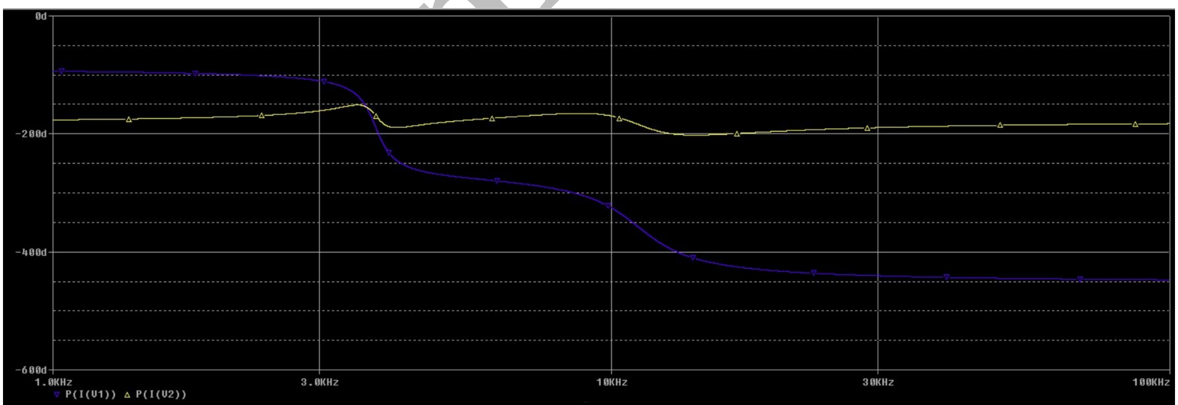


Figure 16: Phase response of the parameters $y_{12}$ and $y_{22}$.

The simulation setup for the impedance and admittance parameters are shown in Figs. 7 and 8, respectively. The magnitude and phase response of each parameter are plotted in Figs. 11-16.

# BONUS QUESTIONS

## Question 8

**Return your answers by filling the LaTeX template of the assignment. If you want to add a circuit schematic, you can draw it directly using TikZ package, or draw it in a secondary application such as Microsoft Visio and then, import it as a figure.**

# EXTRA QUESTIONS

## Question 9

**Feel free to solve the following questions from the book *"Basic Circuit Theory"* by C. Desoer and E. Kuh.**

1. **Chapter 17, question 4.**
2. **Chapter 17, question 5.**
3. **Chapter 17, question 6.**
4. **Chapter 17, question 9.**
5. **Chapter 17, question 10.**
6. **Chapter 17, question 11.**
7. **Chapter 17, question 15.**
8. **Chapter 17, question 16.**