MATHEMATICAL QUESTIONS

Question 1

If the two networks shown in each of Figs. 1 and 2 are equivalent, specify values for L_a, L_b, and L_c. For each equivalent circuit, show that L_a, L_b, and L_c can be non-negative by a proper choice of n

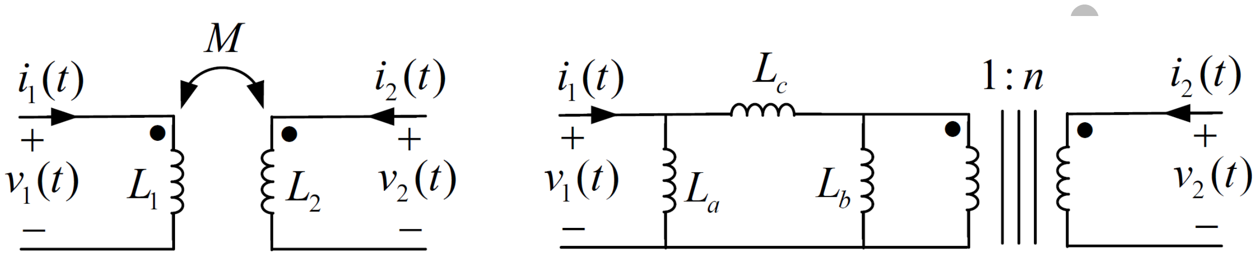

Figure 1: A pair of coupled inductors and its Π equivalent circuit.

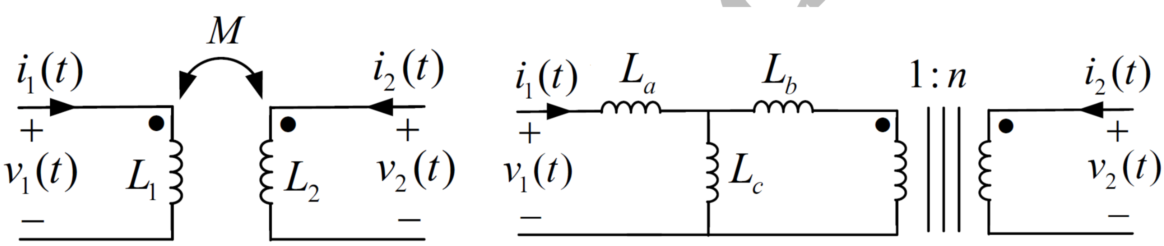

Figure 2: A pair of coupled inductors and its T equivalent circuit.

For Fig. 1, the current-voltage equations ate the ports are

$$i_1(t) = \Gamma_{11} \int_0^t v_1(t')dt' + \Gamma_{12} \int_0^t v_2(t')dt'$$
$$i_2(t) = \Gamma_{21} \int_0^t v_1(t')dt' + \Gamma_{22} \int_0^t v_2(t')dt'$$
$$i_1(t) = \Gamma_a \int_0^t v_1(t')dt' + \Gamma_c \int_0^t [v_1(t') - \frac{v_2(t')}{n}]dt'$$
$$i_2(t) = \frac{1}{n}[\Gamma_b \int_0^t \frac{v_2(t')}{n} dt' + \Gamma_c \int_0^t [\frac{v_2(t')}{n} - v_1(t')]dt']$$

Equating the port equations,

$$\Gamma_a = \Gamma_{11} + n\Gamma_{12}$$
$$\Gamma_b = n^2\Gamma_{22} + n\Gamma_{12}$$
$$\Gamma_c = -n\Gamma_{12}$$

Thus,

$$L_a = \frac{L_1 L_2 - M^2}{L_2 - nM}$$

Question 1 continued on next page...

$$L_b = \frac{L_1L_2 - M^2}{n^2L_1 - nM}$$
$$L_c = \frac{L_1L_2 - M^2}{nM}$$

We know that $L_1L_2 \geq M^2$. To have the inductances non-negative,

$$L_2 - nM \geq 0, \quad n^2L_1 - nM \geq 0 \Rightarrow \frac{M}{L_1} \leq n \leq \frac{L_2}{M}$$

A good choice is $n = \sqrt{\frac{L_2}{L_1}}$.
Similarly, In Fig. 2,

$$v_1(t) = L_1 i_1'(t) + M i_2'(t)$$
$$v_2(t) = M i_1'(t) + L_2 i_2'(t)$$
$$v_1(t) = L_a i_1'(t) + L_c(i_1'(t) + n i_2'(t))$$
$$v_2(t) = n[nL_b i_2'(t) + L_c(i_1'(t) + n i_2'(t))]$$

Thus,

$$L_a = L_1 - \frac{M}{n}$$
$$L_b = \frac{L_2}{n^2} - \frac{M}{n}$$
$$L_c = \frac{M}{n}$$

Again, We know that $L_1L_2 \geq M^2$. To have the inductances non-negative,

$$L_2 - nM \geq 0, \quad n^2L_1 - nM \geq 0 \Rightarrow \frac{M}{L_1} \leq n \leq \frac{L_2}{M}$$

and a feasible choice is $n = \sqrt{\frac{L_2}{L_1}}$.

Question 2

Find the frequency response $H(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)}$ of the double-tuned circuit shown in Fig. 3.

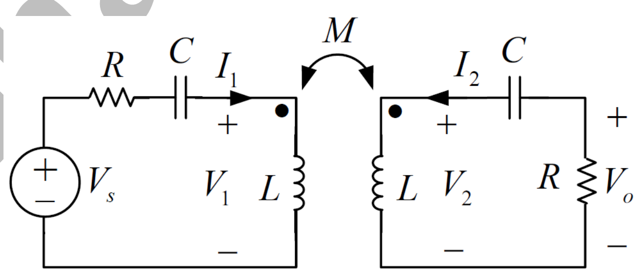

Figure 3: Double-tuned circuit.

Considering the phasor-domain model of coupled inductors

$$V_1 = j\omega L I_1 + j\omega M I_2$$

$$V_2 = j\omega M I_1 + j\omega L I_2$$

KVL in the left side of the circuit yields:

$$-V_s + RI_1 + \frac{1}{j\omega C} I_1 + V_1 = 0 \Rightarrow \big(R + \frac{1}{j\omega C} + j\omega L\big) I_1 + (j\omega M) I_2 = V_s$$

while KVL in the right side of the circuit yields

$$+RI_2 + \frac{1}{j\omega C} I_2 + V_2 = 0 \Rightarrow (j\omega M) I_1 + \big(R + \frac{1}{j\omega C} + j\omega L\big) I_2 = 0$$

Hence,

$$\begin{bmatrix} R + \frac{1}{j\omega C} + j\omega L & j\omega M \\ j\omega M & R + \frac{1}{j\omega C} + j\omega L \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

We have,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{(R + \frac{1}{j\omega C} + jwL)^2 + (\omega M)^2} \begin{bmatrix} (R + \frac{1}{j\omega C} + j\omega L) V_s \\ -j\omega M V_s \end{bmatrix}$$

Finally,

$$\begin{aligned} H(j\omega) &= \frac{V_o(j\omega)}{V_s(j\omega)} \\ &= \frac{-RI_2}{(R + \frac{1}{j\omega C} + j\omega L) I_1 + (j\omega M) I_2} \\ &= \frac{-R \frac{-j\omega M}{(R+\frac{1}{j\omega C}+j\omega L)^2+(\omega M)^2} V_s}{(R + \frac{1}{j\omega C} + j\omega L) \frac{R+\frac{1}{j\omega C}+j\omega L}{(R+\frac{1}{j\omega C}+j\omega L)^2+(\omega M)^2} V_s + (j\omega M) \frac{-j\omega M}{(R+\frac{1}{j\omega C}+j\omega L)^2+(\omega M)^2} V_s} \\ &= \frac{Rj\omega M}{(R + \frac{1}{j\omega C} + jwL)^2 + (\omega M)^2} \\ &= \frac{0.5R}{R + \frac{1}{j\omega C} + j\omega L - j\omega M} - \frac{0.5R}{R + \frac{1}{j\omega C} + jwL + j\omega M} \\ &= \frac{0.5}{1 + jQ_-(\frac{\omega}{\omega_-} - \frac{\omega_-}{\omega})} - \frac{0.5}{1 + jQ_+(\frac{\omega}{\omega_+} - \frac{\omega_+}{\omega})} \end{aligned}$$

, where

$$\omega_+^2 = \frac{1}{LC(1+k)}, \quad Q_+ = \omega_+ \frac{L+M}{R}, \quad \omega_-^2 = \frac{1}{LC(1-k)}, \quad Q_- = \omega_- \frac{L-M}{R}$$

Question 3

For the circuit of Fig. 4, find the input impedance Z_{in} seen from the source terminals.

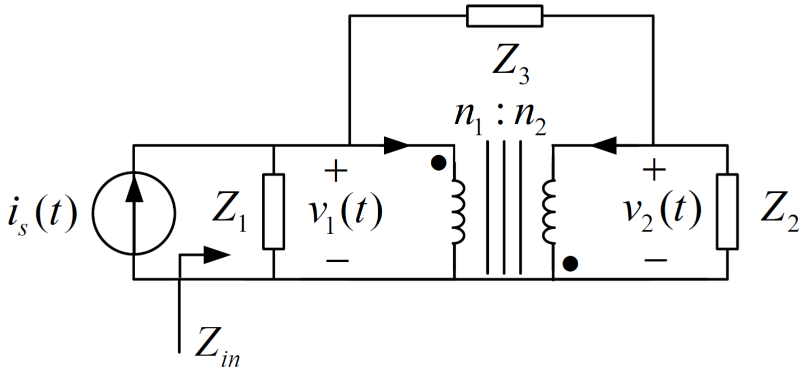

Figure 4: A circuit for which the input impedance is required.

We know that

$$Z_{in} = \frac{V_1}{I_s}$$

Writing a phasor KCL at the node having voltage $v_2(t)$,

$$I_2 = \frac{V_1 - V_2}{Z_3} - \frac{V_2}{Z_2}$$

From the governing equations of the transformer,

$$\frac{V_1}{V_2} = -\frac{n_1}{n_2}$$

So,

$$I_2 = \frac{V_1 + \frac{n_2 V_1}{n_1}}{Z_3} - \frac{-n_2 V_1}{n_1 Z_2}$$

$$I_2 = V_1\left(\frac{n_1 + n_2}{n_1 Z_3} + \frac{n_2}{n_1 Z_2}\right)$$

$$I_2 = V_1\left(\frac{n_1 Z_2 + n_2 Z_2 + n_2 Z_3}{n_1 Z_2 Z_3}\right)$$

On the other hand, the current equation of the transformer gives

$$\frac{I_1}{I_2} = \frac{n_2}{n_1}$$

$$I_1 = \frac{n_2}{n_1} V_1\left(\frac{n_1 Z_2 + n_2 Z_2 + n_2 Z_3}{n_1 Z_2 Z_3}\right)$$

$$I_1 = V_1\left(\frac{n_1 n_2 Z_2 + n_2 n_2 Z_2 + n_2 n_2 Z_3}{n_1 n_1 Z_2 Z_3}\right)$$

Using a KCL at the super node surrounding Z_3,

$$I_s = \frac{V_1}{Z_1} + \frac{V_2}{Z_2} + I_1 + I_2$$

$$I_s = \frac{V_1}{Z_1} + \frac{-n_2 V_1}{n_1 Z_2} + V_1\left(\frac{n_1 n_2 Z_2 + n_2 n_2 Z_2 + n_2 n_2 Z_3}{n_1 n_1 Z_2 Z_3}\right) + V_1\left(\frac{n_1 Z_2 + n_2 Z_2 + n_2 Z_3}{n_1 Z_2 Z_3}\right)$$

Question 3 continued on next page. . .

$$I_s = V_1\left(\frac{1}{Z_1} - \frac{n_2}{n_1 Z_2} + \left(\frac{n_1 n_2 Z_2 + n_2 n_2 Z_2 + n_2 n_2 Z_3}{n_1 n_1 Z_2 Z_3}\right) + \left(\frac{n_1 Z_2 + n_2 Z_2 + n_2 Z_3}{n_1 Z_2 Z_3}\right)\right)$$

$$I_s = V_1 \frac{n_1^2 Z_2 Z_3 - n_1 n_2 Z_1 Z_3 + Z_1(n_1 n_2 Z_2 + n_2^2 Z_2 + n_2^2 Z_3) + n_1 Z_1(n_1 Z_2 + n_2 Z_2 + n_2 Z_3)}{n_1^2 Z_1 Z_2 Z_3}$$

Finally,

$$Z_{in} = \frac{n_1^2 Z_1 Z_2 Z_3}{n_1^2 Z_2 Z_3 - n_1 n_2 Z_1 Z_3 + Z_1(n_1 n_2 Z_2 + n_2^2 Z_2 + n_2^2 Z_3) + n_1 Z_1(n_1 Z_2 + n_2 Z_2 + n_2 Z_3)}$$

which simplifies to

$$Z_{in} = \frac{n_1^2 Z_1 Z_2 Z_3}{n_1^2 Z_2 Z_3 + n_2^2 Z_1 Z_3 + (n_1 + n_2)^2 Z_1 Z_2}$$

Question 4

Calculate I_x and V_2 as labeled in Fig. 5.

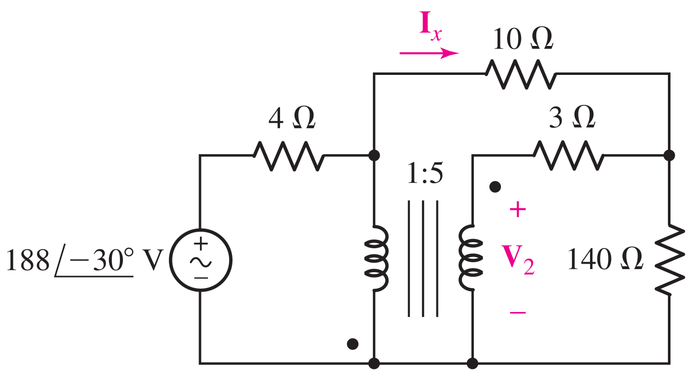

Figure 5: A simple circuit having an ideal transformer.

Let denote the voltage and current of the primary and secondary by V_1, V_2, I_1, and I_2, as shown in Fig. 6. We have,

$$\begin{cases} \frac{V_2}{-V_1} = 5 \\ \frac{-I_2}{I_1} = -\frac{1}{5} \\ V_1 + 4(I_1 + I_x) = 188\angle -30^\circ \\ 3I_2 + V_2 + 140(I_2 - I_x) = 0 \\ -V_1 + 10I_x + 140(I_x - I_2) = 0 \end{cases}$$

Question 4 continued on next page...

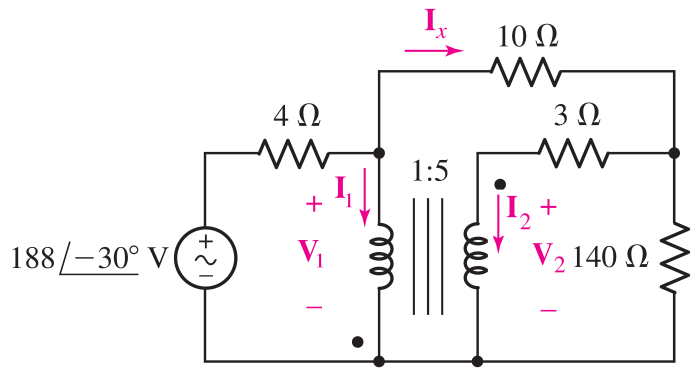

Figure 6: Denoted voltages and currents in the circuit of Fig. 5.

$$\Rightarrow \begin{cases} 5V_1 + V_2 + 0I_1 + 0I_2 + 0I_x = 0 \\ 0V_1 + 0V_2 + I_1 - 5I_2 + 0I_x = 0 \\ V_1 + 0V_2 + 4I_1 + 0I_2 + 4I_x = 162.81 - j94 \\ 0V_1 + V_2 + 0I_1 + 143I_2 - 140I_x = 0 \\ -V_1 + 0V_2 + 0I_1 - 140I_2 + 150I_x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} V_1 = 13.08 + j0.08 = 13.08\angle 0.35^\circ \\ V_2 = -65.42 - j0.4 = 65.42\angle -179.65^\circ \\ I_1 = 31.47 + j0.19 = 31.47\angle 0.35^\circ \\ I_2 = 6.29 + j0.04 = 6.29\angle 0.35^\circ \\ I_x = 5.96 + j0.04 = 5.96\angle 0.35^\circ \end{cases}$$

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Question 5

With respect to the circuit depicted in Fig. 7,

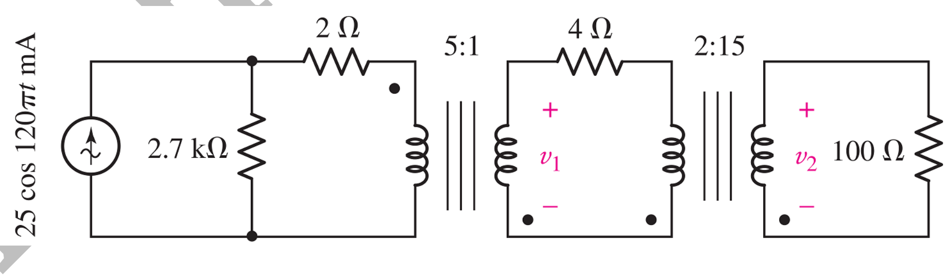

Figure 7: A circuit with two ideal transformers.

Question 5 continued on next page...

(a) Calculate the voltages v_1 and v_2.

The impedance seen from the primary of the transformer $2:15 \equiv 1:7.5$ is $\frac{100}{7.5^2} = 1.778$. In the next stage, the impedance seen from the primary of the $5:1 \equiv 1:0.2$ transformer is $\frac{1.778+4}{0.2^2} = 144.450$. Therefore, the current flowing into the primary of the $5:1$ transformer is $0.025\frac{2700}{2700+144.450+2} = 0.0237$. Further, $V_1 = -0.2 \times (144.450 \times 0.0237) = -0.685$. This gives $V_2 = 7.5 \times (\frac{1.778}{1.778+4})(-0.685) = -1.582$ V. Finally,

$$v_1(t) = -0.685\cos(120\pi t)$$

and

$$v_2(t) = -1.582\cos(120\pi t)$$

.

(b) Compute the average power delivered to each resistor.

$$P_{100} = \frac{1}{2}\frac{1.582^2}{100} = 0.0125 \text{ W}$$

$$P_4 = \frac{1}{2}\frac{(-0.685 - \frac{-1.582}{7.5})^2}{4} = 0.0281 \text{ W}$$

$$P_2 = \frac{1}{2} \times 2 \times 0.0237^2 = 0.000562 \text{ W}$$

$$P_{2700} = \frac{1}{2} \times 2700 \times (0.025 - 0.0237)^2 = 0.00228 \text{ W}$$

SOFTWARE QUESTIONS

Question 6

A real transformer is usually modeled as the circuit of Fig. 7, where L_p is the primary leakage inductance, R_p is the primary copper loss, R_c is the core losses due to eddy currents and hysteresis, L_m is the magnetization inductance, L_s is the secondary leakage inductance, and R_s is the secondary copper loss. Use CircuitLab, which is an online circuit simulation platform, to investigate the impact of L_p, R_p, R_c, L_m, L_s, and R_s on the transformer performance. You may plot the voltages of the primary and secondary versus time to investigate the impact of each item.

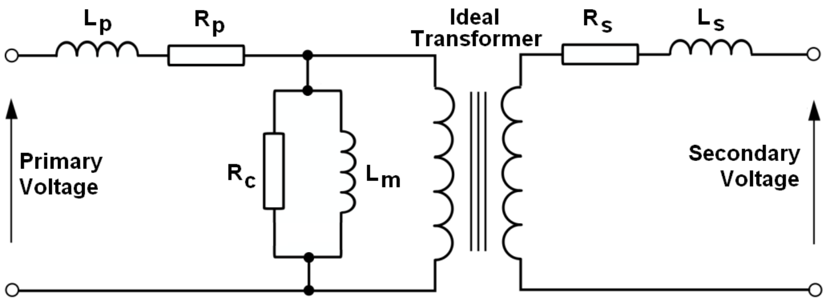

Figure 8: Real transformer equivalent circuits.

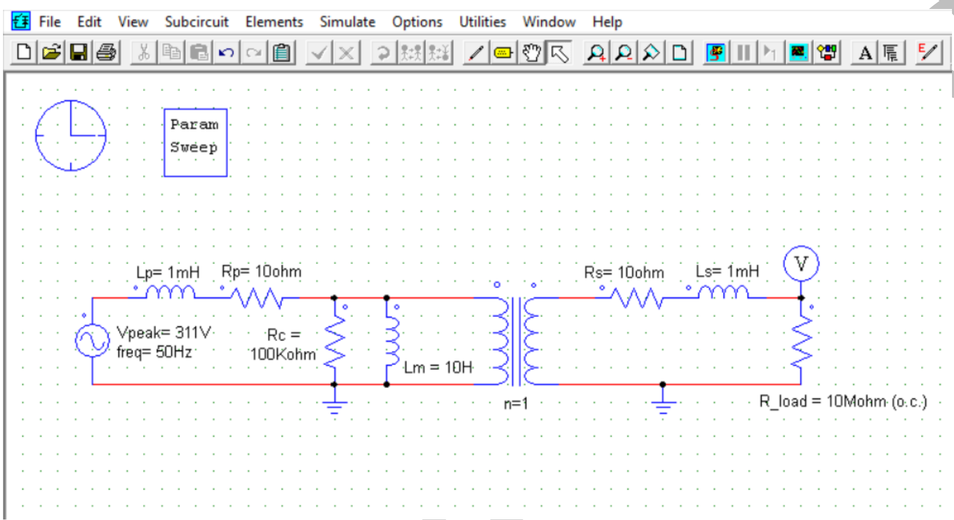

Figure 9: Scematic of the read transformer in the simulator editor.

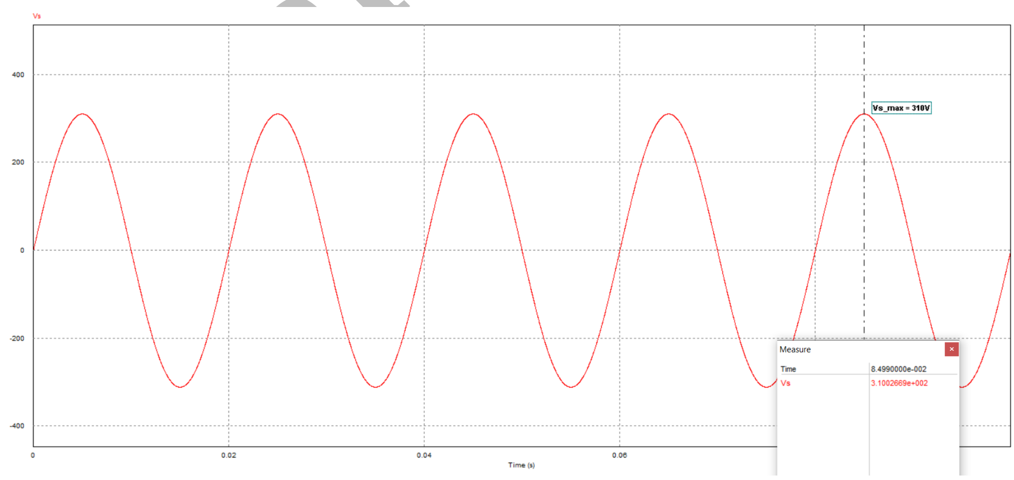

Figure 10: The secondary voltage corresponding to the default values.

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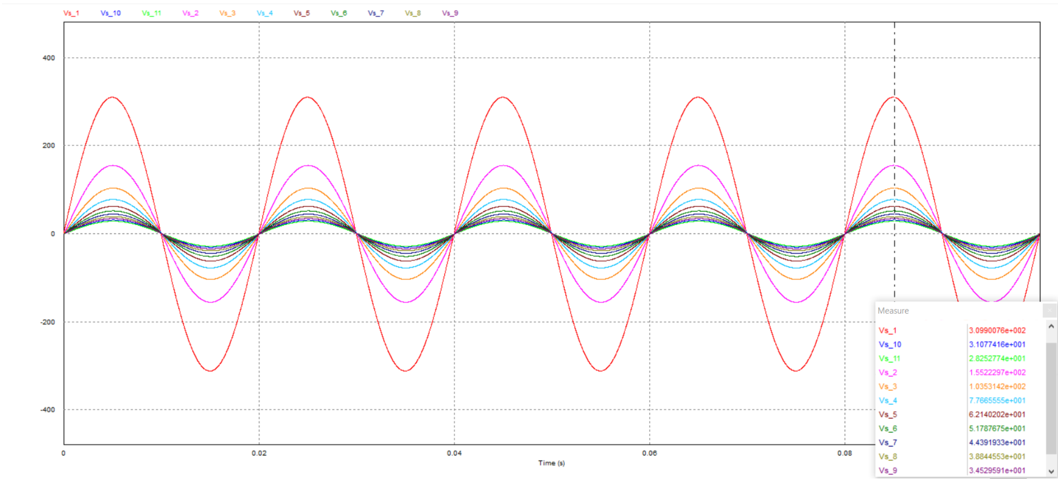

Figure 11: Impact of L_p on the secondary voltage, where $L_p = 0.0001 : 10 : 100$ H.

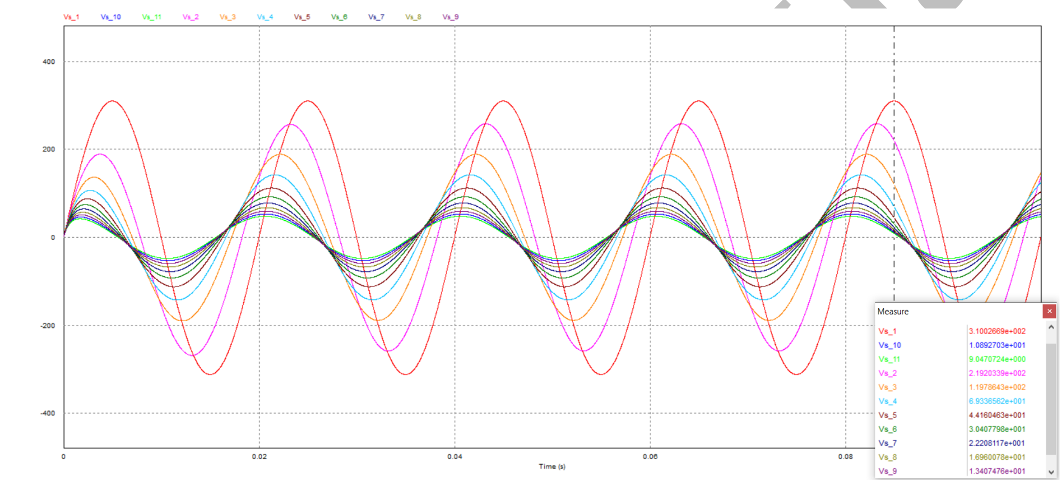

Figure 12: Impact of R_p on the secondary voltage, where $R_p = 0.01 : 2 : 20\ \text{K}\Omega$.

Question 6 continued on next page. . .

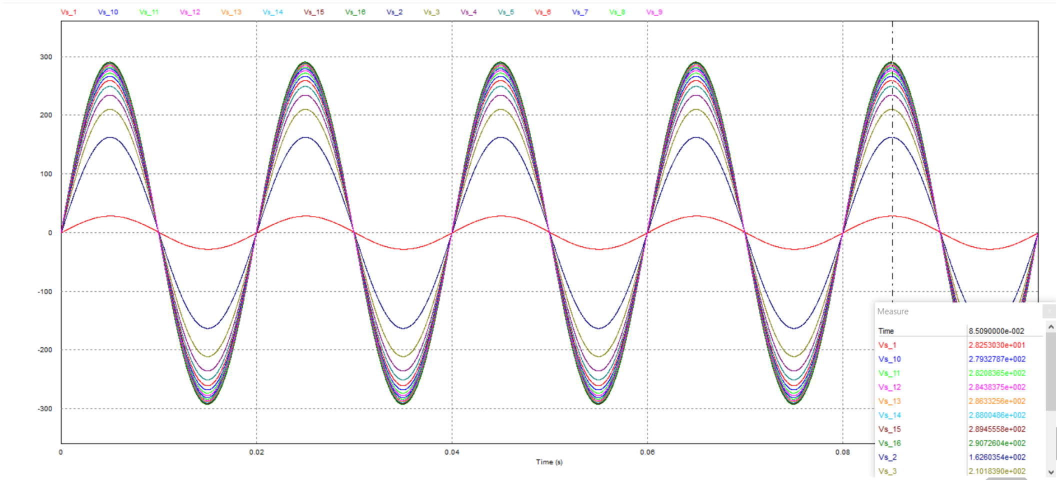

Figure 13: Impact of R_c on the secondary voltage, where $R_c = 1 : 10 : 150\ \Omega$.

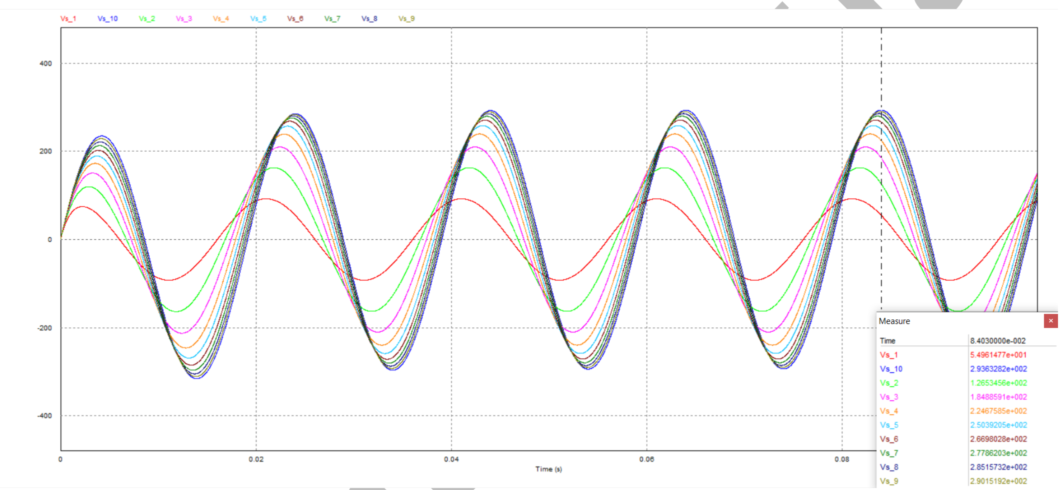

Figure 14: Impact of L_m on the secondary voltage, where $L_m = 10 : 10 : 100$ mH.

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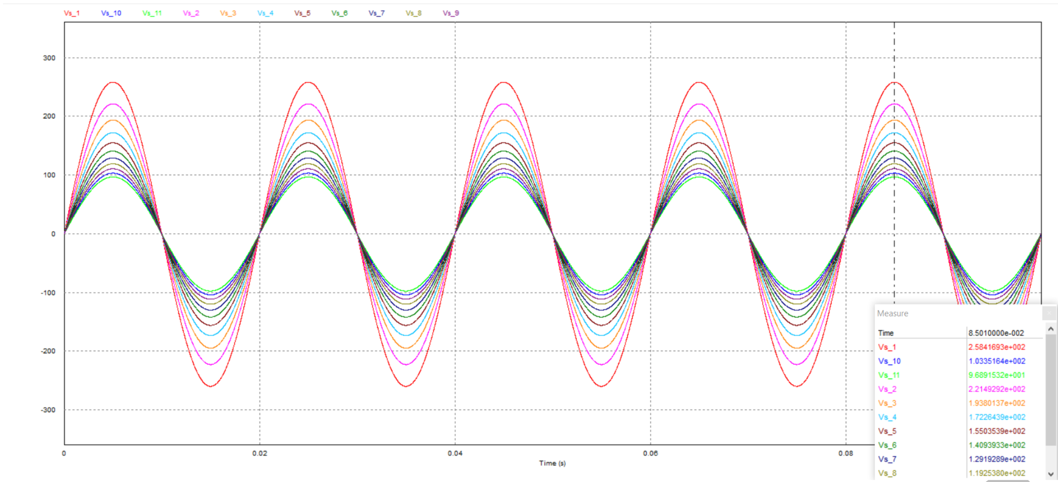

Figure 15: Impact of L_s on the secondary voltage, where $R_s = 10:10:200\ \Omega$.

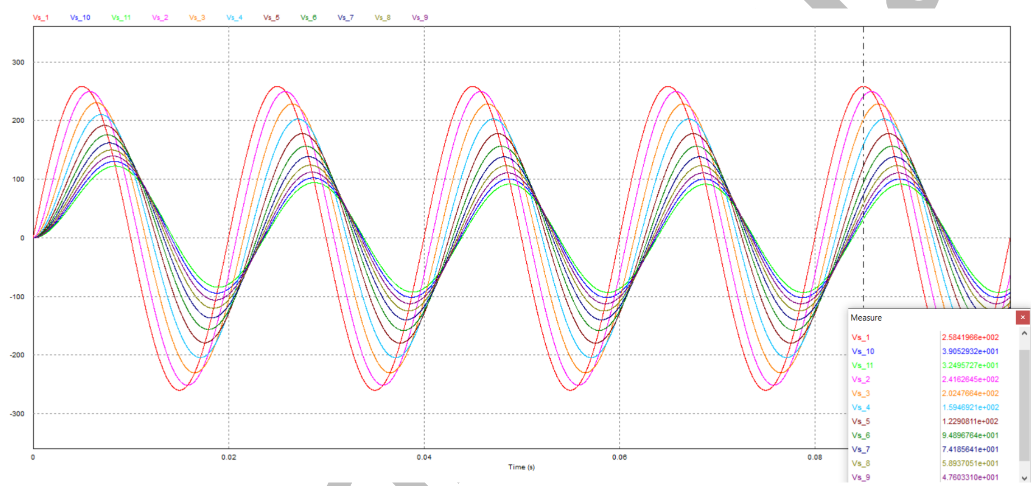

Figure 16: The impact of L_s on the secondary voltage, where $L_s = 1:100:1000$ mH.

The drawn schematic and some default values for the elements are shown in Fig. 9. The default values are chosen such that no considerable impairment is observed for the secondary voltage. Fig. 10 shows the secondary voltage corresponding to the default values. The impact of R_p and L_p on the secondary voltage for an almost open-circuit load are shown in Figs. 11 and 12. Depending on the values of the elements, the secondary voltage may be attenuated, lagged, or distorted with respect to the near-optimal voltage plotted in Fig. 10. Similarly, the impacy of R_c and L_m are illustrated in Figs. 13 and 14. To observe the impact of R_s and L_s, we connect a lower resistance load to the secondary to draw current from the transformer. The obtained results are shown in Figs. 15 and 16.

BONUS QUESTIONS

Question 7

Return your answers by filling the LaTeX template of the assignment. If you want to add a circuit schematic, you can draw it directly using TikZ package, or draw it in a secondary application such as Microsoft Visio and then, import it as a figure.

EXTRA QUESTIONS

Question 8

Feel free to solve the following questions from the book *"Engineering Circuit Analysis"* by W. Hayt, J. Kemmerly, and S. Durbin.

1. **Chapter 13, question 5.**
2. **Chapter 13, question 18.**
3. **Chapter 13, question 21.**
4. **Chapter 13, question 22.**
5. **Chapter 13, question 44.**
6. **Chapter 13, question 48.**