MATHEMATICAL QUESTIONS

Question 1

For the circuit of Fig. 1,

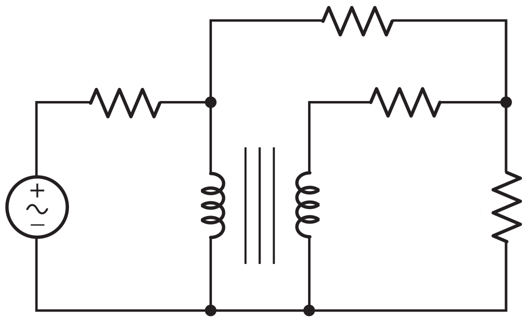
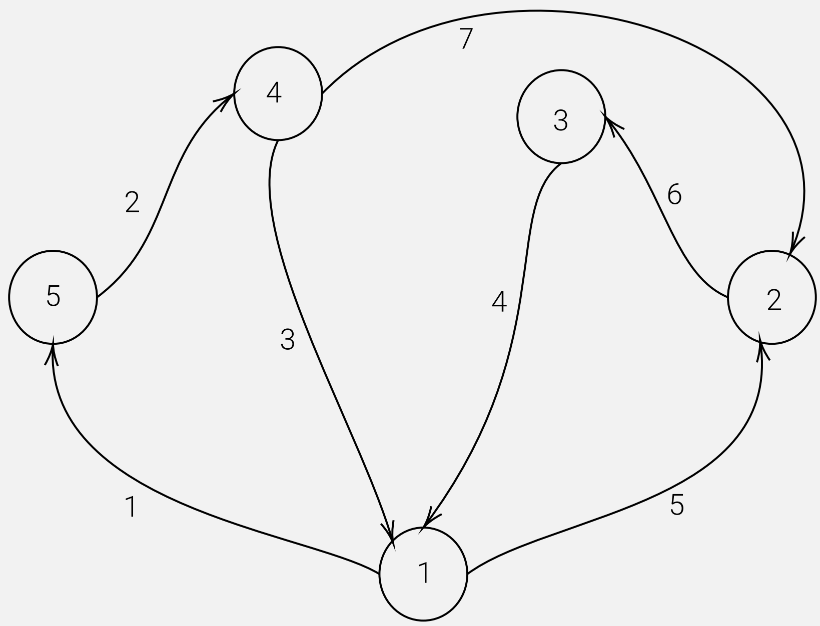


Figure 1: A sample circuit.

(a) Draw the circuit graph.



Question 1 continued on next page...

(b) Find a reduced node-to-branch incident matrix $\mathbf{A}$.

Selecting node 5 as the reference node,

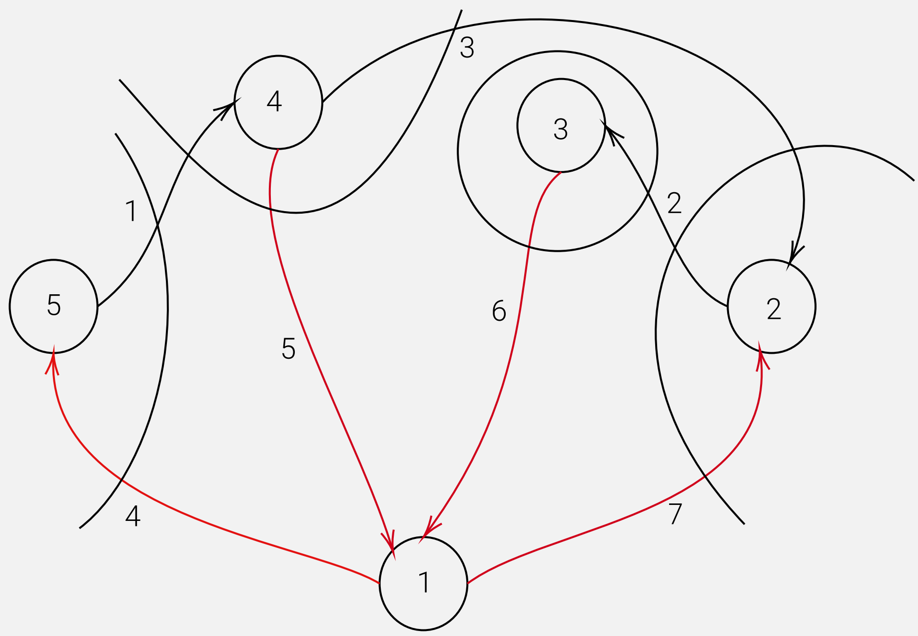
$$A = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Find a reduced mesh-to-branch incident matrix $\mathbf{M}$.

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}$$

(d) Find a fundamental cut-set matrix $\mathbf{Q}$.

First we choose a tree that includes branches 1,3,4, and 5, and then, determine a cut-set for each tree branch.



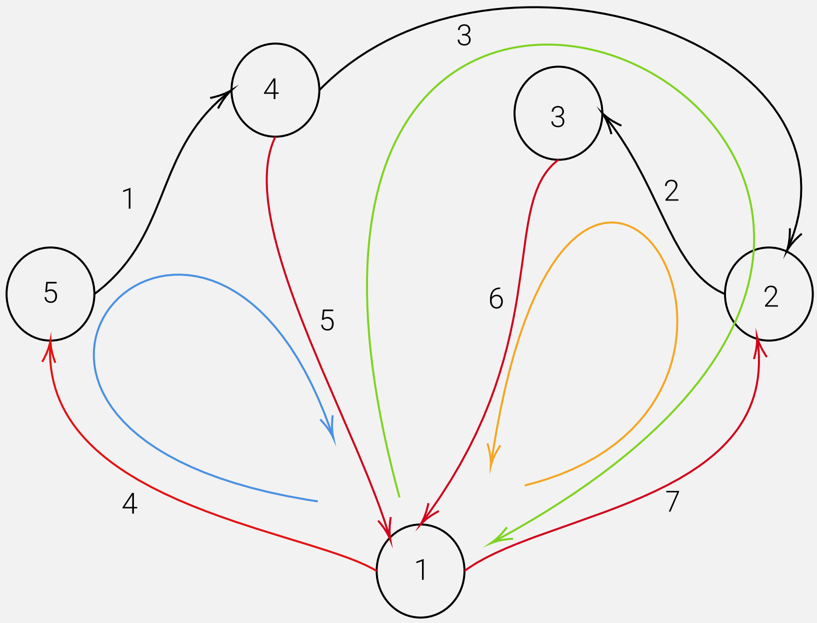
The associative fundamental cut-set matrix Q is

$$Q = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 1 continued on next page…

(e) Find a fundamental loop matrix $\mathbf{B}$.

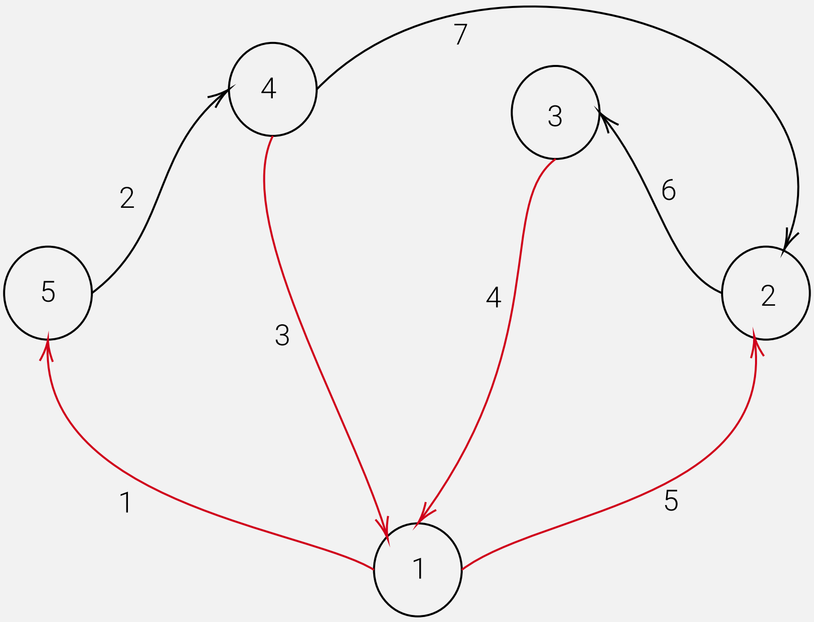
For the same tree highlighted in red,



$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \end{bmatrix}$$

(f) Can you introduce a tree for which the matrices $\mathbf{A}$ *and* $\mathbf{Q}$ *are equal?*

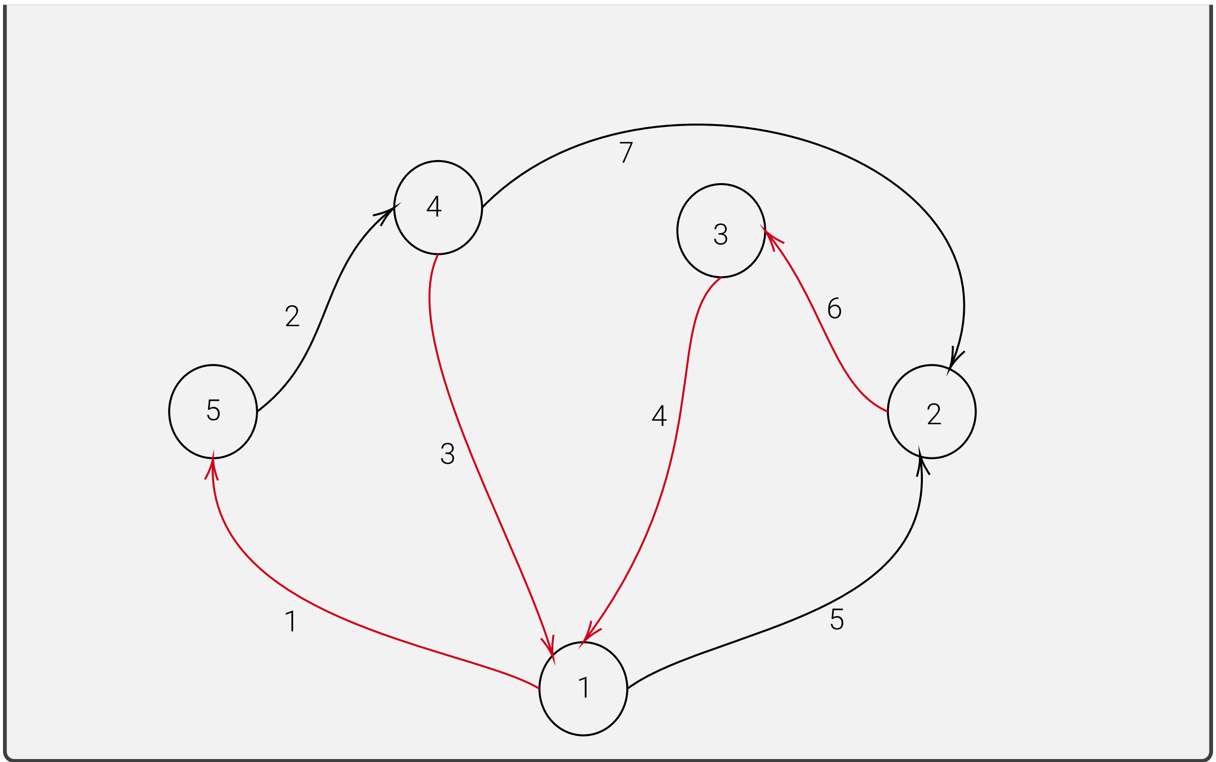
In matrix A, rows are related to nodes of the graph and columns are related to branches in an arbitrary order. In matrix Q, rows are related to cut-sets and columns are related to branches. So if we want matrices A and Q equal, the set of branches that are connected to each node should be a cut-set. Therefore, every node except one of them should be connected to only one branch of tree, so we need a node in graph that is connected to all of the other nodes. Node 1 is the desired node and the desired tree is the set of branches $\{1, 3, 4, 5\}$ shown below.



(g) Can you introduce a tree for which the matrices $\mathbf{M}$ *and* $\mathbf{B}$ *are equal?*

In matrix M, rows are related to meshes of the graph and columns are related to branches in an arbitrary order. In matrix B, rows are related to loops and columns are related to branches. So, if we want matrices B and M equal, every mesh should be a loop. Therefore, every mesh should include only one link and other branches of it should belong to tree. So, the desired tree can be the set of branches $\{1, 3, 4, 6\}$ in the graph below.

Question 1 continued on next page...



Question 2

Prove that the branch voltages of a tree of a given circuit graph provide a set of linearly independent voltages.

Assume that some tree branch voltages are linearly dependent. Then, they should provide a KVL around a loop. This contradicts with the fact that there is no loop over tree.

Question 3

The circuit of Fig. 2 includes LTI resistors and a voltage source. In an experimental measurement, we set $R_2 = 1\,\Omega$, and find that $v_1 = 4$ V, $i_1 = 1$ A, and $v_2 = 1$ V. In a second measurement, we set $R_2 = 2\,\Omega$, and find that $v_1 = 2$ V and $i_1 = 1.2$ A, but we forget to measure v_2. Can you determine the value of v_2 in the second experiment? The inside of the sub-circuit N remains unchanged for the two experiments.

Question 3 continued on next page. . .

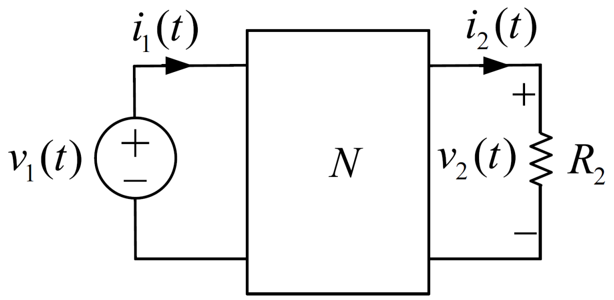


Figure 2: An LTI resistive network with a driving voltage source.

According to the Tellegan's theorem and resistive nature of the network,

$$\Rightarrow \; -v_1\hat{i}_1 + \sum_k v_k\hat{i}_k + v_2\hat{i}_2 = -\hat{v}_1 i_1 + \sum_k \hat{v}_k i_k + \hat{v}_2 i_2 = 0$$
$$\Rightarrow -v_1\hat{i}_1 + \sum_k R_k i_k\hat{i}_k + v_2\hat{i}_2 = -\hat{v}_1 i_1 + \sum_k R_k\hat{i}_k i_k + \hat{v}_2 i_2 = 0$$

So,

$$-v_1\hat{i}_1 + v_2\hat{i}_2 = -\hat{v}_1 i_1 + \hat{v}_2 i_2$$

First measurement yields

$$v_1 = 4\text{ V} \quad i_1 = 1\text{ A} \quad v_2 = 1\text{ V} \quad i_2 = \frac{v_2}{R_2} = 1\text{ A}$$

while for the second measurement,

$$\hat{v}_1 = 2\text{ V} \quad \hat{i}_1 = 1.2\text{ A} \quad \hat{R}_2 = 2\Omega \quad i_2 = \frac{\hat{v}_2}{\hat{R}_2} = \frac{\hat{v}_2}{2}$$

Place the parameters:

$$-4 \times 1.2 + 1 \times \frac{\hat{v}_2}{2} = -2 \times 1 + 1 \times \hat{v}_2 \Rightarrow \hat{v}_2 = -5.6V$$

Question 4

Draw the dual circuit of the circuit shown in Fig. 3 and write at least two dual circuit equations for the two circuits.

Question 4 continued on next page...

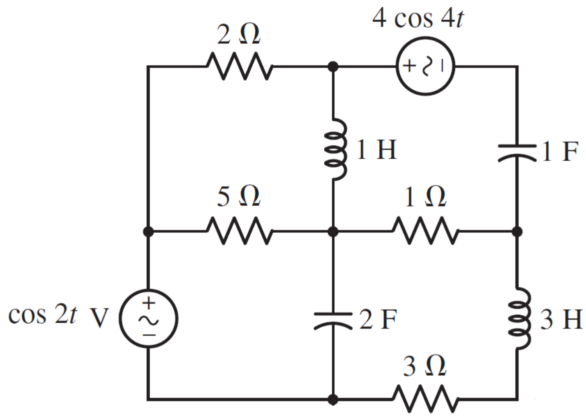
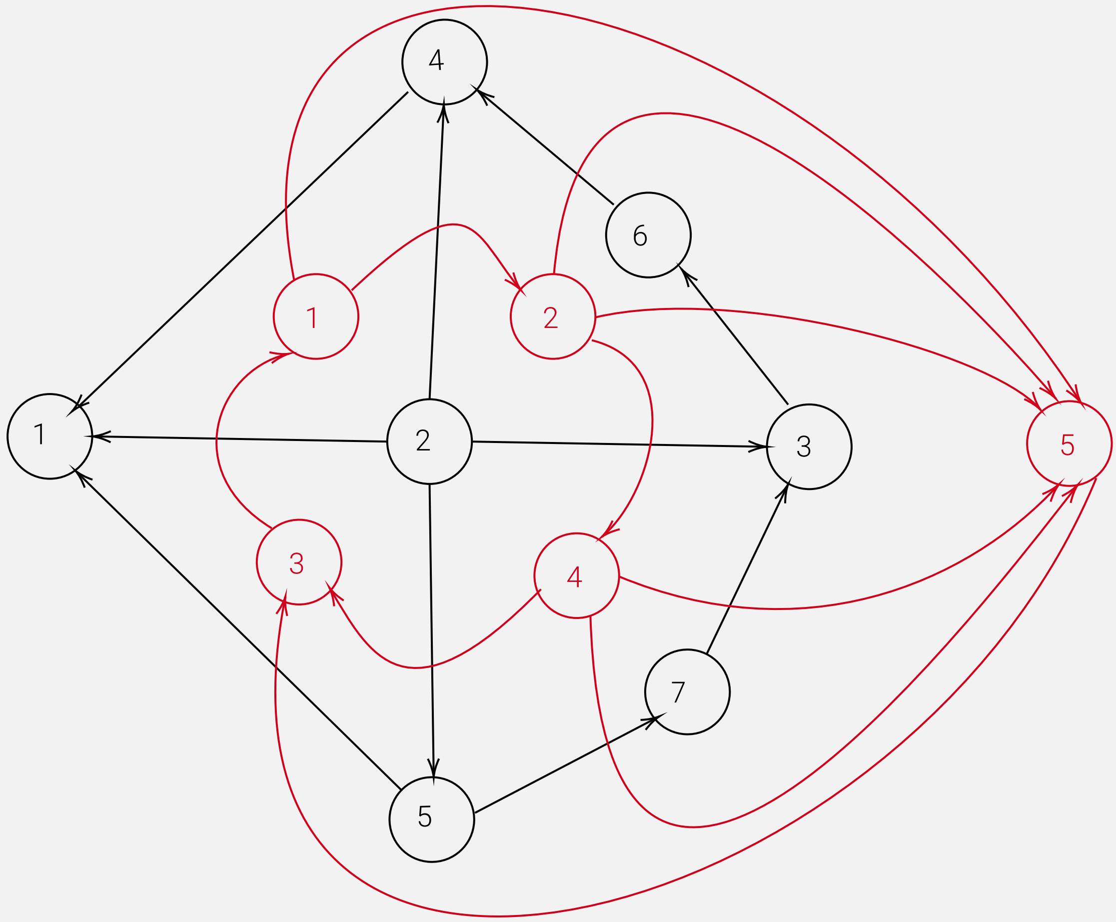
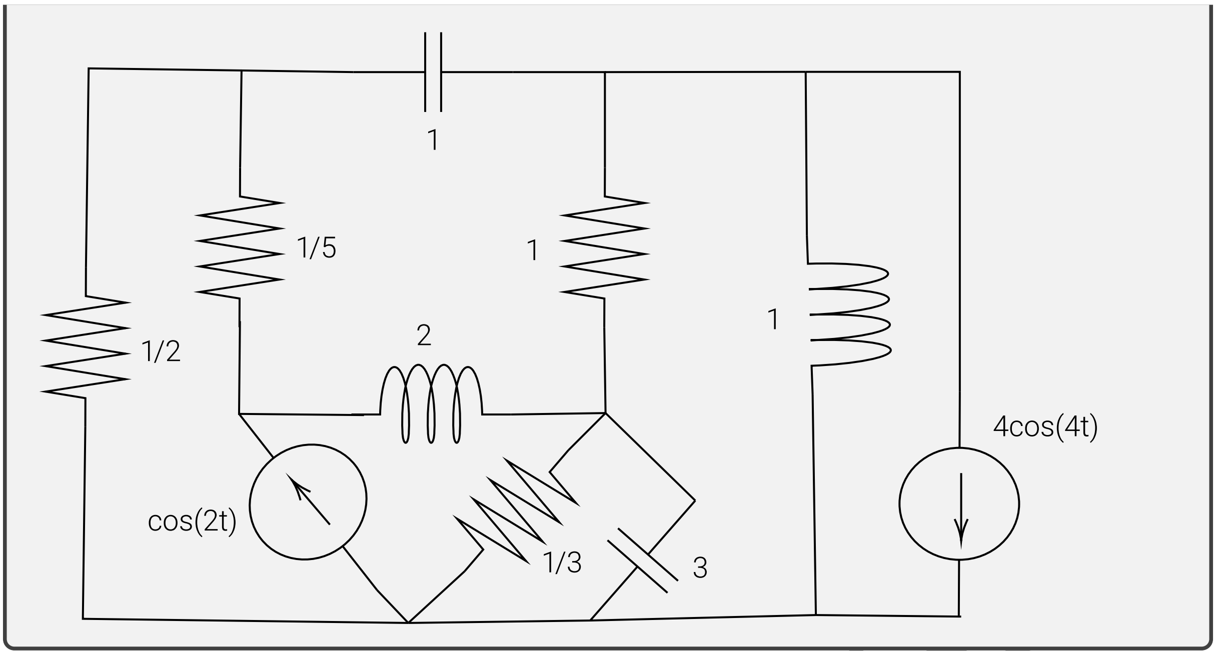


Figure 3: A circuit for which the dual network is required.

The circuit graph and its corresponding dual graph are as follows.



The dual circuit is drawn below.



Question 5

Write the KCL and KVL equations corresponding to the fundamental cut sets and loops of the circuit graph shown in Fig. 4 having the highlighted tree.

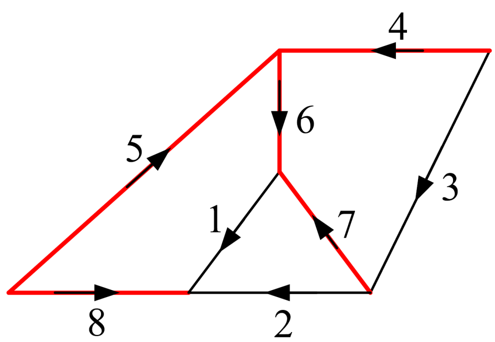


Figure 4: A circuit graph and one of its associated trees.

According to Fig. 5, we have
KCL:

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix}, \quad \boldsymbol{Qj} = \boldsymbol{0}, \quad \begin{cases} j_3 + j_4 = 0 \\ -j_1 - j_2 + j_5 = 0 \\ -j_1 - j_2 + j_3 + j_6 = 0 \\ j_2 - j_3 + j_7 = 0 \\ j_1 + j_2 + j_8 = 0 \end{cases}$$

Question 5 continued on next page...

$$\boldsymbol{Q^T} = \begin{bmatrix} 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix}$$

$$\boldsymbol{v} = \boldsymbol{Q}^T\boldsymbol{e}, \quad \begin{cases} v_1 = -e_2 - e_3 + e_5 \\ v_2 = -e_2 - e_3 + e_4 + e_5 \\ v_3 = e_1 + e_3 - e_4 \\ v_4 = e_1 \\ v_5 = e_2 \\ v_6 = e_3 \\ v_7 = e_4 \\ v_8 = e_5 \end{cases}$$

KVL:

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix},$$

$$\boldsymbol{Bv} = \boldsymbol{0}, \quad \begin{cases} v_1 + v_5 + v_6 - v_8 = 0 \\ v_2 + v_5 + v_6 - v_7 - v_8 = 0 \\ v_3 - v_4 - v_6 + v_7 = 0 \end{cases}$$

$$\boldsymbol{B}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \quad \boldsymbol{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix}, \quad \boldsymbol{j} = \boldsymbol{B}^T\boldsymbol{i}, \quad \begin{cases} j_1 = i_1 \\ j_2 = i_2 \\ j_3 = i_3 \\ j_4 = -i_3 \\ j_5 = i_1 + i_2 \\ j_6 = i_1 + i_2 - i_3 \\ j_7 = -i_2 + i_3 \\ j_8 = -i_1 - i_2 \end{cases}$$

Question 5 continued on next page...

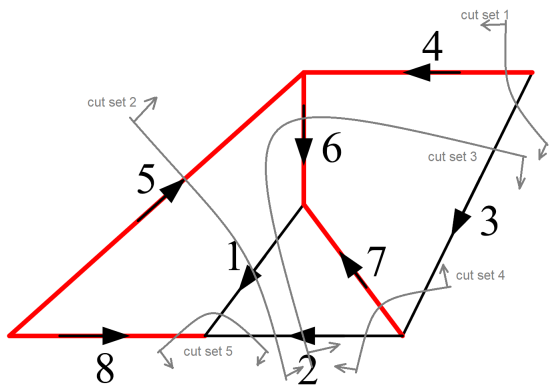


Figure 5: Cut-sets for the highlighted tree branches.

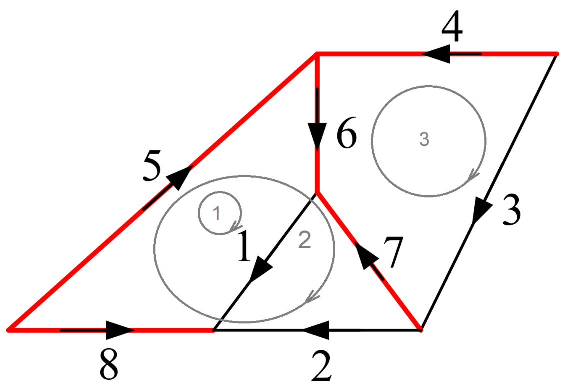


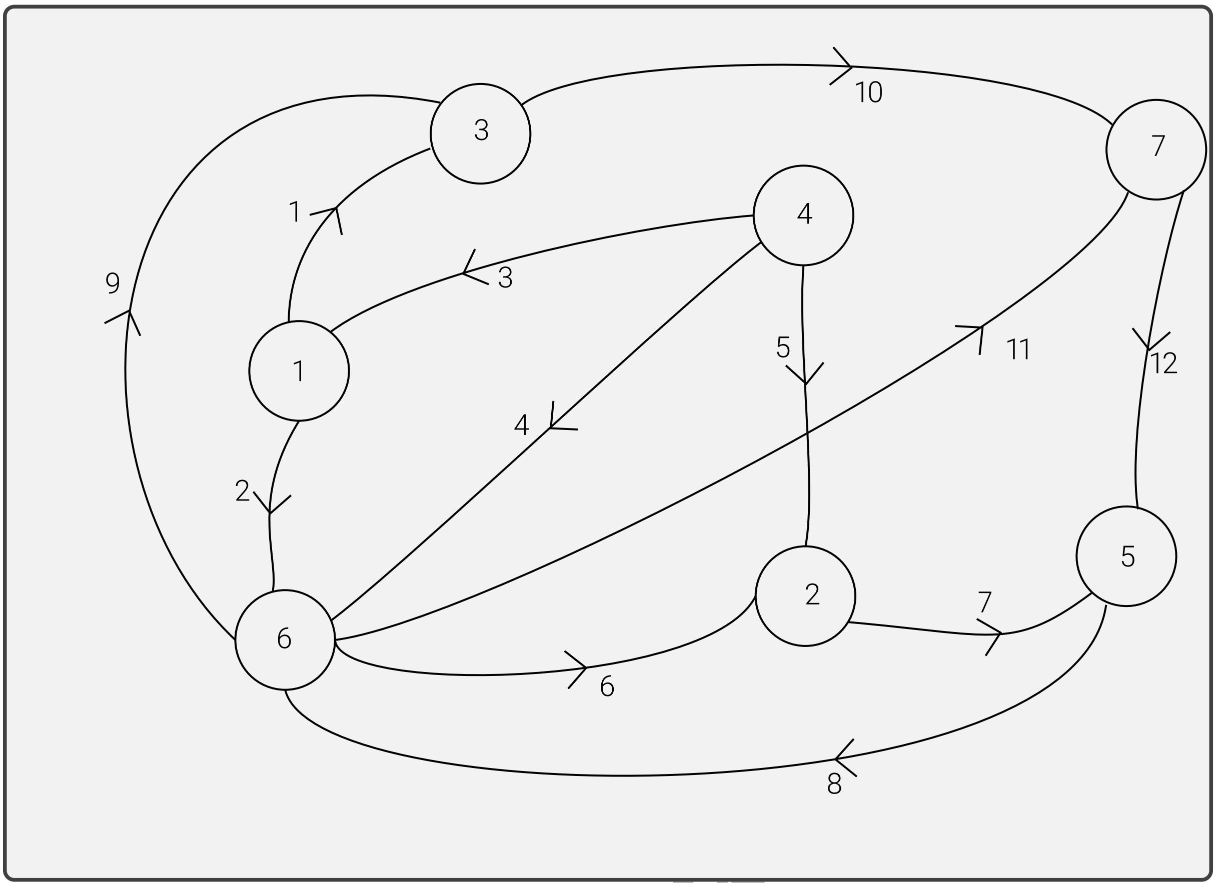
Figure 6: Loops for the highlighted link branches.

Question 6

Draw a directed graph whose node-to-branch incidence matrix $\mathbf{A}_a$ is given by

$$\mathbf{A}_a = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

Question 6 continued on next page...



SOFTWARE QUESTIONS

Question 7

Dijkstra's conventional algorithm is a systematic method to find the shortest path between two given nodes of a weighted graph. However, a more common variant of the algorithm fixes a single node as the reference node and finds shortest paths from the source to all other nodes in the graph, producing a shortest-path tree. Implement Dijkstra's algorithm as a MATLAB function and use it to find a tree of a given connected circuit graph.
Note: A circuit graph is a special weighted graph, where all the edges have a same weight.
Note: A graph can be represented by a matrix. In fact, for the graph $G(\mathbf{N} = \{1, 2, \cdots, n\}, \mathbf{E})$ with n node, the representing matrix of the graph is $A_{n\times n|} = [a_{ij}]$, where a_{ij} is 1 if $(i, j) \in \mathbf{E}$, and 0 otherwise.

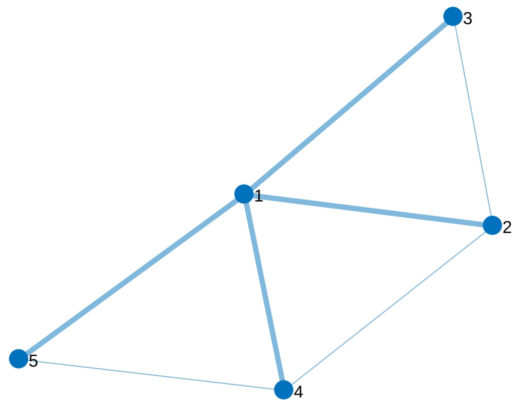


Figure 7: A graph and one of its trees.

Here is a MATLAB function that finds the spanning tree of a circuit graph.

```
%Minimum Spanning Tree Algorithm
function [route, tree]=mst(top) %top=topology matrix, start= initial node
[x y]=size(top);
if ne(x,y) % check if matrix is square
    fprintf('enter square matrix');
    return;end
for i=1 : size(top,1) %assigning large cost(inf) to not connected edges
    for j=1 : size(top,1)
        if top(i,j)==0; top(i,j)=inf;
        end
    end
end
route=zeros(x-1,3); %initialize route matrix (first node, second node, cost)
C=(1); %initial node
C_N=(1:x); % all nodes
C_N(:,1)=[]; %removing selected node
for k = 2:x
    counter=0;
    min=inf; %can be set to infinity
    for i=C
        count=0;
        for j=C_N
            count=count+1;
            if min>top(i,j)
                min=top(i,j);
                s=i;e=j;counter=count; %s=start node e=end node counter=node to remove
            end
        end
    end
    C(end+1)=e; % add node
    C_N(:,counter)=[]; % remove added none
    route(k-1,:)=[s e min]; % [start_node end_node cost]
end

% make the tree matrix from its routes
tree=zeros(size(top));
for i=1:size(route,1)
    tree(route(i,1), route(i,2))=1;
end


end
```

You may use the following mfile to call the developed function and see its results.

Question 7 continued on next page. . .

```
clear all
clc

% sample circuit graph
top = [0 1 1 1 1;
       1 0 1 1 0;
       1 1 0 0 0;
       1 1 0 0 1;
       1 0 0 1 0];

% find the tree
[route, tree]=mst(top);

% show the tree
showTree(top, tree)
```

, where the function below is used to show the graph and its tree.

```
function showTree(ingraph, intree)

% convert the input graph to matlab graphs
sg = [];
dg = [];
wg = [];
for i=1:size(ingraph,1)
    for j=i:size(ingraph,2)
        if (ingraph(i,j)~=0)
            sg = [sg i];
            dg = [dg j];
            wg = [wg 1];
        end
    end
end
G = graph(sg,dg,wg);

% convert the input tree to matlab graphs
st = [];
dt = [];
wt = [];
for i=1:size(intree,1)
    for j=i:size(intree,2)
        if (intree(i,j)~=0)
            st = [st i];
            dt = [dt j];
            wt = [wt 1];
        end
    end
end
T = graph(st,dt,wt);

% plot graph
p = plot(G);
% hightlight tree
highlight(p,T)

end
```

Sample output of the codes are shown in Fig. 7.

BONUS QUESTIONS

Question 8

Return your answers by filling the LaTeX template of the assignment. If you want to add a circuit schematic, you can draw it directly using TikZ package, or draw it in a secondary application such as Microsoft Visio and then, import it as a figure.

EXTRA QUESTIONS

Question 9

Feel free to solve the following questions from the book *"Basic Circuit Theory"* by C. Desoer and E. Kuh.

1. **Chapter 9, question 1.**
2. **Chapter 9, question 3.**
3. **Chapter 9, question 4.**
4. **Chapter 9, question 9.**