MATHEMATICAL QUESTIONS

Question 1

Find the unidirectional Laplace transform of the following functions.

(a) $f(t) = 2|K|e^{-at}\cos(\beta t + \angle K)u(t)$.

$$g(t) = \cos(\beta t + \angle K) = \cos(\beta t)\cos(\angle K) - \sin(\beta t)\sin(\angle K)$$

$$\Rightarrow G(s) = \mathcal{L}\{\cos(\beta t + \angle K)\} = \tfrac{s}{s^2+\beta^2}\cos(\angle K) - \tfrac{\beta}{s^2+\beta^2}\sin(\angle K)$$

$$f(t) = 2|K|e^{-at}g(t) \rightarrow F(s) = 2|K|G(s+a)$$

$$\Rightarrow F(s) = 2|K|\left(\tfrac{s+a}{(s+a)^2+\beta^2}\cos(\angle K) - \tfrac{\beta}{(s+a)^2+\beta^2}\sin(\angle K)\right)$$

$$= 2|K|\,\tfrac{(s+a)\cos(\angle K) - \beta\sin(\angle K)}{(s+a)^2+\beta^2} = \tfrac{K}{s+a-j\beta} + \tfrac{K^*}{s+a+j\beta}$$

(b) $f(t) = 2|K|te^{-at}\cos(\beta t + \angle K)u(t)$.

$$g(t) = 2|K|e^{-at}\cos(\beta t + \angle K)u(t)$$

$$f(t) = tg(t) \rightarrow F(s) = -G'(s) = \tfrac{K}{(s+a-j\beta)^2} + \tfrac{K^*}{(s+a+j\beta)^2}$$

(c) $f(t) = g(t)u(t), \quad g(t) = at[u(t) - u(t-a)], g(t-a) = g(t)$.

Since $g(t)$ is periodic with period a,

$$F(s) = \frac{\int_0^a ate^{-st}dt}{1-e^{-sa}} = \frac{a(-ase^{-as} - e^{-as} + 1)}{s^2}\frac{1}{1-e^{-sa}} = a\frac{-as - 1 + e^{as}}{s^2(e^{as}-1)}$$

(d) $f(t) = e^{-at^2}$.

$$F(s) = \mathcal{L}\left\{e^{-at^2}\right\} = \int_0^\infty e^{-at^2 - st}dt = \int_0^\infty e^{-(at^2+st)}dt = \int_0^\infty e^{-(at^2+st+\frac{s^2}{4a}-\frac{s^2}{4a})}dt$$

$$= e^{\frac{s^2}{4a}}\int_0^\infty e^{-(\sqrt{a}t+\frac{s}{2\sqrt{a}})^2}dt = \frac{e^{\frac{s^2}{4a}}}{\sqrt{a}}\int_{\frac{s}{2\sqrt{a}}}^\infty e^{-\tau^2}d\tau = e^{\frac{s^2}{4a}}\sqrt{\frac{\pi}{4a}}\frac{2}{\sqrt{\pi}}\int_{\frac{s}{2\sqrt{a}}}^\infty e^{-\tau^2}d\tau = e^{\frac{s^2}{4a}}\sqrt{\frac{\pi}{4a}}\text{erfc}(\tfrac{s}{2\sqrt{a}})$$

Question 2 continued on next page. . .

Question 2

Find the inverse unidirectional Laplace transform of the following functions.

(a) $F(s) = a\frac{-as-1+e^{as}}{s^2(e^{as}-1)}$.

$$F(s) = a\frac{-as-1+e^{as}}{s^2(e^{as}-1)} \times \frac{e^{-as}}{e^{-as}} = \frac{X(s)}{1-e^{-as}}, \quad X(s) = a\frac{1-(as+1)e^{-as}}{s^2}$$

$$X(s) = \frac{a}{s^2} - \frac{a^2}{s}e^{-as} - \frac{a}{s^2}e^{-as} \Rightarrow x(t) = atu(t) - a^2u(t-a) - a(t-a)u(t-a) = at[u(t)-u(t-a)]$$

$$\Rightarrow f(t) = x(t), t \in [0, a], \quad f(t-a) = f(t)$$

(b) $F(s) = \frac{1}{s(s+1)^2(s^2+1)^2}$.

$$F(s) = \frac{1}{s(s+1)^2(s^2+1)^2} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2} + \frac{d^*}{s+j} + \frac{d}{s-j} + \frac{e^*}{(s+j)^2} + \frac{e}{(s-j)^2}$$

$$a = sF(s)|_{s=0} = 1$$

$$b = ((s+1)^2F(s))'|_{s=-1} = -\frac{5s^4+6s^2+1}{s^2(s^2+1)^4}|_{s=-1} = -\frac{3}{4}$$

$$c = (s+1)^2F(s)|_{s=-1} = -\frac{1}{4}$$

$$d = ((s-j)^2F(s))'|_{s=j} = -\frac{5s^4+8(j+1)s^3+12js^2-4(1-j)s-1}{s^2(s+1)^4(s+j)^4}|_{s=j} = \frac{3j-1}{8}$$

$$e = (s-j)^2F(s)|_{s=j} = \frac{1}{8}$$

$$\Rightarrow f(t) = [1 - \frac{3}{4}e^{-t} - \frac{1}{4}te^{-t} - \frac{1}{4}\cos(t) - \frac{3}{4}\sin(t) + \frac{1}{4}t\cos(t)]u(t)$$

(c) $F(s) = \frac{s}{(s^2+2s+2)^3}$.

$$F(s) = \frac{s}{(s^2+2s+2)^3} = \frac{a}{s+1-j} + \frac{b}{(s+1-j)^2} + \frac{c}{(s+1-j)^3} + \frac{a^*}{s+1+j} + \frac{b^*}{(s+1+j)^2} + \frac{c^*}{(s+1+j)^3}$$

$$a = \frac{1}{2}((s+1-j)^3F(s))''|_{s=j-1} = \frac{1}{2}\frac{-2(s+1+j)^4-4(s+1+j)^3(1+j-2s)}{(s+1+j)^8}|_{s=j-1} = \frac{3j}{16}$$

$$b = ((s+1-j)^3F(s))'|_{s=j-1} = \frac{(s+1+j)^3-3(s+1+j)^2s}{(s+1+j)^6}|_{s=j-1} = \frac{3-j}{16}$$

$$c = (s+1-j)^3F(s)|_{s=j-1} = \frac{-1-j}{8}$$

$$\Rightarrow f(t) = \frac{1}{8}e^{-t}\big[(t^2+t-3)\sin(t) - (t-3)t\cos(t)\big]u(t)$$

Question 3 continued on next page...

Question 3

Calculate the time-domain mesh currents for the circuit of Fig. 1.

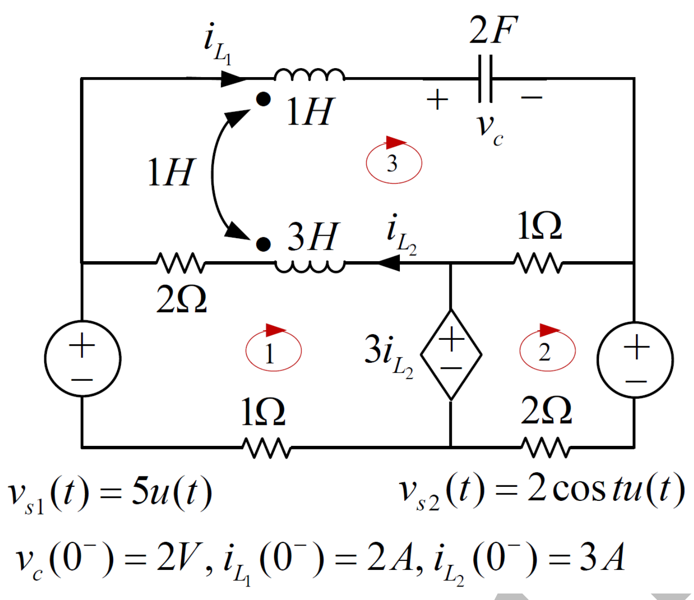


Figure 1: A coupled circuit for which the mesh currents are required.

We now that

$$V_c(s) = \frac{1}{cs} I s + \frac{v_c(0^-)}{s}$$

$$V_{L_1}(s) = L_1 s I_{L_1}(s) - M s I_{L_2}(s) - L_1 i_{L_1}(0^-) + M i_{L_2}(0^-)$$

$$V_{L_2}(s) = -L_2 s I_{L_2}(s) + M s I_{L_1}(s) + L_2 i_{L_2}(0^-) - M i_{L_1}(0^-))$$

So, we have

$$V_c(s) = \frac{1}{2s} I_3(s) + \frac{2}{s}$$

$$I_{L_1} = I_3(s), \quad I_{L_2} = I_3(s) - I_1(s)$$

$$V_{L_1}(s) = sI_3(s) - s(I_3(s) - I_1(s)) - 2 + 3 = sI_1(s) + 1$$

$$V_{L_2}(s) = -3s(I_3(s) - I_1(s)) + sI_3(s) + 3 \times 3 - 2 = 3sI_1(s) - 2sI_3(s) + 7$$

$$v_{s_1}(s) = \mathcal{L}(5u(t)) = \frac{5}{s}$$

$$v_{s_2}(s) = \mathcal{L}(2\cos(t)u(t)) = \frac{2s}{s^2+1}$$

Now, we can write KVL equations for the meshes as

$$-\frac{5}{s} + 2(I_1(s) - I_3(s)) + (3sI_1(s) - 2sI_3(s) + 7) + 3(I_3(s) - I_1(s)) + I_1(s) = 0$$

$$(sI_1(s) + 1)(\frac{1}{2s} I_3(s) + \frac{2}{s}) + (I_3(s) - I_2(s)) - (3sI_1(s) - 2sI_3(s) + 7) + 2(I_3(s) - I_1(s)) = 0$$

Question 3 continued on next page. . .

$$\frac{2s}{s^2+1} + 2I_2(s) - 3(I_3(s) - I_1(s)) + (I_2(s) - I_3(s)) = 0$$

Simplify the equations,

$$(-3s)I_1(s) + (2s-1)I_3(s) = -\frac{5}{s} + 7$$

$$(-s-2)I_1(s) - I_2(s) + (\frac{1}{2s} + 2s + 3)I_3(s) = -\frac{2}{s} + 6$$

$$I_1(s) + 3I_2(s) - 4I_3(s) = -\frac{2s}{s^2+1}$$

Solving the equations,

$$I_1(s) = \frac{-33s^4 - 63s^3 + 33s^2 - 55s + 62}{s(33s^4 + 16s^3 + 43s^2 + 16s + 10)}$$

$$I_2(s) = \frac{77s^4 - 107s^3 + 137s^2 - 83s + 46}{s(33s^4 + 16s^3 + 43s^2 + 16s + 10)}$$

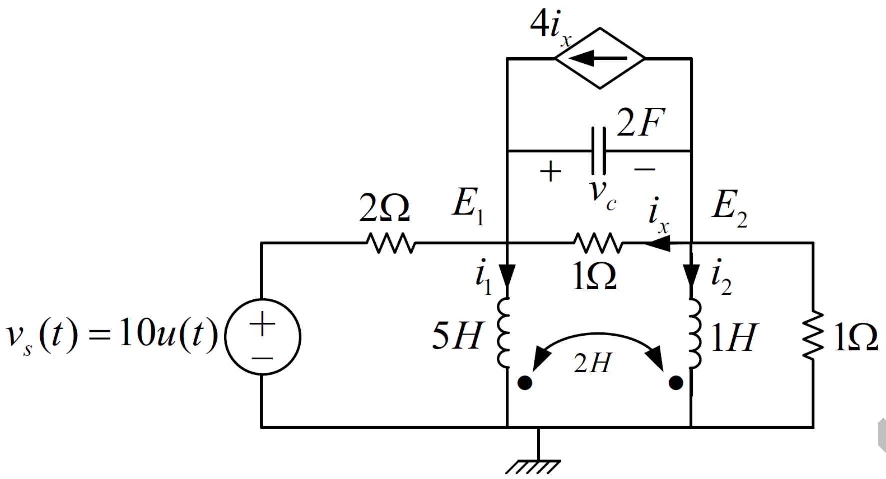
$$I_3(s) = \frac{66s^4 - 88s^3 + 116s^2 - 76s + 50}{s(33s^4 + 16s^3 + 43s^2 + 16s + 10)}$$

Finally, taking the inverse Laplace transform,

$$\begin{aligned} i_1(t) = &-0.153\sin(t) - 0.280\cos(t) \\ &- e^{-0.242t}\left[5.967\sin(0.494t) + 6.920\cos(0.494t)\right] + 6.200 \\ i_2(t) = &-0.418\sin(t) - 0.899\cos(t) \\ &- e^{-0.242t}\left[8.675\sin(0.494t) + 1.368\cos(0.494t)\right] + 4.600 \\ i_3(t) = &-0.352\sin(t) - 0.245\cos(t) \\ &- e^{-0.242t}\left[7.998\sin(0.494t) + 2.755\cos(0.494t)\right] + 5.000 \end{aligned}$$

Question 4

Obtain the time-domain node voltages for the circuit of Fig. 2.



$$v_c(0^-) = 2V,\ i_1(0^-) = 2A,\ i_2(0^-) = 3A$$

$$L = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \Gamma = L^{-1} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

Figure 2: A coupled circuit for which the node voltages are required.

We know that

$$\begin{cases} -I_1(s) = -\frac{1}{s}E_1(s) - \frac{-2}{s}E_2(s) - \frac{2}{s} \\ -I_2(s) = -\frac{-2}{s}E_1(s) - \frac{5}{s}E_2(s) - \frac{3}{s} \\ I_C(s) = 2s(E_1(s) - E_2(s)) - 2 \times 2 \\ V_s(s) = \frac{10}{s} \\ I_x(s) = \frac{E_2 - E_1}{1} \end{cases}$$

Using KCL at the nodes,

$$\frac{E_1 - \frac{10}{s}}{2} + \frac{1}{s}E_1 + \frac{-2}{s}E_2 + \frac{2}{s} + \frac{E_1 - E_2}{1} - 4\frac{E_2 - E_1}{1} + 2s(E_1 - E_2) - 4 = 0$$

$$\frac{E_2}{1} + \frac{-2}{s}E_1 + \frac{5}{s}E_2 + \frac{3}{s} + \frac{E_2 - E_1}{1} + 4\frac{E_2 - E_1}{1} - 2s(E_1 - E_2) + 4 = 0$$

Simplifying the equations,

$$E_1\left(\frac{11}{2} + \frac{1}{s} + 2s\right) + E_2\left(-5 - \frac{2}{s} - 2s\right) = \frac{3}{s} + 4$$

$$E_1\left(-5 - \frac{2}{s} - 2s\right) + E_2\left(6 + \frac{5}{s} + 2s\right) = -\frac{3}{s} - 4$$

Solving the equations,

$$E_1 = -\frac{2\left(-4s - 3\right)\left(s + 3\right)}{6s^3 + 24s^2 + 27s + 2} = -\frac{\left(-4s - 3\right)\left(0.33s + 1\right)}{s^3 + 4s^2 + 4.5s + 0.33}$$

$$E_2 = \frac{\left(-4s - 3\right)\left(s - 2\right)}{6s^3 + 24s^2 + 27s + 2} = \frac{\left(-0.67s - 0.5\right)\left(s - 2\right)}{s^3 + 4s^2 + 4.5s + 0.33}$$

Question 4 continued on next page. . .

Finally, taking the inverse Laplace transform,

$$e_1(t) = -0.673e^{-0.0787t} - e^{-1.961t}\big[0.647\cos(0.589t) - 1.752\sin(0.589t)\big]$$
$$e_2(t) = 0.239e^{-0.0787t} - e^{-1.961t}\big[0.909\cos(0.589t) - 2.981\sin(0.589t)\big]$$

Question 5

Find an expression for $v(t)$ valid for all times in the circuit of Fig. 3.

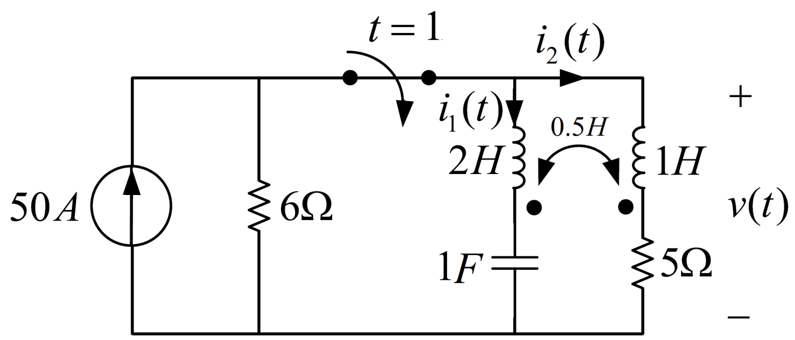


Figure 3: A circuit with a switch opened at $t = 1$.

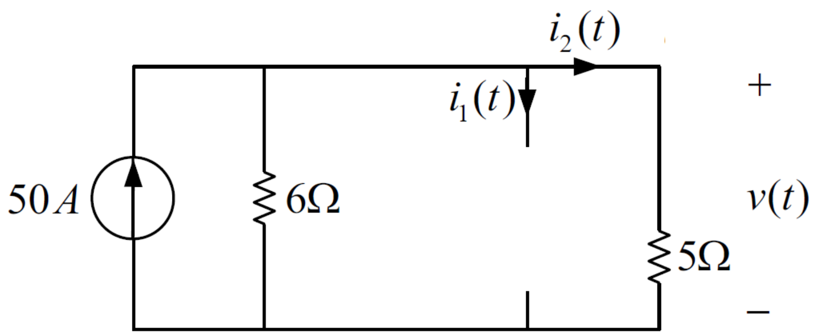


Figure 4: The circuit of Fig. 3 for $t < 1$.

Question 5 continued on next page...

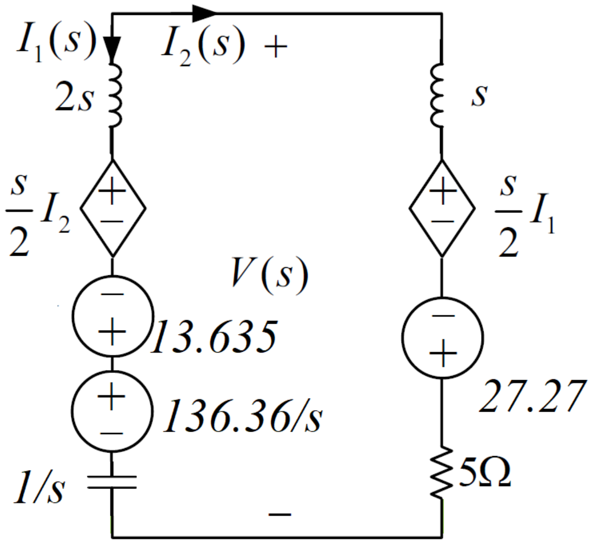


Figure 5: The circuit of Fig. 3 for $t > 1$.

We analyze the circuit by assuming that the switch opens at $t = 0$. Then, we shift the response to $t = 1$. Before $t = 0$, we have the circuit 4 and therefore,

$$i_1(0^-) = 0$$
$$i_2(0^-) = 50\frac{6}{6+5} = 27.27$$
$$v(0^-) = 5i_2(0^-) = 136.36$$
$$v_c(0^-) = (0^-) = 136.36$$

When the switch opens at $t = 0$, we get the circuit shown in Fig. 5. So,

$$\begin{cases} sI_2 + 0.5sI_1 - 27.27 + 5I_2 + \frac{I_2}{s} - \frac{136.36}{s} + 13.635 - 0.5sI_2 + 2sI_2 = 0 \\ I_1 = -I_2 \end{cases}$$

$$I_2(s) = \frac{s}{2s^2+5s+1}\frac{13.635s+136.36}{s} = \frac{13.635s+136.36}{2s^2+5s+1}$$

$$V(s) = sI_2 - 0.5sI_2 + 27.27 + 5I_2 = I_2 + 27.27 = \frac{(0.5s+5)(13.635s+136.36)}{2s^2+5s+1} + 27.27$$

$$v(t) = 30.67875\delta(t) + 256.72502e^{-\frac{5t}{4}}\sinh(1.03077t) + 59.655625e^{-\frac{5t}{4}}\cosh(1.03077t)$$

Finally,

$$v(t) = \begin{cases} 136.36 & , \quad t < 1 \\ 30.68\delta(t-1) + e^{-\frac{5(t-1)}{4}}\left[256.73\sinh(1.03(t-1)) + 59.66\cosh(1.03(t-1))\right] & , \quad t \geq 1 \end{cases}$$

SOFTWARE QUESTIONS

Question 6

Use AC analysis of PSpice to investigate the frequency response $H(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)}$ **of the double-tuned circuit shown in Fig. 6. Analyze the impact of each parameter on the frequency response.**

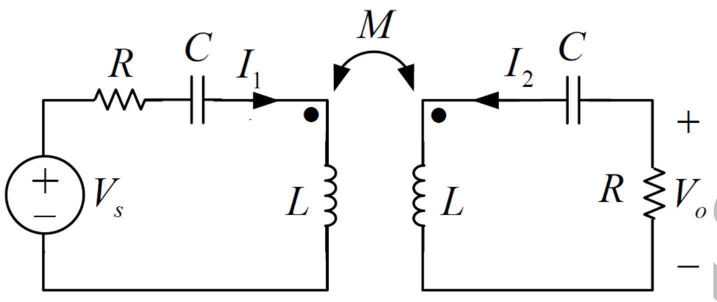


Figure 6: Double-tuned circuit.

> The frequency responses of the circuit for a sinusoidal input with amplitude 10 are shown in Figs. 8-15. In each figure, a circuit parameter is changed and its impact on the magnitude or phase of the frequency response is shown.

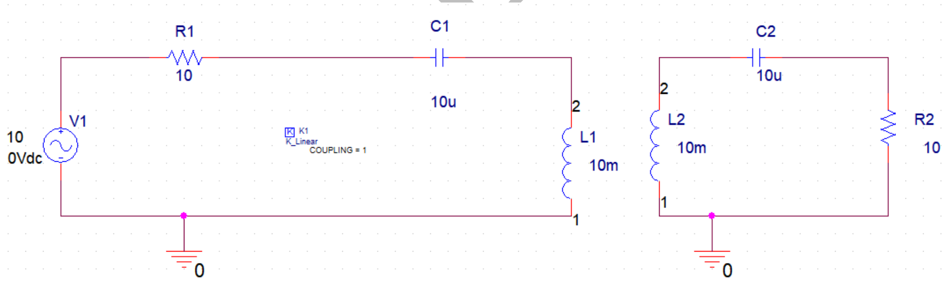


Figure 7: Schematic of the double-tuned circuit in PSpice.

Question 6 continued on next page...

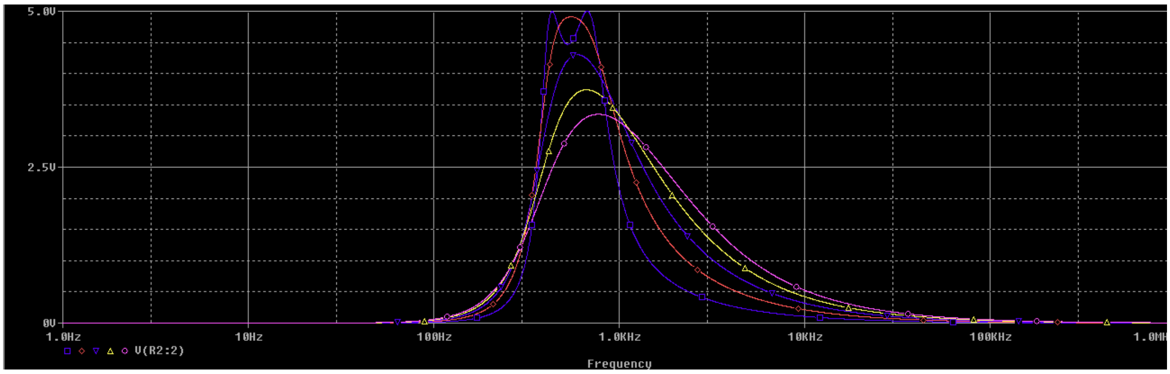


Figure 8: Frequency response magnitude for $R = 10, 20, 30, 40, 50\,\Omega$. Increasing R, the bandwidth of the magnitude response increases.

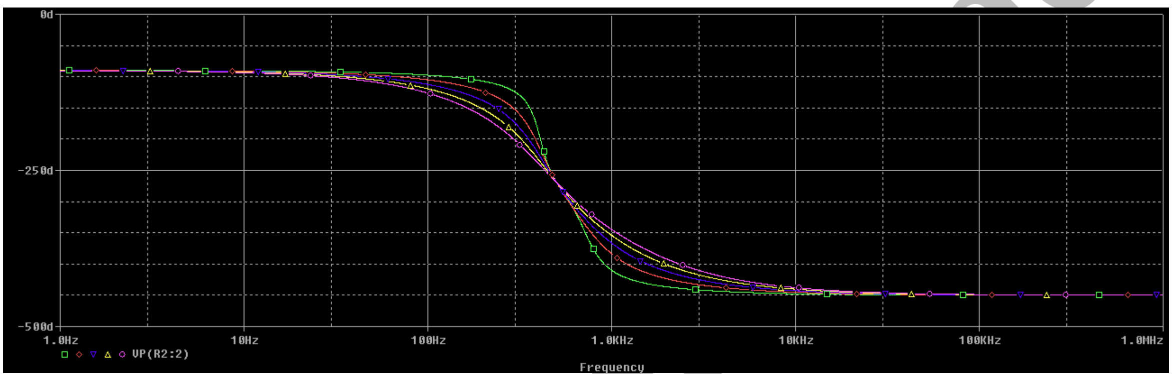


Figure 9: Frequency response phase for $R = 10, 20, 30, 40, 50\ \Omega$. Increasing R, the smoothness of the phase response increases.

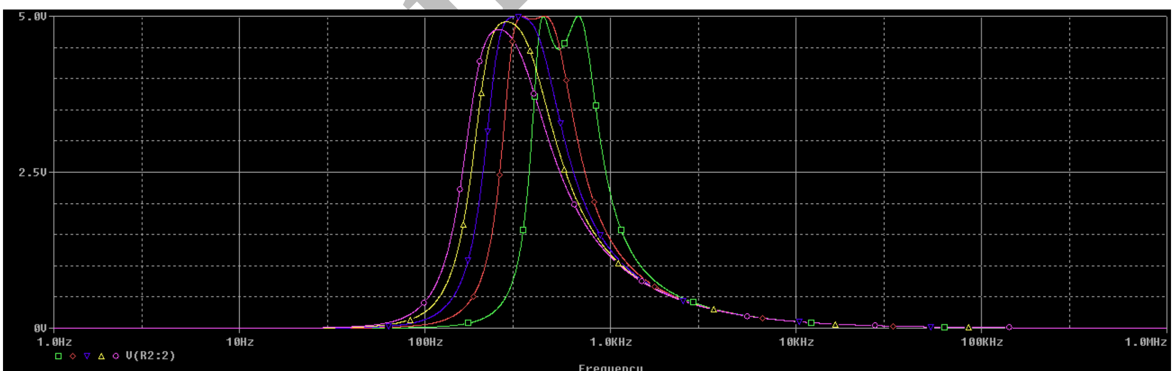


Figure 10: Frequency response magnitude for $C = 10, 20, 30, 4050\ \mu$F. Increasing C, the central frequency of the magnitude response decreases.

Question 6 continued on next page…

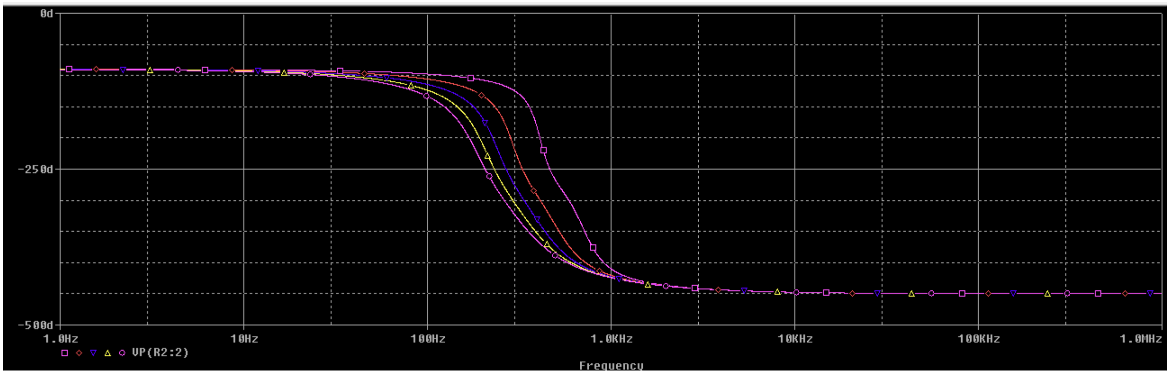


Figure 11: Frequency response phase or $C = 10, 20, 30, 40, 50$ μF. Increasing C, the turning frequency of the phase response decreases.

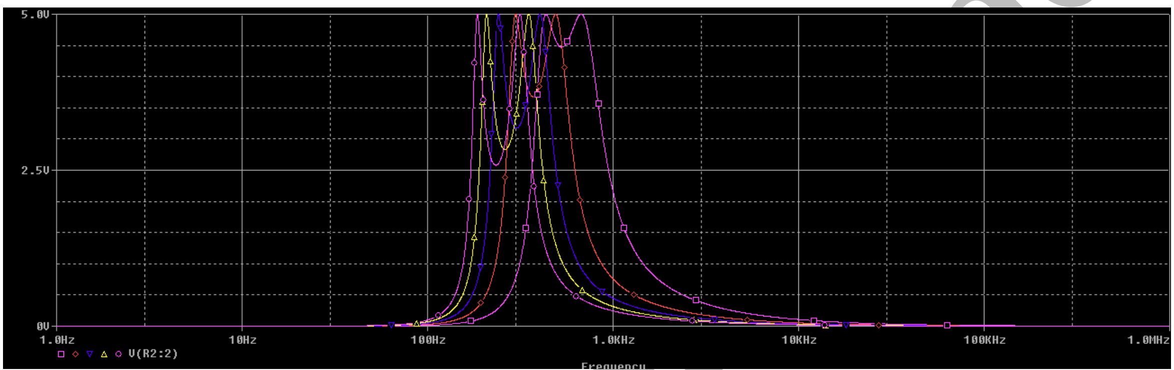


Figure 12: Frequency response magnitude for $L = 10, 20, 30, 40, 50$ mH. Increasing L, the central frequency of the magnitude response decreases.

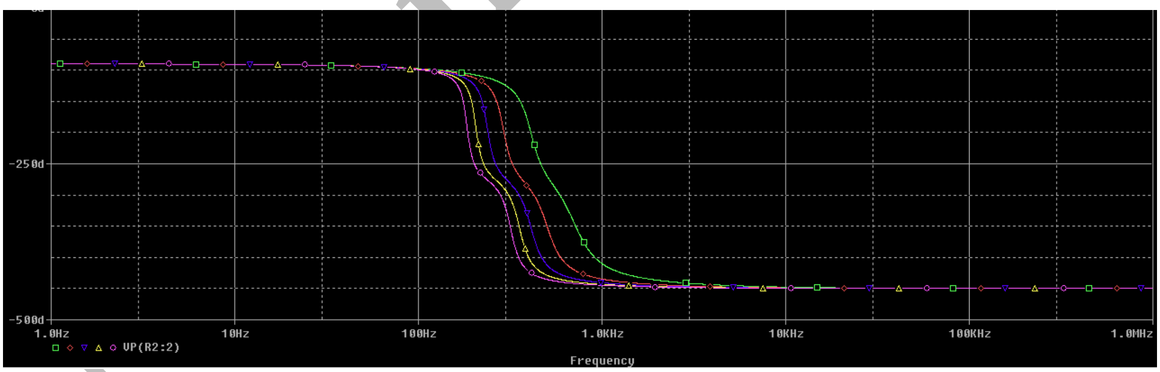


Figure 13: Frequency response phase for for $L = 10, 20, 30, 40, 50$ mH. Increasing L, the turning frequency of the phase response decreases.

Question 6 continued on next page...

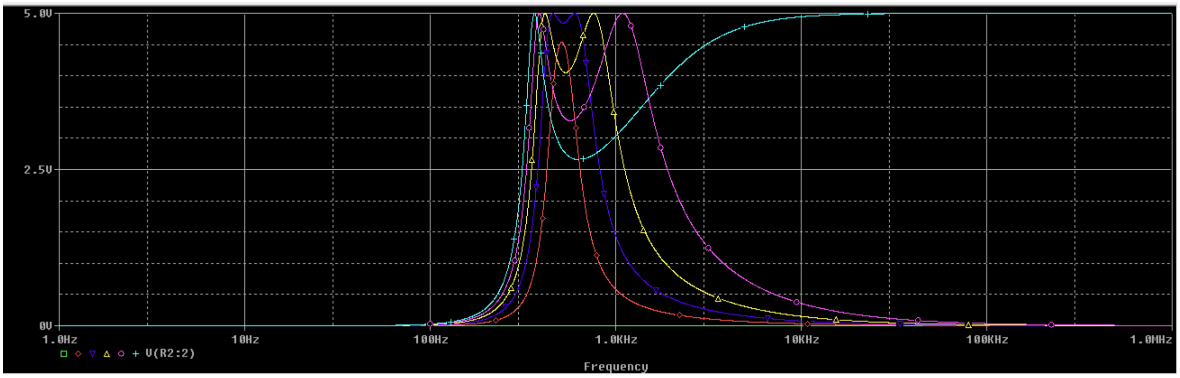


Figure 14: Frequency response magnitude for $k = 0, 0.2, 0.4, 0.6, 0.8, 1$. Increasing k, the bandwidth of the magnitude response increases.

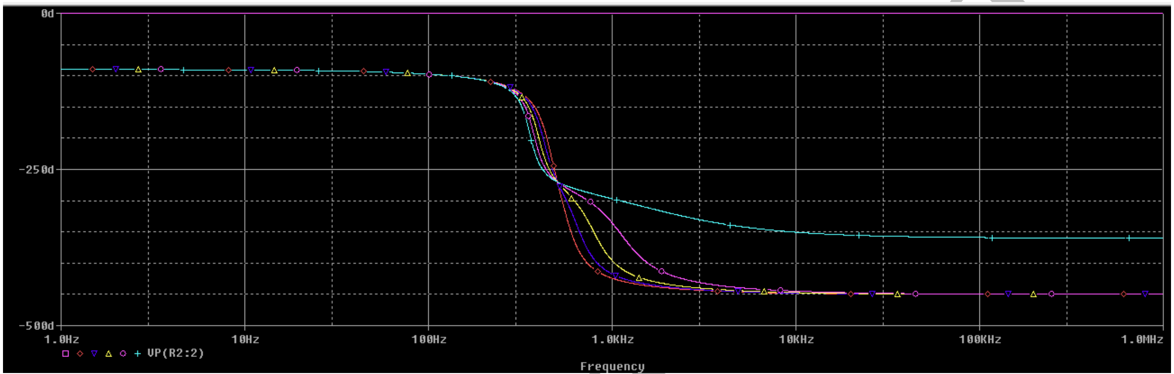


Figure 15: Frequency response phase for $k = 0, 0.2, 0.4, 0.6, 0.8, 1$. Increasing k, the turning point of the phase response almost remains unchanged.

BONUS QUESTIONS

Question 7

Return your answers by filling the LaTeXtemplate of the assignment. If you want to add a circuit schematic, you can draw it directly using TikZ package, or draw it in a secondary application such as Microsoft Visio and then, import it as a figure.

EXTRA QUESTIONS

Question 8

Feel free to solve the following questions from the book *"Engineering Circuit Analysis"* by W. Hayt, J. Kemmerly, and S. Durbin.

1. **Chapter 15, question 14.**
2. **Chapter 15, question 19.**
3. **Chapter 15, question 22.**
4. **Chapter 15, question 24.**
5. **Chapter 15, question 27.**