MATHEMATICAL QUESTIONS

Question 1

For the circuit of Fig. 1,

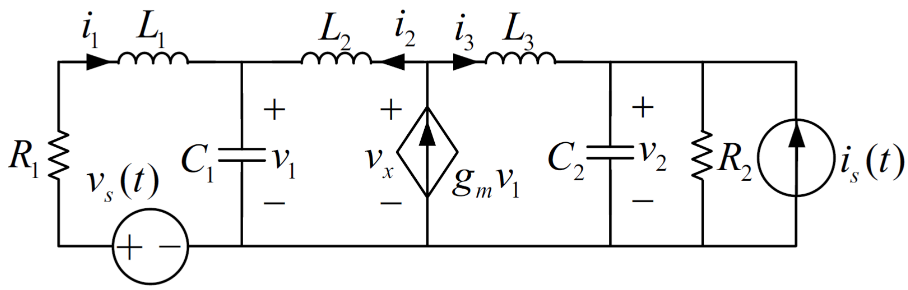

Figure 1: A circuit for which the state equations are required.

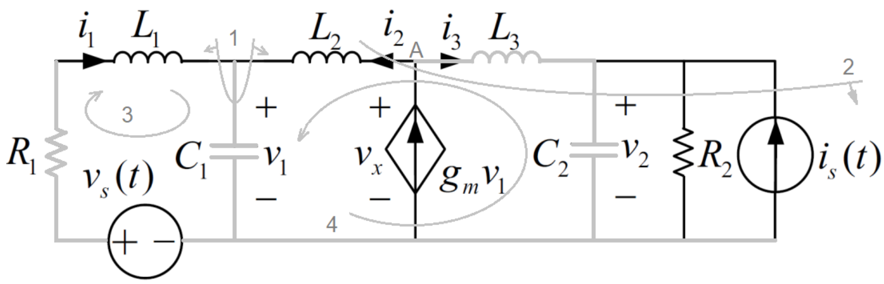

Figure 2: Cut-sets and loops of the circuit shown in Fig. 1.

(a) Write the matrix form of the state equations.

In this circuit, there are three inductors and two capacitors. So it seems that we have a 5^{th} order system. But by writing a KCL at node A shown in Fig. 2,

$$i_2 + i_3 - g_m v_1 = 0$$

So, the currents of L_2 and L_3 and the voltage of C_1 are dependent. So, the order of the circuit and the number of state variables is four. Choosing the state vector and input vector as

$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_1(t) \\ i_2(t) \end{bmatrix}, \quad \boldsymbol{W}(t) = \begin{bmatrix} v_s(t) \\ i_s(t) \end{bmatrix}$$

and noting to the highlighted cut sets and loops in Fig. 2, we have

$$\begin{cases} C_1\frac{dv_1(t)}{dt} - i_1(t) - i_2(t) = 0 \\ C_2\frac{dv_2(t)}{dt} + i_2(t) - g_m v_1(t) + \frac{v_2(t)}{R_2} - i_s(t) = 0 \\ L_1\frac{di_1(t)}{dt} + v_1(t) - v_s(t) + R_1 i_1(t) = 0 \\ L_2\frac{di_2(t)}{dt} + v_1(t) - v_2(t) - L_3\frac{d[g_m v_1(t) - i_2(t)])}{dt} = 0 \end{cases}$$

$$\frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & \frac{1}{C_1} \\ \frac{g_m}{C_2} & \frac{-1}{R_2C_2} & 0 & \frac{-1}{C_2} \\ \frac{-1}{L_1} & 0 & \frac{-R_1}{L_1} & 0 \\ \frac{-1}{L_2+L_3} & \frac{1}{L_2+L_3} & \frac{L_3g_m}{C_1(L_2+L_3)} & \frac{L_3g_m}{C_1(L_2+L_3)} \end{bmatrix}\boldsymbol{X}(t) + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{C_2} \\ \frac{1}{L_1} & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{W}(t)$$

(b) Express $v_x(t)$ in terms of the state and input vectors.

$$\begin{aligned} v_x(t) = L_2\frac{di_2}{dt} + v_1(t) &= \boldsymbol{C}^T\boldsymbol{X}(t) + \boldsymbol{D}^T\boldsymbol{w}(t) \\ &= \begin{bmatrix} \frac{-L_2}{L_2+L_3}+1 & \frac{L_2}{L_2+L_3} & \frac{L_2L_3g_m}{C_1(L_2+L_3)} & \frac{L_2L_3g_m}{C_1(L_2+L_3)} \end{bmatrix}\boldsymbol{X}(t) + \begin{bmatrix} 0 & 0 \end{bmatrix}\boldsymbol{W}(t) \\ &= \begin{bmatrix} \frac{L_3}{L_2+L_3} & \frac{L_2}{L_2+L_3} & \frac{L_2L_3g_m}{C_1(L_2+L_3)} & \frac{L_2L_3g_m}{C_1(L_2+L_3)} \end{bmatrix}\boldsymbol{X}(t) + \begin{bmatrix} 0 & 0 \end{bmatrix}\boldsymbol{W}(t) \end{aligned}$$

Question 2

For the circuit of Fig. 3,

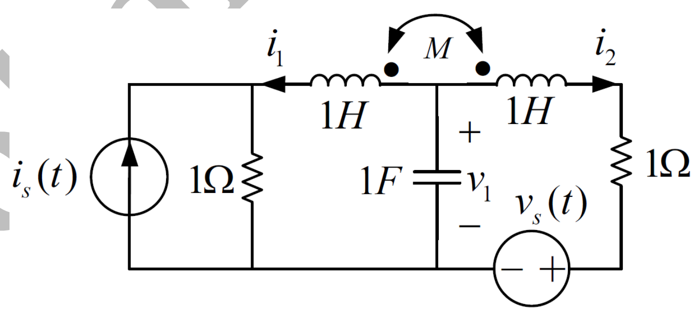

Figure 3: A coupled circuit for which the state equations are required.

(a) *Write the state equations if $M = 0.5$.*

Let the state and input vectors be

$$\boldsymbol{X} = \begin{bmatrix} i_1(t) \\ i_2(t) \\ v_1(t) \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} i_s(t) \\ v_s(t) \end{bmatrix}$$

We have,

$$\begin{cases} \frac{di_1(t)}{dt} + \frac{1}{2}\frac{di_2(t)}{dt} + i_1(t) + i_s(t) - v_1(t) = 0 \\ \frac{di_2(t)}{dt} + \frac{1}{2}\frac{di_1(t)}{dt} + i_2(t) + v_s(t) - v_1(t) = 0 \\ \frac{dv_1(t)}{dt} + i_1(t) + i_2(t) = 0 \end{cases}$$

So,

$$\frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} \frac{-4}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-4}{3} & \frac{2}{3} \\ -1 & -1 & 0 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} \frac{-4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-4}{3} \\ 0 & 0 \end{bmatrix} \boldsymbol{W}$$

(b) *Write the state equations if $M = 1$.*

We have,

$$\begin{cases} \frac{di_1(t)}{dt} + \frac{di_2(t)}{dt} + i_1(t) + i_s(t) - v_1(t) = 0 \\ \frac{di_2(t)}{dt} + \frac{di_1(t)}{dt} + i_2(t) + v_s(t) - v_1(t) = 0 \\ \frac{dv_1(t)}{dt} + i_1(t) + i_2(t) = 0 \end{cases}$$

Clearly,

$$i_2(t) = i_1(t) + i_s(t) - v_s(t)$$

Which shows that $i_2(t)$ depends on $i_1(t)$ and it cannot be among the state variables. Now, let the state and input vectors be

$$\boldsymbol{X} = \begin{bmatrix} i_1(t) \\ v_1(t) \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} i_s(t) \\ v_s(t) \end{bmatrix}$$

We have

$$\frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ -2 & 0 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} i_s(t) \\ v_s(t) \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{di_s(t)}{dt} \\ \frac{dv_s(t)}{dt} \end{bmatrix}$$

$$\frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ -2 & 0 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix} \boldsymbol{W} + \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \frac{d\boldsymbol{W}}{dt}$$

(c) *Find the transfer functions $H_V(s) = \frac{V_1(s)}{V_s(s)}\big|_{I_s(s)=0}$ and $H_I(s) = \frac{V_1(s)}{I_s(s)}\big|_{V_s(s)=0}$ if $M = 1$.*

We know,

$$v_1(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{X}(t) \Rightarrow V_1(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{X}(s)$$

Assuming $i_s(0) = v_s(0) = 0$,

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{W}(s) + \boldsymbol{B'}s\boldsymbol{W}(s) \Rightarrow \boldsymbol{X}(s) = (sI - A)^{-1}(\boldsymbol{B} + s\boldsymbol{B'})\boldsymbol{W}(s)$$

So,

$$V_1(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+\frac{1}{2} & \frac{-1}{2} \\ 2 & s \end{bmatrix}^{-1} \begin{bmatrix} -\frac{s+1}{2} & \frac{s}{2} \\ -1 & 1 \end{bmatrix} \boldsymbol{W}(s)$$

$$V_1(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+\frac{1}{2} & \frac{-1}{2} \\ 2 & s \end{bmatrix}^{-1} \left(\begin{bmatrix} -\frac{s+1}{2} \\ -1 \end{bmatrix} I_s(s) + \begin{bmatrix} \frac{s}{2} \\ 1 \end{bmatrix} V_s(s) \right)$$

We have,

$$H_v(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+\frac{1}{2} & \frac{-1}{2} \\ 2 & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{s}{2} \\ 1 \end{bmatrix} = \frac{1}{2s^2+s+2}$$

$$H_I(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+\frac{1}{2} & \frac{-1}{2} \\ 2 & s \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1+s}{2} \\ -1 \end{bmatrix} = \frac{1}{2s^2+s+2}$$

(d) Solve the state equations if $M = 1$, $i_s(t) = u(t)$, $v_s(t) = 0$, and the initial state vector $\boldsymbol{X}_0$ is an all-one vector.

$$s\boldsymbol{X}(s) = \boldsymbol{X}_0 + \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{W}(s) + \boldsymbol{B}'s\boldsymbol{W}(s) \Rightarrow \boldsymbol{X}(s) = (sI - A)^{-1}(\boldsymbol{X}_0 + \boldsymbol{B} + s\boldsymbol{B}')\boldsymbol{W}(s)$$

$$X(s) = \begin{bmatrix} s+\frac{1}{2} & \frac{-1}{2} \\ 2 & s \end{bmatrix}^{-1} \left(\begin{bmatrix} -\frac{1+s}{2} & \frac{s}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$X(s) = \begin{bmatrix} \frac{(s-1)(s+1)}{s(2+2s^2+s)} \\ \frac{2s^2-3s+1}{s(2+2s^2+s)} \end{bmatrix}$$

$$i_1(t) = \left[-\frac{1}{2} + e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}t}{4}\right) \right] u(t)$$

$$v_1(t) = \left[\frac{1}{2} + \frac{1}{2} e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}t}{4}\right) - \frac{\sqrt{15} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}t}{4}\right)}{2} \right] u(t)$$

Question 3

Write the state equations for the linear time-varying RLC circuit shown in Fig. 4, where the element values are $R(t)$, $L(t)$, and $C(t)$.

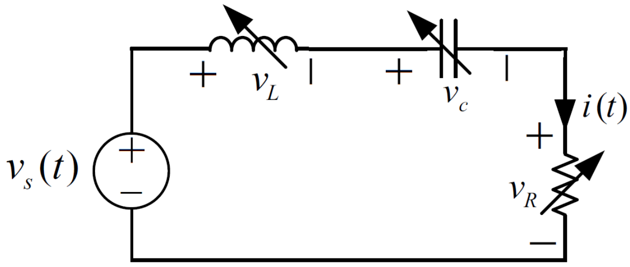

Figure 4: A linear time-varying RLC circuit.

Capacitor charge $q(t)$ and inductor flux $\phi(t)$ can be used as state variables in the circuit. We know that

$$v_L(t) = \frac{d\phi(t)}{dt}, v_C(t) = \frac{q(t)}{C(t)}, i_R(t) = i_L(t) = \frac{\phi(t)}{L(t)}$$

Writing a KVL for the circuit loop,

$$v_L(t) + v_C(t) + v_R(t) - v_s(t) = 0 \Rightarrow \frac{d\phi(t)}{dt} = -\frac{q(t)}{C(t)} - R(t)\frac{\phi(t)}{L(t)} + v_s(t)$$

On the other hand, KCL gives

$$i_c(t) = i_L(t) \Rightarrow \frac{dq(t)}{dt} = \frac{\phi(t)}{L(t)}$$

Let

$$\boldsymbol{X}(t) = \begin{bmatrix} \phi(t) \\ q(t) \end{bmatrix}, \quad \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} \phi(0) \\ q(0) \end{bmatrix}$$

So,

$$\frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} -\frac{R(t)}{L(t)} & -\frac{1}{C(t)} \\ \frac{1}{L(t)} & 0 \end{bmatrix} \boldsymbol{X}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_s(t)$$

SOFTWARE QUESTIONS

Question 4

Write a MATLAB function that plots the approximated state trajectory corresponding to the state equation

$$\boldsymbol{X}'(t) = \boldsymbol{A}\boldsymbol{X}(t), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$$

. Compare the state trajectories for a certain coefficient matrix A, a certain initial state vector $\boldsymbol{X}_0$, and different values of the time step Δt.

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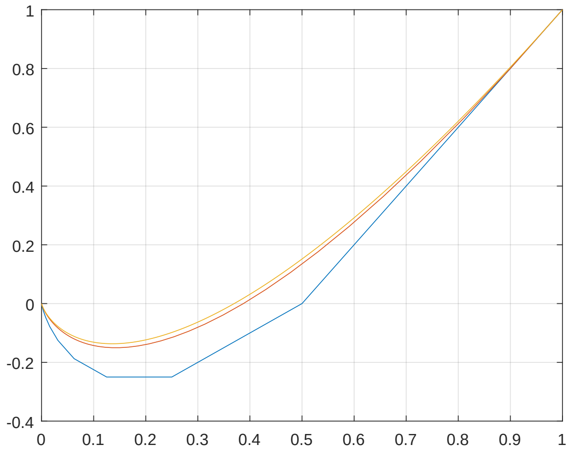

Figure 5: A sample state trajectory for different values of Δt. the blue, red, and yellow curves correspond to $\Delta t = 0.5, 0.1, 0.01$, respectively.

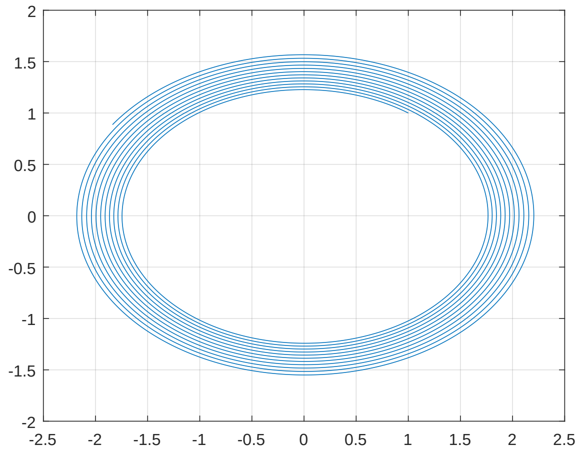

Figure 6: A sample 2D state trajectory.

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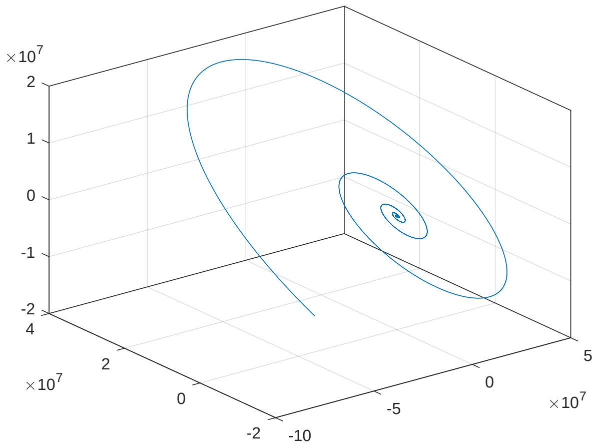

Figure 7: A sample 3D state trajectory.

Here is a MATLAB function that computes the trajectory points.

```
function X = state_trajectory(A, X0, Dt, Np)

% Np: number of points
% Dt: time step
% X0: initial state vector
% A: coefficient matrix

IpADt = eye(size(A))+Dt*A;
X = zeros([size(X0) Np]);
X(:,1) = X0;
for i=2:Np
    X(:,i) = IpADt*X(:,i-1);
end

end
```

You may use the following mfile to call the developed function and see its results.

```
clear all
close all

% test of accuracy
A = [-1 0;-1 -1];
X0 = [1;1];

figure
hold on
for Dt=[0.5 0.1 0.01]
     X = state_trajectory(A, X0, Dt, 1000);
     plot(X(1,:),X(2,:));
end
grid on
box on

% 1D trajectory
A = [-0.1];
X0 = [1];
X = state_trajectory(A, X0, Dt, 10000);
figure
plot(X(1,:), ones(size(X(1,:))));
grid on
box on

```

Question 4 continued on next page...

```
% 2D trajectory
A = [0  -1;0.5  0];
X0 = [1;1];
X = state_trajectory(A, X0, Dt, 10000);
figure
plot(X(1,:),X(2,:));
grid on
box on

% 3D trajectory
A = [0 -1 -1;0.5 0 0.5;0.25 0.25 0];
X0 = [1;1;1];
X = state_trajectory(A, X0, Dt, 10000);
figure
plot3(X(1,:),X(2,:),X(3,:));
grid on
box on

% 4D trajectory
A = [0 -1 -1 -1;0.5 0 0.5 0.5;0.25 0.25 0 0.25;0.125 0.125 0.125 0];
X0 = [1;1;1;1];
X = state_trajectory(A, X0, Dt, 1000);
figure
plot3(X(1,:),X(2,:),X(3,:));
grid on
box on
figure
plot3(X(1,:),X(2,:),X(4,:));
grid on
box on
figure
plot3(X(1,:),X(3,:),X(4,:));
grid on
box on
figure
plot3(X(2,:),X(3,:),X(4,:));
grid on
box on
```

As illustrated in Fig. 5, the accuracy of the state trajectory improves by reducing the time step Δt. A sample 2D state trajectory is shown in Fig. 6 while Fig. 7 represents a sample 3D state trajectory.

BONUS QUESTIONS

Question 5

Return your answers by filling the LaTeXtemplate of the assignment. If you want to add a circuit schematic, you can draw it directly using TikZ package, or draw it in a secondary application such as Microsoft Visio and then, import it as a figure.

EXTRA QUESTIONS

Question 6

eel free to solve the following questions from the book *"Basic Circuit Theory"* by C. Desoer and E. Kuh.

1. **Chapter 12, question 3.**
2. **Chapter 12, question 4.**
3. **Chapter 12, question 5.**
4. **Chapter 12, question 7.**