MATHEMATICAL QUESTIONS

Question 1

For the circuit of Fig. 1,

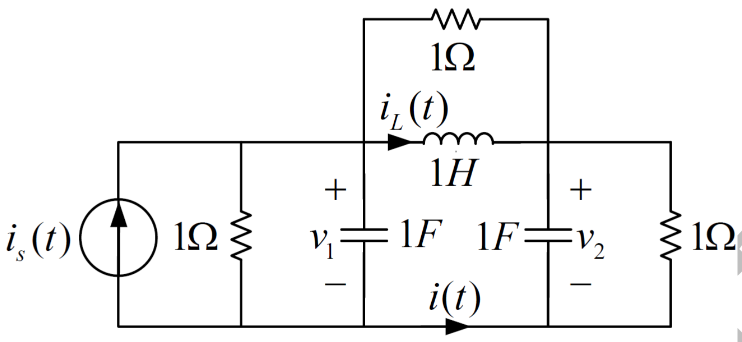

Figure 1: A circuit for which the natural frequencies are required.

(a) Find the natural frequencies of the node voltages.

Using nodal analysis for the unforced equivalent circuit,

$$\frac{V_1 - V_2}{1} + \frac{V_1 - V_2}{s} + \frac{V_1}{1} + \frac{V_1}{\frac{1}{s}} = v_1(0^-) - \frac{i_L(0^-)}{s}$$

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_1}{s} + \frac{V_2}{1} + \frac{V_2}{\frac{1}{s}} = v_2(0^-) + \frac{i_L(0^-)}{s}$$

So, we have

$$(1 + \frac{1}{s} + 1 + s)V_1 - (1 + \frac{1}{s})V_2 = v_1(0^-) - \frac{i_L(0^-)}{s}$$

$$-(1 + \frac{1}{s})V_1 + (1 + \frac{1}{s} + 1 + s)V_2 = v_2(0^-) + \frac{i_L(0^-)}{s}$$

Solving the equations,

$$V_1 = \frac{(s+1)v_1(0^-) + v_2(0^-) - i_L(0^-)}{(s+2)(s+1)} \Rightarrow s = -1, -2$$

$$V_2 = \frac{(s+1)v_2(0^-) + v_1(0^-) + i_L(0^-)}{(s+2)(s+1)} \Rightarrow s = -1, -2$$

Question 1 continued on next page. . .

(b) Find the natural frequencies of the circuit.

$$\boldsymbol{Y}_n(s) = \begin{bmatrix} 2+\frac{1}{s}+s & -(1+\frac{1}{s}) \\ -(1+\frac{1}{s}) & 2+\frac{1}{s}+s \end{bmatrix}$$

$$\Delta_n(s) = \text{det}\boldsymbol{Y}_n(s) = \frac{s^3+4s^2+5s+2}{s} = 0 \Rightarrow s = -2, -1, -1$$

Question 2

Calculate the natural frequencies of the circuits shown in Fig. 2.

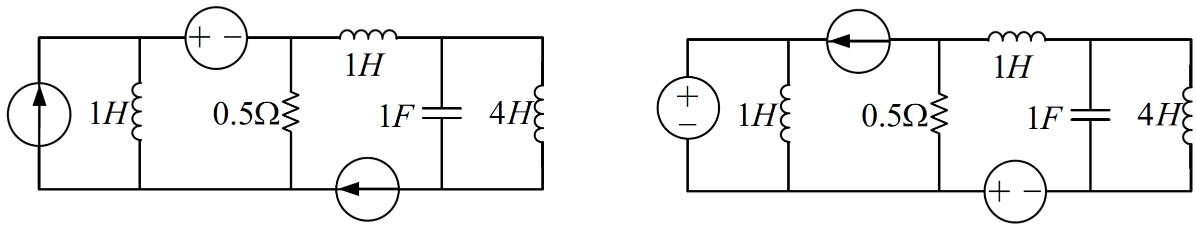

Figure 2: Two circuits with different types of independent sources.

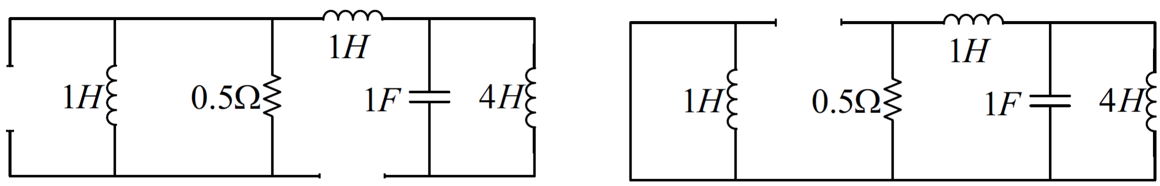

Figure 3: Circuits of Fig. 2 in the zero-input state.

The unforced circuits are shown in Fig. 3. For the right circuit, there is an inductive loop in the left part of the circuit that results in a zero natural frequency $s_1 = 0$. For the right part,

$$Y_n(s) = \begin{bmatrix} 2+\frac{1}{s} & \frac{-1}{s} \\ \frac{-1}{s} & \frac{1}{s}+s+\frac{1}{4s} \end{bmatrix}$$

So,

$$\Delta_n(s) = \text{det}Y_n(s) = \frac{8s^3+4s^2+10s+1}{4s^2} = 0 \Rightarrow s_2 \approx -0.10339, s_3, s_4 \approx -0.1983 \pm j1.08151$$

Hence, all the natural frequencies are

$$s_1 = 0, s_2 \approx -0.10339, s_3, s_4 \approx -0.1983 \pm j1.08151$$

Question 2 continued on next page...

For the left circuit, The natural frequencies for the LC part is

$$s_1, s_2 = \pm\frac{j}{\sqrt{LC}} = \pm\frac{j}{2}$$

And for the RL part,

$$\tau = \frac{L}{R} = 2 \Rightarrow s_3 = \frac{-1}{\tau} = \frac{-1}{2}$$

Hence, all the natural frequencies are

$$s_1, s_2 = \pm\frac{j}{2}, s_3 = \frac{-1}{2}$$

Question 3

For the circuit of Fig. 4,

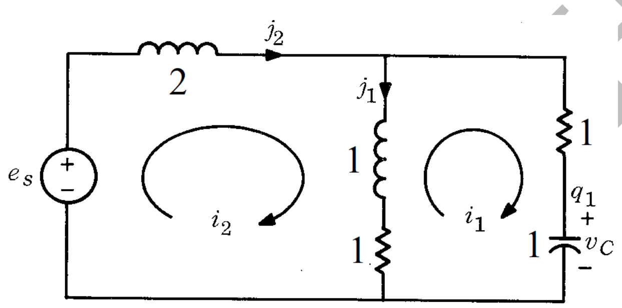

Figure 4: A circuit for which the minimal diffrential equation of j_2 is required.

(a) Find the minimal differential equation of j_2.

Mesh analysis yields

$$\begin{cases} i_1 + D^{-1}i_1 + v_c(0^-) + i_1 - i_2 + D(i_1 - i_2) = 0 \\ 2Di_2 + D(i_2 - i_1) + i_2 - i_1 - e_s = 0 \end{cases}$$

So,

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} : \begin{bmatrix} D + 2 + D^{-1} & -D - 1 \\ -D - 1 & 3D + 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_c(0^-) \\ e_s \end{bmatrix}$$

Replacing $i_1 = Dq_1$ with the q_1 in the equations,

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} : \begin{bmatrix} D^2 + 2D + 1 & -(D + 1) \\ -D(D + 1) & 3D + 1 \end{bmatrix} \begin{bmatrix} q_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_c(0^-) \\ e_s \end{bmatrix}$$

Manipulating the matrix with the elementary row operations,

$$\begin{bmatrix} E_1 \\ E_2 + E_1 \end{bmatrix} : \begin{bmatrix} D^2 + 2D + 1 & -(D + 1) \\ D + 1 & 2D \end{bmatrix} \begin{bmatrix} q_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_c(0^-) \\ e_s - v_c(0^-) \end{bmatrix}$$

Question 3 continued on next page...

$$\left[\begin{matrix} E_1-(D+1)(E_2+E_1) \\ (E_2+E_1) \end{matrix}\right] : \left[\begin{matrix} 0 & -(2D^2+3D+1) \\ D+1 & 2D \end{matrix}\right]\left[\begin{matrix} q_1 \\ i_2 \end{matrix}\right] = \left[\begin{matrix} -(D+1)e_s \\ e_s - v_c(0^-) \end{matrix}\right]$$

$$\left[\begin{matrix} (E_2+E_1) \\ E_1-(D+1)(E_2+E_1) \end{matrix}\right] : \left[\begin{matrix} D+1 & 2D \\ 0 & -(2D^2+3D+1) \end{matrix}\right]\left[\begin{matrix} q_1 \\ i_2 \end{matrix}\right] = \left[\begin{matrix} e_s - v_c(0^-) \\ -(D+1)e_s \end{matrix}\right]$$

Since $j_2 = i_2$, we have the corresponding minimal differential equation as

$$(2D^2+3D+1)j_2 = (D+1)e_s$$

(b) *Find the natural frequencies of* j_2.

The characteristic equation of the minimal differential equation of j_2 gives the natural frequencies. So,

$$2s^2+3s+1=0 \Rightarrow s_1=-1, s_2=-\frac{1}{2}$$

Question 4

Calculate the natural frequencies of the circuits shown in Fig. 5. How many zero natural frequencies does each circuit have?

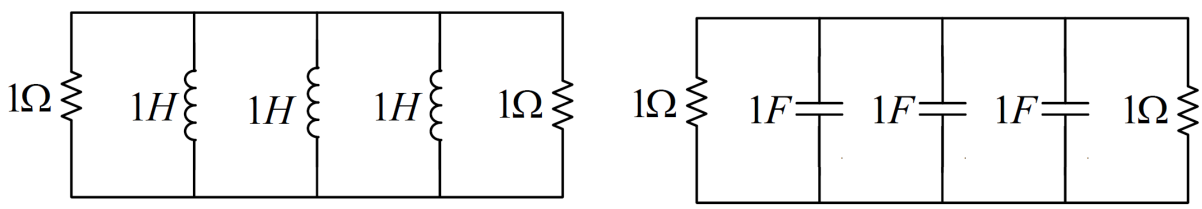

Figure 5: Two circuits with different types of energy storage elements.

For the right RC circuit, there are 3 capacitors and 2 independent capacitive loop, so the circuit has $3-2=1$ nonzero natural frequency. The nonzero natural frequency is

$$\Delta_n(s) = \text{det}[\boldsymbol{Y}_n] = |s+s+s+1+1| = 3s+2 = 0 \Rightarrow s = -\frac{2}{3}$$

For the left RL circuit, there are 3 inductors and 2 independent inductive loops. So, there are 3 natural frequencies including 2 zero and 1 nonzero natural frequencies. The nonzero natural frequency is obtained as

$$\Delta_n(s) = \text{det}[\boldsymbol{Y}_n] = |\frac{1}{s}+\frac{1}{s}+\frac{1}{s}+1+1| = \frac{3+2s}{s} = 0 \Rightarrow s = -\frac{3}{2}$$

Question 5

For the circuit shown in Fig. 6,

Question 5 continued on next page...

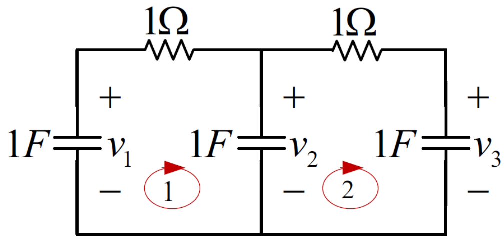

Figure 6: A circuit with three energy storage elements.

(a) Find the natural frequencies using the governing state equations.

We use capacitor voltages as the state variables.

$$\begin{cases} \frac{dv_1(t)}{dt} = v_2(t) - v_1(t) \\ \frac{dv_2(t)}{dt} = v_3(t) - v_2(t) + v_1(t) - v_2(t) = v_3(t) - 2v_2(t) + v_1(t) \\ \frac{dv_3(t)}{dt} = v_2(t) - v_3(t) \end{cases}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_1(0^-) \\ v_2(0^-) \\ v_3(0^-) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{X}(t)$$

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s+1 & -1 & 0 \\ -1 & s+2 & -1 \\ 0 & -1 & s+1 \end{bmatrix}$$

$\det[s\boldsymbol{I} - \boldsymbol{A}] = (s+1)[(s+2)(s+1)-1] - (s+1) = s^3 + 4s^2 + 3s = s(s+1)(s+3) \Rightarrow s = 0, -1, -3$

(b) Introduce a set of initial conditions for which only one natural frequency exist in zero-input response of the state variables.

If the initial conditions is parallel to the eigen vectors of matrix $\boldsymbol{A}$, only one natural frequency appears in the state variables. The eigen vectors are

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

and the zero-input response is

$$\boldsymbol{X}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}_0 = K_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + K_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-3t}$$

Question 5 continued on next page. . .

If we set $\boldsymbol{X}(0) = K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, only natural frequency 0 appears in the response. To show this, we find the unknown K_1, K_2, K_3 as

$$\boldsymbol{X}_0 = K_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + K_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This yields

$$\begin{cases} K_1 - K_2 + K_3 = K \\ K_1 - 2K_3 = K \\ K_1 + K_2 + K_3 = K \end{cases} \Rightarrow K_1 = K, \quad K_2 = K_3 = 0$$

So,

$$\boldsymbol{X}(t) = K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Similarly, if $\boldsymbol{X}(0) = K \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, the response only includes natural frequency -1. Finally, $\boldsymbol{X}(0) = K \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ causes the response to have only natural frequency -3.

SOFTWARE QUESTIONS

Question 6

Write a MATLAB function that finds the minimal differential equation and natural frequencies corresponding to the last variable in the matrix differential equation

$$\boldsymbol{A}(D)\boldsymbol{X} = \boldsymbol{F}$$

.

Note 1: Here, we have a matrix differential equation, where the elements of $\boldsymbol{A}(D)$ are polynomials of the differentiation operator $D^n, n \geq 0$. No element contains the integral operator D^{-1}.

Note 2: A polynomial can be expressed as a vector, i.e.,

$$\sum_{i=0}^{n} p_i D^i \equiv \begin{bmatrix} p_n & p_{n-1} & \cdots & p_1 & p_0 \end{bmatrix}$$

Question 6 continued on next page. . .

Here is a MATLAB function that computes the minimal differential equation and the corresponding natural frequencies.

```
function [natFreq, minDifEqu]=MDEC(A)

% the numbers in (-epsilon,epsilon) are assumed zero
epsilon = 1e-6;

for j=1:size(A,2)-1
    % manipulate column j
    while (sum(abs(cell2mat(A(j+1:end,j)'))) > epsilon)
        % find the minimum degree polynomial
        minDegRow = Inf;
        minDeg = Inf;
        for i = j:size(A,1)
            tmpDeg = length(cell2mat(A(i,j)))-1;
            if (tmpDeg < minDeg)
                minDegRow = i;
                minDeg = tmpDeg;
            end
        end

        % swap the polynomials
        tmpRow = A(minDegRow,:);
        A(minDegRow,:) = A(j,:);
        A(j,:) = tmpRow;

        % do elementry row operations
        for i = j+1:size(A,1)
            qTmp = deconv(cell2mat(A(i,j)), cell2mat(A(j,j)));
            for k=j:size(A,2)
                mulTmp = conv(qTmp, cell2mat(A(j,k)));
                eltTmp = [zeros(1,max([length(mulTmp) length(cell2mat(A(i,k)))])-length(cell2mat(A(i,k)))) cell2mat(A(i,k))]-[zeros(1,max([length(mulTmp) length(cell2mat(A(i,k)))])-length(mulTmp)) mulTmp];
                if (sum(abs(eltTmp))<epsilon)
                        eltTmp = 0;
                else
                    eltTmp=eltTmp(find(eltTmp~=0):end);
                end
                A(i,k) = {eltTmp};
            end
        end


    end
end

% print minimal differential equation
minDifEqu = cell2mat(A(end,end));
% calculate natural frequencies
natFreq = roots(minDifEqu);

end
```

You may use the following mfile to call the developed function and see its results.

```
% example 1
A = cell(2);
A(1,1) = {[1 2 1]};
A(1,2) = {[-1 -1]};
A(2,1) = {[-1 -1 0]};
A(2,2) ={[3 1]};
[natFreq, minDifEqu] = MDEC(A)

% example 2
A = cell(3);
A(1,1,:) = {[1 3 1]};
A(1,2,:) = {[-1]};
```

Question 6 continued on next page. . .

```
A(1,3,:) = {[-3]};
A(2,1,:) = {[-1]};
A(2,2,:) = {[1 3 1]};
A(2,3,:) = {[-3]};
A(3,1,:) = {[-3 0]};
A(3,2,:) = {[-3 0]};
A(3,3,:) = {[1 6]};

[natFreq, minDifEqu] = MDEC(A)
```

BONUS QUESTIONS

Question 7

Return your answers by filling the LaTeX template of the assignment. If you want to add a circuit schematic, you can draw it directly using TikZ package, or draw it in a secondary application such as Microsoft Visio and then, import it as a figure.

EXTRA QUESTIONS

Question 8

eel free to solve the following questions from the book *"Basic Circuit Theory"* by C. Desoer and E. Kuh.

1. **Chapter 14, question 1.**
2. **Chapter 14, question 2.**
3. **Chapter 14, question 4.**
4. **Chapter 14, question 5.**
5. **Chapter 14, question 6.**