State Equations

Mohammad Hadi

mohammad.hadi@sharif.edu

@MohammadHadiDastgerdi

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Overview

State Equations

State Equations

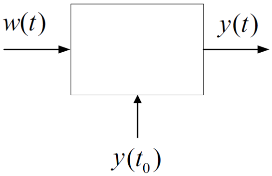


Figure: An LTI circuit with an input and initial conditions. State equations allow to solve a system of 1st-order linear differential equations instead of solving an nth order linear differential equation. State equations are very useful for analysis of time-varying circuits.

- nth-order linear differential equation:
  $\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{l=0}^{m} b_l w^{(l)}(t), \quad y(0^-), y'(0^-), \cdots, y^{(n-1)}(0^-)$
- System of 1st-order linear equations:
  $$\begin{cases} x_1'(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t) + b_1 w(t) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x_n'(t) = a_{1n}x_1(t) + \cdots + a_{nn}x_n(t) + b_n w(t) \end{cases}$$
- State equations: $\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t) + \boldsymbol{B}w(t), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$
- Output response: $y(t) = \boldsymbol{C}^T\boldsymbol{X}(t) + dw(t)$

State Equations

Definition (Network state)

A set of data qualifies to be called the state of a network if it satisfies two conditions

- For any time t_0, the state at time t_0 and the inputs from t_0 on determine uniquely the state at any time $t > t_0$.
- For any time t, the state at time t and the inputs at time t (and sometimes some of their derivatives) determine uniquely every network variable at time t.

The components of the state are called state variables.

State Equations for LTI Circuits

State Variables

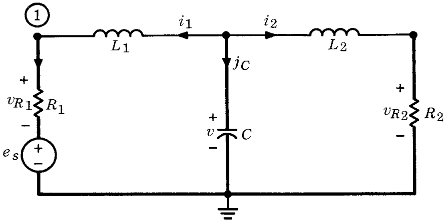


Figure: A sample suitable tree for writing state equations.

- State variables: Independent capacitor voltages and inductor currents.
- Number of state variables: Number of independent energy storage elements.
- Number of state variables: Circuit order.

State Equations

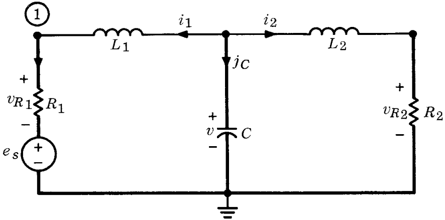


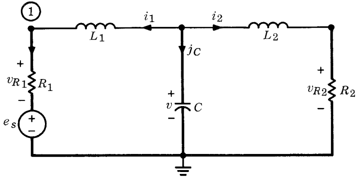
Figure: A sample suitable tree for writing state equations.

- State variables: Independent capacitor voltages and inductor currents.
- Proper tree: Capacitors in tree branches and inductors in link branches.
- State equations: KCL in fundamental cut sets and KVL in fundamental loops.
- State equations: $\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t) + \boldsymbol{B}\boldsymbol{W}(t), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$
- Output response: $y(t) = \boldsymbol{C}^T\boldsymbol{X}(t) + \boldsymbol{D}^T\boldsymbol{W}(t)$

State Equations

Example (State equations for a circuit with three state variables)

The first voltage node of the circuit below can be given in terms of state variables.



$$Cv' = -i_1 - i_2$$
$$i_1 = v_{R_1}/R_1$$
$$i_2 = v_{R_2}/R_2$$

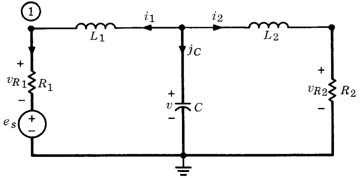
$$L_1 i_1' = v - v_{R_1} - e_s$$
$$L_2 i_1' = v - v_{R_2}$$

$$\begin{cases} Cv' = -i_1 - i_2 \\ L_1 i_1' = v - R_1 i_1 - e_s \\ L_2 i_1' = v - R_2 i_2 \end{cases} \Rightarrow \begin{bmatrix} v' \\ i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \begin{bmatrix} v \\ i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} e_s$$

State Equations

Example (State equations for a circuit with three state variables (cont.))

The first voltage node of the circuit below can be given in terms of state variables.

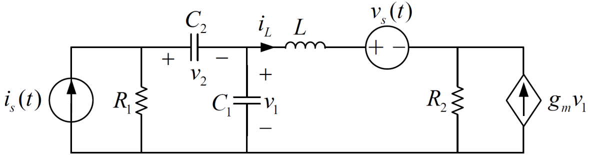


$$\boldsymbol{X}(t) = \begin{bmatrix} v(t) \\ i_1(t) \\ i_2(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v(0) \\ i_1(0) \\ i_2(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} e_s$$

$$e_1 = v_{R_1} + e_s = R_1 i_1 + e_s = \begin{bmatrix} 0 & R_1 & 0 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 1 \end{bmatrix} e_s$$

Example (State equations for a circuit with two inputs)

The capacitor current i_2 in the circuit below can be given in terms of state variables.



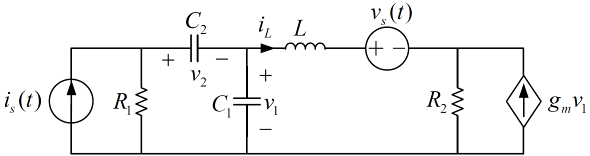
$$C_1 v_1' + v_{R_1}/R_1 + i_L - i_s = 0$$
$$C_2 v_2' + v_{R_1}/R_1 - i_s = 0$$
$$L i_L' + v_s + v_{R_2} - v_1 = 0$$

$$v_{R_1} = v_1 + v_2$$
$$v_{R_2} = R_2(i_L + g_m v_1)$$

$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_L(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_1(0) \\ v_2(0) \\ i_L(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} -\frac{1}{R_1 C_1} & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} \\ -\frac{1}{R_1 C_2} & -\frac{1}{R_1 C_2} & 0 \\ \frac{1-g_m R_2}{L} & 0 & -\frac{R_2}{L} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 & \frac{1}{C_1} \\ 0 & \frac{1}{C_2} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$

Example (State equations for a circuit with two inputs (cont.))

The capacitor current i_2 in the circuit below can be given in terms of state variables.



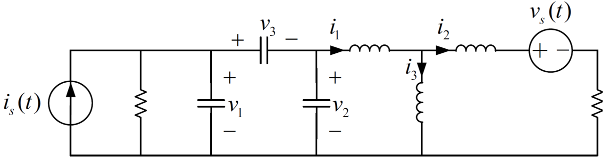
$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_L(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_1(0) \\ v_2(0) \\ i_L(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} -\frac{1}{R_1 C_1} & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} \\ -\frac{1}{R_1 C_2} & -\frac{1}{R_1 C_2} & 0 \\ \frac{1-g_m R_2}{L} & 0 & -\frac{R_2}{L} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 & \frac{1}{C_1} \\ 0 & \frac{1}{C_2} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$

$$\boldsymbol{W}(t) = \begin{bmatrix} v_s(t) \\ i_s(t) \end{bmatrix} \Rightarrow i_2 = C_2 v_2' = -\frac{v_1 + v_2}{R_1} + i_s = \begin{bmatrix} -\frac{1}{R_1} & -\frac{1}{R_1} & 0 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{W}$$

State Equations

Example (State equations for a circuit with dependent energy storage elements)

All the energy storage elements are not among the state variables of the circuit below whose element have unit values.



$$\begin{cases} v_1' + v_1 - i_s + (v_1 - v_2)' = 0 \\ v_2' + i_1 - (v_1 - v_2)' = 0 \\ i_1' + (i_1 - i_2)' - v_2 = 0 \\ i_2' + v_s + i_2 - (i_1 - i_2)' = 0 \end{cases}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_1(t) \\ i_2(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_1(0) \\ v_2(0) \\ i_1(0) \\ i_2(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} -\frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & -\frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \\ -\frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$

State Equations for LTV circuits

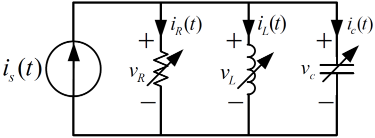
State Equations

- State variables: Independent capacitor voltages and inductor currents.
- State variables: Independent capacitor charges and inductor fluxes.
- Number of state variables: Number of independent energy storage elements.
- Proper tree: Capacitors in tree branches and inductors in link branches.
- State equations: KCL in fundamental cut sets and KVL in fundamental loops.
- State equations: $\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}(t)\boldsymbol{X}(t) + \boldsymbol{B}(t)\boldsymbol{W}(t), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$
- Output response: $y(t) = \boldsymbol{C}^T(t)\boldsymbol{X}(t) + \boldsymbol{D}^T(t)\boldsymbol{W}(t)$

State Equations

Example (State equations for an LTV circuit)

Capacitor voltage and inductor current can be used as state variables in the LTV circuit below.

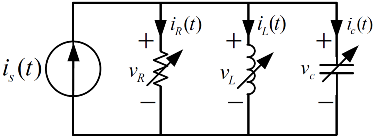


$$\begin{cases} \frac{dq(t)}{dt} + \frac{v_C(t)}{R(t)} + i_L(t) - i_s(t) = 0 \\ \frac{d\phi(t)}{dt} - v_C(t) = 0 \end{cases} , \quad \begin{cases} \frac{dq(t)}{dt} = C(t)\frac{dv_C c(t)}{dt} + v_C(t)\frac{C(t)}{dt} \\ \frac{d\phi(t)}{dt} = L(t)\frac{di_L(t)}{dt} + i_L(t)\frac{L(t)}{dt} \end{cases}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} -\frac{1}{R(t)C(t)} - \frac{C'(t)}{C(t)} & -\frac{1}{C(t)} \\ \frac{1}{L(t)} & -\frac{L'(t)}{L(t)} \end{bmatrix} \boldsymbol{X}(t) + \begin{bmatrix} -\frac{1}{C(t)} \\ 0 \end{bmatrix} i_s(t)$$

Example (State equations for an LTV circuit)

Capacitor charge and inductor flux can be used as state variables in the LTV circuit below.



$$\begin{cases} \frac{dq(t)}{dt} + \frac{v_C(t)}{R(t)} + i_L(t) - i_s(t) = 0 \\ \frac{d\phi(t)}{dt} - v_C(t) = 0 \end{cases} \Rightarrow \begin{cases} \frac{dq(t)}{dt} + \frac{q(t)}{c(t)R(t)} + \frac{\phi(t)}{L(t)} - i_s(t) = 0 \\ \frac{d\phi(t)}{dt} - \frac{q(t)}{C(t)} = 0 \end{cases}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} q(0) \\ \phi(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} -\frac{1}{R(t)C(t)} & -\frac{1}{L(t)} \\ \frac{1}{C(t)} & 0 \end{bmatrix} \boldsymbol{X}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_s(t)$$

State Equations for NTV circuits

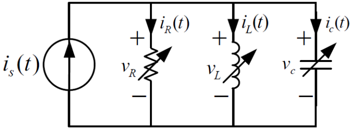
State Equations

- State variables: Independent capacitor charges and inductor fluxes.
- Number of state variables: Number of independent energy storage elements.
- State equations: $\frac{d}{dt}\boldsymbol{X}(t) = f(\boldsymbol{X}(t), \boldsymbol{W}(t), t), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$
- Output response: $y(t) = h(\boldsymbol{X}(t), \boldsymbol{W}(t), t)$

State Equations

Example (State equations for an NTI circuit)

Capacitor voltage and inductor current can be used as state variables in the NTI circuit below.



$$\begin{cases} i_R = f_1(v_R) \\ q = f_2(v_c) \\ \varphi = f_3(i_L) \end{cases}$$

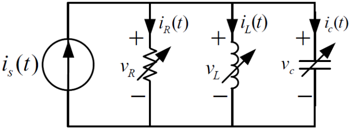
$$\begin{cases} \frac{dq(t)}{dt} + i_R(t) + i_L(t) - i_s(t) = 0 \\ \frac{d\phi(t)}{dt} - v_C(t) = 0 \end{cases} , \quad \begin{cases} \frac{dq(t)}{dt} = \frac{dq}{dv_C}\frac{dv_C}{dt} = f_2'(v_C)\frac{dv_C}{dt} \\ \frac{d\phi(t)}{dt} = \frac{d\phi}{di_L}\frac{di_L}{dt} = f_3'(i_L)\frac{di_L}{dt} \end{cases}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} \frac{-f_1(v_C(t)) - i_L(t) + i_s(t)}{f_2'(v_C(t))} \\ \frac{v_C(t)}{f_3'(i_L(t))} \end{bmatrix}$$

State Equations

Example (State equations for a an NTI circuit)

Capacitor charge and inductor flux can be used as state variables in the NTI circuit below.



$$\begin{cases} i_R = f_1(v_R) \\ v_c = f_2(q) \\ i_L = f_3(\varphi) \end{cases}$$

$$\begin{cases} \frac{dq(t)}{dt} + i_R(t) + i_L(t) - i_s(t) = 0 \\ \frac{d\phi(t)}{dt} - v_C(t) = 0 \end{cases} , \quad \begin{cases} \frac{dq(t)}{dt} + f_1(f_2(q(t))) + f_3(\phi(t)) - i_s(t) = 0 \\ \frac{d\phi(t)}{dt} - f_2(q(t)) = 0 \end{cases}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} q(0) \\ \phi(0) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} -f_1(f_2(q(t))) - f_3(\phi(t)) + i_s(t) \\ f_2(q(t)) \end{bmatrix}$$

Solving State Equations

Single State Equation

- Single state equation: $x'(t) = ax(t) + bw(t), x(0)$
- Zero-input response: $x_{zi}(t) = x(0)e^{at}, t \geq 0$
- Zero-state response: $x_{zs}(t) = \int_0^t e^{a(t-t')} bw(t')dt', t \geq 0$
- Complete response: $x_{cs}(t) = x_{zi}(t) + x_{zs}(t), t \geq 0$

System of State Equations

- Single state equation: $\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t) + \boldsymbol{B}\boldsymbol{W}(t), \quad \boldsymbol{X}(0)$
- Zero-input response: $\boldsymbol{X}_{zi}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}(0), t \geq 0$
- Zero-state response: $\boldsymbol{X}_{zs}(t) = \int_0^t e^{\boldsymbol{A}(t-t')}\boldsymbol{B}\boldsymbol{W}(t')dt', t \geq 0$
- Complete response: $\boldsymbol{X}_{cs}(t) = \boldsymbol{X}_{zi}(t) + \boldsymbol{X}_{zs}(t), t \geq 0$
- Matrix exponential: $e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \frac{1}{3!}\boldsymbol{A}^3t^3 + \cdots$

Exponential Matrix

- Eigen values and vectors: $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \Rightarrow \lambda_1, \cdots, \lambda_n, \quad \boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$
- Diagonal decomposition:

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_n \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\Rightarrow \boldsymbol{A} = \boldsymbol{U}\Lambda\boldsymbol{U}^{-1} \Rightarrow e^{\boldsymbol{A}t} = \boldsymbol{U}e^{\Lambda t}\boldsymbol{U}^{-1}, \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Exponential Matrix

- Zero-input response: $\frac{d}{dt}\boldsymbol{X}_{zi}(t) = \boldsymbol{A}\boldsymbol{X}_{zi}(t), \boldsymbol{X}_{zi}(0) = \boldsymbol{X}_0 \Rightarrow \boldsymbol{X}_{zi}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}_0$
- Laplace-domain zero-input response:
  $s\boldsymbol{X}_{zi}(s) - \boldsymbol{X}_0 = A\boldsymbol{X}_{zi}(s) \Rightarrow \boldsymbol{X}_{zi}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{X}_0$
- Laplace-domain zero-input response: $\boldsymbol{X}_{zi}(s) = \mathcal{L}\left\{e^{\boldsymbol{A}t}\right\}\boldsymbol{X}_0$
- Laplace transform of matrix exponential: $\mathcal{L}\left\{e^{\boldsymbol{A}t}\right\} = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$
- Matrix exponential: $e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right\}$

Example (Solving state equation)

Matrix diagonal decomposition can be used to solve the matrix state equation below.

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{AX}(t), \quad \boldsymbol{A} = \begin{bmatrix} -0.5 & 1 \\ 1 & -3 \end{bmatrix}$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = (-0.5 - \lambda)(-3 - \lambda) + 1 = 0 \Rightarrow \lambda = -1, -2.5$$

$$\boldsymbol{Au} = \lambda \boldsymbol{u} \Rightarrow \boldsymbol{Au} = -\boldsymbol{u} \Rightarrow \boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{Au} = \lambda \boldsymbol{u} \Rightarrow \boldsymbol{Au} = -2.5\boldsymbol{u} \Rightarrow \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{U}\Lambda\boldsymbol{U}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2.5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$e^{\boldsymbol{A}t} = \boldsymbol{U}e^{\Lambda t}\boldsymbol{U}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2.5t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4e^{-t} - e^{-2.5t} & -2e^{-t} + 2e^{-2.5t} \\ 2e^{-t} - 2e^{-2.5t} & -e^{-t} + 4e^{-2.5t} \end{bmatrix}$$

$$\boldsymbol{X}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}_0 = \frac{1}{3} \begin{bmatrix} 4e^{-t} - e^{-2.5t} & -2e^{-t} + 2e^{-2.5t} \\ 2e^{-t} - 2e^{-2.5t} & -e^{-t} + 4e^{-2.5t} \end{bmatrix} \boldsymbol{X}_0$$

Example (Solving state equation)

Laplace transform can be used to solve the matrix state equation below.

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t), \quad \boldsymbol{A} = \begin{bmatrix} -0.5 & 1 \\ 1 & -3 \end{bmatrix}$$

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s+0.5 & 1 \\ -1 & s+3 \end{bmatrix}^{-1} \right\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+2.5)} & \frac{-1}{(s+1)(s+2.5)} \\ \frac{1}{(s+1)(s+2.5)} & \frac{s+0.5}{(s+1)(s+2.5)} \end{bmatrix} \right\}$$

$$\boldsymbol{X}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}_0 = \frac{1}{3}\begin{bmatrix} 4e^{-t} - e^{-2.5t} & -2e^{-t} + 2e^{-2.5t} \\ 2e^{-t} - 2e^{-2.5t} & -e^{-t} + 4e^{-2.5t} \end{bmatrix}\boldsymbol{X}_0$$

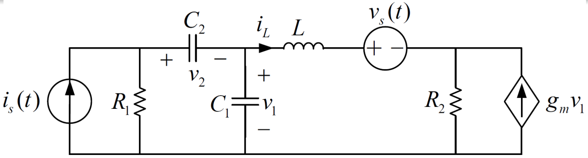
Response Equation

- State equations: $\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{AX}(t) + \boldsymbol{B}w(t), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$
- Laplace-domain state equations:
$s\boldsymbol{X}(s) - \boldsymbol{X}_0 = \boldsymbol{AX}(s) + \boldsymbol{B}W(s) \Rightarrow \boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}W(s) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{X}_0$
- Single-input Output response: $y(t) = \boldsymbol{C}^T\boldsymbol{X}(t) + dw(t)$
- Laplace transform of Output response:
$Y(s) = \boldsymbol{C}^T\boldsymbol{X}(s) + dW(s) = \boldsymbol{C}^T\left[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}W(s) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{X}_0\right] + dW(s)$
- Transfer function: $H(s) = \frac{Y(s)}{W(s)}\Big|_{\boldsymbol{X}_0=0} = \boldsymbol{C}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + d$

Single-input Response Equation

Example (Transfer function)

Transfer function can be found by the matrices involved in state equations



$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_L(t) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} -\frac{1}{R_1C_1} & -\frac{1}{R_1C_1} & -\frac{1}{C_1} \\ -\frac{1}{R_1C_2} & -\frac{1}{R_1C_2} & 0 \\ \frac{1-g_mR_2}{L} & 0 & -\frac{R_2}{L} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{L} \end{bmatrix} v_s + \begin{bmatrix} \frac{1}{C_1} \\ \frac{1}{C_2} \\ 0 \end{bmatrix} i_s$$

$$\boldsymbol{W}(t) = \begin{bmatrix} v_s(t) \\ i_s(t) \end{bmatrix} \Rightarrow i_2 = C_2 v_2' = \begin{bmatrix} -\frac{1}{R_1} & -\frac{1}{R_1} & 0 \end{bmatrix} \boldsymbol{X} + 0 \times v_s(t) + 1 \times i_s(t)$$

$$H_1(s) = \frac{I_2(s)}{V_s(s)}\Big|_{I_s(s)=0} = \begin{bmatrix} -\frac{1}{R_1} & -\frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} s+\frac{1}{R_1C_1} & \frac{1}{R_1C_1} & \frac{1}{C_1} \\ \frac{1}{R_1C_2} & s+\frac{1}{R_1C_2} & 0 \\ -\frac{1-g_mR_2}{L} & 0 & s+\frac{R_2}{L} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{L} \end{bmatrix} + 0$$

$$H_2(s) = \frac{I_2(s)}{I_s(s)}\Big|_{V_s(s)=0} = \begin{bmatrix} -\frac{1}{R_1} & -\frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} s+\frac{1}{R_1C_1} & \frac{1}{R_1C_1} & \frac{1}{C_1} \\ \frac{1}{R_1C_2} & s+\frac{1}{R_1C_2} & 0 \\ -\frac{1-g_mR_2}{L} & 0 & s+\frac{R_2}{L} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{C_1} \\ \frac{1}{C_2} \\ 0 \end{bmatrix} + 1$$

Approximated Solution

- General state equations: $\frac{d}{dt}\boldsymbol{X}(t) = f(\boldsymbol{X}(t)), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$
- Approximated solution for general state equations:

$$\boldsymbol{X}(\Delta t) \approx \boldsymbol{X}(0) + \frac{d}{dt}\boldsymbol{X}(t)\Big|_{t=0}\Delta t = \boldsymbol{X}(0) + f(\boldsymbol{X}(0))\Delta t$$

$$\boldsymbol{X}(2\Delta t) \approx \boldsymbol{X}(\Delta t) + f(\boldsymbol{X}(\Delta t))\Delta t$$

$$\vdots$$

$$\boldsymbol{X}((k+1)\Delta t) \approx \boldsymbol{X}(k\Delta t) + f(\boldsymbol{X}(k\Delta t))\Delta t$$

- Approximated solution for LTI state equations:

$$\boldsymbol{X}((k+1)\Delta t) \approx \boldsymbol{X}(k\Delta t) + f(\boldsymbol{X}(k\Delta t))\Delta t = \boldsymbol{X}(k\Delta t) + \boldsymbol{A}\boldsymbol{X}(k\Delta t)\Delta t$$

$$\boldsymbol{X}((k+1)\Delta t) \approx (\boldsymbol{I} + \boldsymbol{A}\Delta t)\boldsymbol{X}(k\Delta t)$$

Example (State trajectory)

State trajectory can be plotted using approximated numerical methods.

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{AX}(t), \quad \boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ -1 & -3 \end{bmatrix}, \quad \boldsymbol{X}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s+1 & 0 \\ -1 & s+3 \end{bmatrix}^{-1} \right\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{-1}{(s+1)(s+3)} & \frac{1}{s+3} \end{bmatrix} \right\}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\boldsymbol{A}t}\boldsymbol{X}_0 = \begin{bmatrix} e^{-t} & 0 \\ -0.5e^{-t} + 0.5e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -0.5e^{-t} + 0.5e^{-3t} \end{bmatrix} \Rightarrow x_2 = \frac{-x_1 + x_1^3}{2}$$

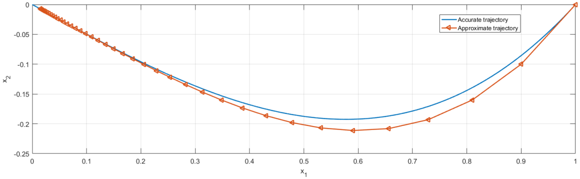
$$\Delta t = 0.1 \Rightarrow \boldsymbol{I} + \boldsymbol{A}\Delta t = \begin{bmatrix} 0.9 & 0 \\ -0.1 & 0.7 \end{bmatrix} \Rightarrow \boldsymbol{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{X}(0.1) = \begin{bmatrix} 0.9 & 0 \\ -0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9 \\ -0.1 \end{bmatrix}$$

$$\boldsymbol{X}(0.2) = \begin{bmatrix} 0.81 \\ -0.16 \end{bmatrix}, \boldsymbol{X}(0.3) = \begin{bmatrix} 0.729 \\ -0.193 \end{bmatrix}, \cdots, \boldsymbol{X}(0.2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

State Trajectory

Example (State trajectory)

State trajectory can be plotted using approximated numerical methods.



State Trajectory

Example (State trajectory)

Natural frequencies involved in the response can be removed if the initial condition equal eigen vectors.

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t), \quad \boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ -1 & -3 \end{bmatrix}, \quad \boldsymbol{X}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ -0.5e^{-t} + 0.5e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ae^{-t} \\ -0.5ae^{-t} + (0.5a + b)e^{-3t} \end{bmatrix}$$

$$\Rightarrow x_2 = -0.5x_1 + 0.5x_1^3 + \frac{b}{a}x_1^3$$

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ -1 & -3 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1, -2, \quad \boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{X}_0 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \boldsymbol{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix} \Rightarrow x_1 = 0, x_2 \in [0, 1]$$

The End