Systematic Analysis

Mohammad Hadi

mohammad.hadi@sharif.edu

@MohammadHadiDastgerdi

Fall 2021

Overview

Circuit Analysis

Circuit Analysis

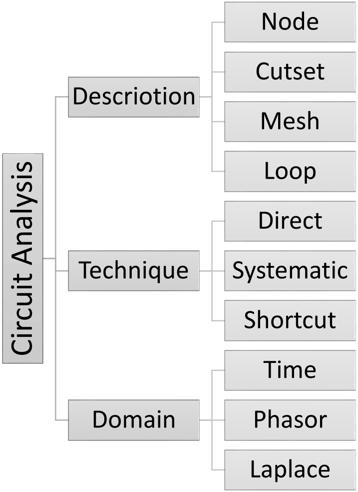


Figure: Different descriptions, techniques, and domains of circuit analysis.

Circuit Analysis

- Knowns
  - Network topology
  - Element characteristic equations
  - Independent sources
  - Initial conditions
- Unknowns
  - b branch voltages
  - b branch currents
- Equations
  - $n_t - 1$ independent KCLs
  - $b - n_t + 1$ independent KVLs
  - b branch characterizing equations

Circuit Analysis

- Vectors
  - Branch voltages vector: $\boldsymbol{V}(t) = \begin{bmatrix} v_1(t) & v_2(t) & \cdots & v_b(t) \end{bmatrix}^T$
  - Branch currents vector: $\boldsymbol{J}(t) = \begin{bmatrix} j_1(t) & j_2(t) & \cdots & j_b(t) \end{bmatrix}^T$
  - Tree branch (node) voltages vector:
    $\boldsymbol{E}(t) = \begin{bmatrix} e_1(t) & e_2(t) & \cdots & e_n(t) \end{bmatrix}^T, \quad n = n_t - 1$
  - Link branch (mesh) currents vector:
    $\boldsymbol{I}(t) = \begin{bmatrix} i_1(t) & i_2(t) & \cdots & i_l(t) \end{bmatrix}^T, \quad l = b - n_t + 1$
- Laplace-domain Descriptions
  - Node-bases: $\begin{cases} \boldsymbol{AJ}(s) = 0 \\ \boldsymbol{V}(s) = \boldsymbol{A}^T\boldsymbol{E}(s) \end{cases} \Rightarrow \boldsymbol{Y}_n(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s) \Rightarrow \boldsymbol{E}(s) \Rightarrow \boldsymbol{V}(s) = \boldsymbol{A}^T\boldsymbol{E}(s)$
  - Cutset-bases: $\begin{cases} \boldsymbol{QJ}(s) = 0 \\ \boldsymbol{V}(s) = \boldsymbol{Q}^T\boldsymbol{E}(s) \end{cases} \Rightarrow \boldsymbol{Y}_q(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s) \Rightarrow \boldsymbol{E}(s) \Rightarrow \boldsymbol{V}(s) = \boldsymbol{Q}^T\boldsymbol{E}(s)$
  - Mesh-bases: $\begin{cases} \boldsymbol{MV}(s) = 0 \\ \boldsymbol{J}(s) = \boldsymbol{M}^T\boldsymbol{I}(s) \end{cases} \Rightarrow \boldsymbol{Z}_m(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s) \Rightarrow \boldsymbol{I}(s) \Rightarrow \boldsymbol{J}(s) = \boldsymbol{M}^T\boldsymbol{I}(s)$
  - Loop-bases: $\begin{cases} \boldsymbol{BV}(s) = 0 \\ \boldsymbol{J}(s) = \boldsymbol{B}^T\boldsymbol{I}(s) \end{cases} \Rightarrow \boldsymbol{Z}_l(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s) \Rightarrow \boldsymbol{I}(s) \Rightarrow \boldsymbol{J}(s) = \boldsymbol{B}^T\boldsymbol{I}(s)$

Branch Model

Branch Model

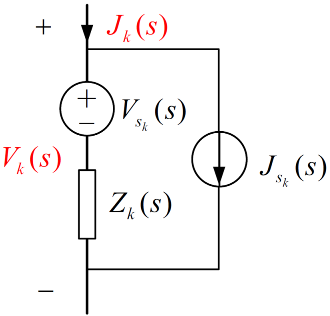


Figure: Two equivalent models for a branch. The voltage and current equations are not suitable for describing alone current and voltage sources, respectively. The voltage and current equations suit mesh/loop and node/cutset analysis techniques, respectively.

$$V_k(s) = V_{s_k}(s) + Z_k(s)(J_k(s) - J_{s_k}(s)) = Z_k(s)J_k(s) + V_{s_k}(s) - Z_k(s)J_{s_k}(s)$$

$$J_k(s) = J_{s_k}(s) + Y_k(s)(V_k(s) - V_{s_k}(s)) = Y_k(s)V_k(s) + J_{s_k}(s) - Y_k(s)V_{s_k}(s)$$

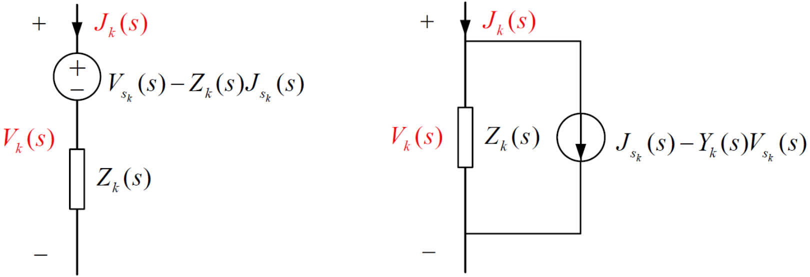


Figure: Simplified series and parallel models for a branch.

$$V_k(s) = V_{s_k}(s) + Z_k(s)(J_k(s) - J_{s_k}(s)) = Z_k(s)J_k(s) + V_{s_k}(s) - Z_k(s)J_{s_k}(s)$$

$$J_k(s) = J_{s_k}(s) + Y_k(s)(V_k(s) - V_{s_k}(s)) = Y_k(s)V_k(s) + J_{s_k}(s) - Y_k(s)V_{s_k}(s)$$

Source Transformation

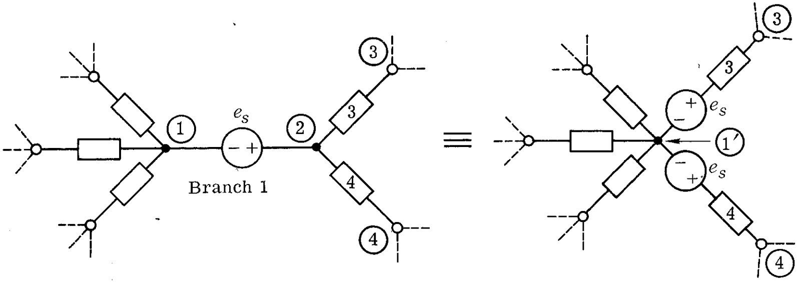


Figure: For any given network, voltage source transformation can be used to modify the network such that each voltage source is connected in series with an element which is not a source. This transformation is suitable for node/cutset analysis.

Source Transformation

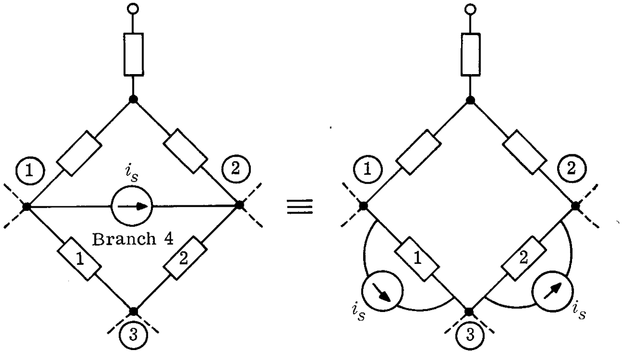


Figure: For any given network, current source transformation can be used to modify the network such that each current source is connected in parallel with an element which is not a source. This transformation is suitable for mesh/loop analysis.

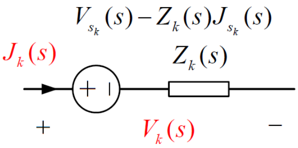


Figure: Branch impedance matrix is diagonal if there is no coupled element or dependent source in the circuit. Impedance representation suits mesh and loop analysis, where alone current sources should be transformed.

$$\begin{cases} V_1(s) = Z_1(s)J_1(s) + V_{s_1}(s) - Z_1(s)J_{s_1}(s) \\ V_2(s) = Z_2(s)J_2(s) + V_{s_2}(s) - Z_2(s)J_{s_2}(s) \\ \vdots \qquad\qquad\qquad \vdots \\ V_b(s) = Z_b(s)J_b(s) + V_{s_b}(s) - Z_b(s)J_{s_b}(s) \end{cases}$$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_b(s) \end{bmatrix} = \begin{bmatrix} Z_1(s) & 0 & \cdots & 0 \\ 0 & Z_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_b(s) \end{bmatrix} \begin{bmatrix} J_1(s) \\ J_2(s) \\ \vdots \\ J_b(s) \end{bmatrix} + \begin{bmatrix} V_{s_1}(s) \\ V_{s_2}(s) \\ \vdots \\ V_{s_b}(s) \end{bmatrix} - \begin{bmatrix} Z_1(s) & 0 & \cdots & 0 \\ 0 & Z_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_b(s) \end{bmatrix} \begin{bmatrix} J_{s_1}(s) \\ J_{s_2}(s) \\ \vdots \\ J_{s_b}(s) \end{bmatrix}$$

$$\boldsymbol{V}(s) = \boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)$$

$$\boldsymbol{V}(s) = \boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \bar{\boldsymbol{V}}_s(s)$$

Branch Matrices

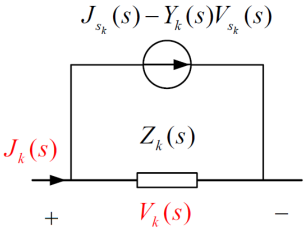


Figure: Branch admittance matrix is diagonal if there is no coupled element or dependent source in the circuit. Admittance representation suits node and cut set analysis, where alone voltage sources should be transformed.

$$\begin{cases} J_1(s) = Y_1(s)V_1(s) + J_{s_1}(s) - Y_1(s)V_{s_1}(s) \\ J_2(s) = Y_2(s)V_2(s) + J_{s_2}(s) - Y_2(s)V_{s_2}(s) \\ \vdots \qquad\qquad\qquad \vdots \\ J_b(s) = Y_b(s)V_b(s) + J_{s_b}(s) - Y_b(s)V_{s_b}(s) \end{cases}$$

$$\begin{bmatrix} J_1(s) \\ J_2(s) \\ \vdots \\ J_b(s) \end{bmatrix} = \begin{bmatrix} Y_1(s) & 0 & \cdots & 0 \\ 0 & Y_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_b(s) \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_b(s) \end{bmatrix} + \begin{bmatrix} J_{s_1}(s) \\ J_{s_2}(s) \\ \vdots \\ J_{s_b}(s) \end{bmatrix} - \begin{bmatrix} Y_1(s) & 0 & \cdots & 0 \\ 0 & Y_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_b(s) \end{bmatrix} \begin{bmatrix} V_{s_1}(s) \\ V_{s_2}(s) \\ \vdots \\ V_{s_b}(s) \end{bmatrix}$$

$$\boldsymbol{J}(s) = \boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)$$

$$\boldsymbol{J}(s) = \boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \bar{\boldsymbol{J}}_s(s)$$

Node Analysis

Systematic Approach

- Method:

$$
\begin{aligned}
&\boldsymbol{J}(s) = \boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)\\
&\boldsymbol{AJ}(s) = 0 \Rightarrow \boldsymbol{A}[\boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)] = 0\\
&\boldsymbol{V}(s) = \boldsymbol{A}^T\boldsymbol{E}(s) \Rightarrow \boldsymbol{A}[\boldsymbol{Y}_b(s)\boldsymbol{A}^T\boldsymbol{E}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)] = 0\\
&[\boldsymbol{AY}_b(s)\boldsymbol{A}^T]\boldsymbol{E}(s) = \boldsymbol{AY}_b(s)\boldsymbol{V}_s(s) - \boldsymbol{AJ}_s(s)\\
&\boldsymbol{Y}_n(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s), \quad \boldsymbol{Y}_n(s) = \boldsymbol{AY}_b(s)\boldsymbol{A}^T, \boldsymbol{I}_s(s) = \boldsymbol{AY}_b(s)\boldsymbol{V}_s(s) - \boldsymbol{AJ}_s(s)
\end{aligned}
$$

- Steps:

$$
\begin{aligned}
&\boldsymbol{E}(s) = \boldsymbol{Y}_n^{-1}(s)\boldsymbol{I}_s(s)\\
&\boldsymbol{V}(s) = \boldsymbol{A}^T\boldsymbol{E}(s)\\
&\boldsymbol{J}(s) = \boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)
\end{aligned}
$$

- Conditions:
  - $\det[\boldsymbol{Y}_n(s)] \neq 0$
  - No alone voltage source as a branch.

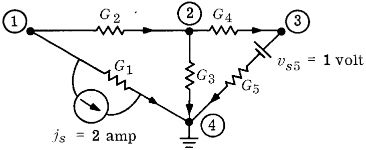
Shortcut Approach

- LTI network without coupling, dependent sources, alone voltage sources, and initial conditions
  - Node admittance matrix $\boldsymbol{Y}_n(s) = \boldsymbol{A}\boldsymbol{Y}_b(s)\boldsymbol{A}^T$
    - Diagonal elements: Sum of the addimtances connected to node i, i.e., $[\boldsymbol{Y}_n(s)]_{ii} = \sum_{k=1}^{b}(a_{ik})^2 Y_k(s)$
    - Off-diagonal elements: Negative sum of the addimtances connected to nodes i and j, i.e., $[\boldsymbol{Y}_n(s)]_{ij} = \sum_{k=1}^{b} a_{ik} Y_k(s) a_{jk}$
  - Current source vector: $\boldsymbol{I}_s(s) = \boldsymbol{A}\boldsymbol{Y}_b(s)\boldsymbol{V}_s(s) - \boldsymbol{A}\boldsymbol{J}_s(s)$
    - Vector elements: Sum of the currents flowing to node i, i.e., $[\boldsymbol{I_s}(s)]_i = \sum_{k=1}^{b} a_{ik}[Y_k(s)V_{s_k}(s) - J_{s_k}(s)]$
- Coupled elements: Add the corresponding off-diagonal elements to $\boldsymbol{Y}_n(s)$
- Dependent sources: Consider them as independent sources
- Alone voltage source: Use supernode or source transformation
- Initial conditions: Model them using parallel independent sources

Node Analysis

Example (Systematic node analysis for a resistive network)

Systematic node analysis can be used for the resistive network below.



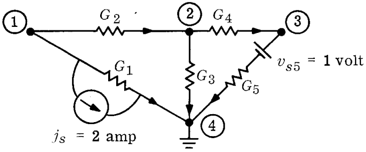
G_1 = 2 mhos
G_2 = 1 mho
G_3 = 3 mhos
G_4 = 1 mho
G_5 = 1 mho

$$\boldsymbol{Aj} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v} = \boldsymbol{A}^T \boldsymbol{e} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example (Systematic node analysis for a resistive network)

Systematic node analysis can be used for the resistive network below.



G_1 = 2 mhos
G_2 = 1 mho
G_3 = 3 mhos
G_4 = 1 mho
G_5 = 1 mho

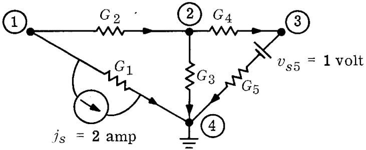
$$\boldsymbol{Y}_n = \boldsymbol{A}\boldsymbol{Y}_b\boldsymbol{A}^T = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{i}_s = \boldsymbol{A}\boldsymbol{Y}_b\boldsymbol{v}_s - \boldsymbol{A}\boldsymbol{j}_s = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{Y}_n\boldsymbol{e} = \boldsymbol{i}_s$$

$$\boldsymbol{e} = \boldsymbol{Y}_n^{-1}\boldsymbol{i}_s = \frac{1}{25}\begin{bmatrix} 9 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 14 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -17 \\ -1 \\ 12 \end{bmatrix}, \quad \boldsymbol{v} = \boldsymbol{A}^T\boldsymbol{e} = \frac{1}{25}\begin{bmatrix} -17 \\ -16 \\ -1 \\ -13 \\ 12 \end{bmatrix}, \quad \boldsymbol{j} = \frac{1}{25}\begin{bmatrix} 16 \\ -16 \\ -3 \\ -13 \\ -13 \end{bmatrix}$$

Node Analysis

Example (Shortcut node analysis for a resistive network)

Shortcut node analysis can be used for the resistive network below.



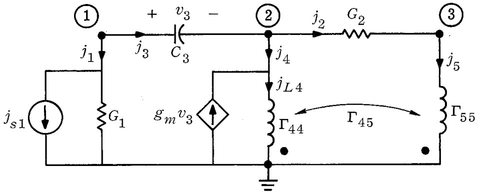
G_1 = 2 mhos
G_2 = 1 mho
G_3 = 3 mhos
G_4 = 1 mho
G_5 = 1 mho

$$\boldsymbol{Y}_n\boldsymbol{e} = \boldsymbol{i}_s \Rightarrow \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{e} = \boldsymbol{Y}_n^{-1}\boldsymbol{i}_s = \frac{1}{25}\begin{bmatrix} 9 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 14 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -17 \\ -1 \\ 12 \end{bmatrix}, \quad \boldsymbol{v} = \boldsymbol{A}^T\boldsymbol{e} = \frac{1}{25}\begin{bmatrix} -17 \\ -16 \\ -1 \\ -13 \\ 12 \end{bmatrix}, \quad \boldsymbol{j} = \frac{1}{25}\begin{bmatrix} 16 \\ -16 \\ -3 \\ -13 \\ -13 \end{bmatrix}$$

Example (Shortcut node analysis for an LTI circuit)

Shortcut node analysis can be used for the LTI circuit below.

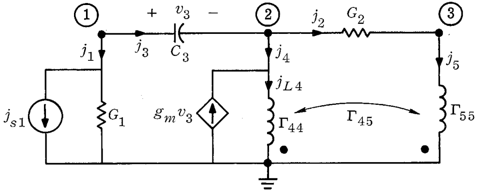


$$\begin{bmatrix} G_1 + C_3 D & -C_3 D & 0 \\ -C_3 D & C_3 D + \Gamma_{44} D^{-1} + G_2 & -G_2 \\ 0 & -G_2 & \Gamma_{55} D^{-1} + G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ -j_{L4}(0) + g_m(e_1 - e_2) - \Gamma_{45} D^{-1} e_3 \\ -j_{L5}(0) - \Gamma_{45} D^{-1} e_2 \end{bmatrix}$$

$$\begin{bmatrix} G_1 + C_3 D & -C_3 D & 0 \\ -C_3 D - g_m & C_3 D + \Gamma_{44} D^{-1} + G_2 + g_m & -G_2 + \Gamma_{45} D^{-1} \\ 0 & -G_2 + \Gamma_{45} D^{-1} & \Gamma_{55} D^{-1} + G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ -j_{L4}(0) \\ -j_{L5}(0) \end{bmatrix}$$

Example (Shortcut node analysis for an LTI circuit (cont.))

Shortcut node analysis can be used for the LTI circuit below.

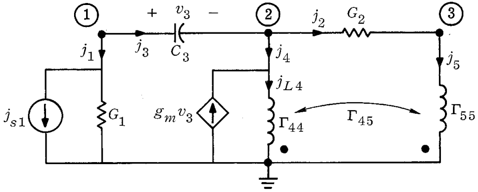


$$\begin{bmatrix} G_1 + C_3 D & -C_3 D^2 & 0 \\ -C_3 D - g_m & C_3 D^2 + \Gamma_{44} + G_2 D + g_m D & -G_2 D + \Gamma_{45} \\ 0 & -G_2 D + \Gamma_{45} & \Gamma_{55} + G_2 D \end{bmatrix} \begin{bmatrix} e_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -j_{s1} \\ -j_{L4}(0) \\ -j_{L5}(0) \end{bmatrix}$$

$$\begin{cases} e_1(0) = \frac{1}{G_1}[-j_{s1}(0) - j_{L4}(0) + g_m v_3(0) - j_{L5}(0)] \\ \phi_2(0) = L_{44} j_{L4}(0) + L_{45} j_{L5}(0), \quad \phi_2'(0) = e_2(0) = e_1(0) - v_3(0) \\ \phi_3(0) = L_{45} j_{L4}(0) + L_{55} j_{L5}(0), \quad \phi_3'(0) = e_3(0) = e_2(0) - j_5(0)/G_2 \end{cases}$$

Example (Shortcut node analysis for an LTI circuit)

Shortcut node analysis can be used for the LTI circuit below.



$$\begin{bmatrix} G_1 + C_3 s & -C_3 s & 0 \\ -C_3 s & C_3 s + \frac{\Gamma_{44}}{s} + G_2 & -G_2 \\ 0 & -G_2 & \frac{\Gamma_{55}}{s} + G_2 \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} = \begin{bmatrix} -J_{s1}(s) + C_3 v_3(0) \\ -\frac{j_{L4}(0)}{s} + g_m(E_1(s) - E_2(s)) - \frac{\Gamma_{45}}{s} E_3(s) - C_3 v_3(0) \\ -\frac{j_{L5}(0)}{s} - \frac{\Gamma_{45}}{s} E_2(s) \end{bmatrix}$$

$$\begin{bmatrix} G_1 + C_3 s & -C_3 s & 0 \\ -C_3 s - g_m & C_3 s + \frac{\Gamma_{44}}{s} + G_2 + g_m & -G_2 + \frac{\Gamma_{45}}{s} \\ 0 & -G_2 + \frac{\Gamma_{45}}{s} & \frac{\Gamma_{55}}{s} + G_2 \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} = \begin{bmatrix} -J_{s1}(s) + C_3 v_3(0) \\ -\frac{j_{L4}(0)}{s} - C_3 v_3(0) \\ -\frac{j_{L5}(0)}{s} \end{bmatrix}$$

Node Analysis

- From time domain:
  - To phasor domain: Make the initial conditions zero and replace D with $j\omega$.
  - To Laplace domain: Specify the inductive initial conditions in Laplace domain, add the capacitive initial conditions, and replace D with s.
- From phsor domain:
  - To time domain: Add the initial conditions and replace $j\omega$ with D.
  - To Laplace domain: Add the initial conditions and replace $j\omega$ with s.
- From Laplace domain:
  - To phasor domain: Make the initial conditions zero and replace s with $j\omega$.
  - To time domain: Specify the inductive initial conditions in time domain, make the capacitive initial conditions zero, and replace s with D.

Mesh Analysis

Systematic Approach

- Method:

$$
\begin{aligned}
&\boldsymbol{V}(s) = \boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)\\
&\boldsymbol{M}\boldsymbol{V}(s) = 0 \Rightarrow \boldsymbol{M}[\boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)] = 0\\
&\boldsymbol{J}(s) = \boldsymbol{M}^T\boldsymbol{I}(s) \Rightarrow \boldsymbol{M}[\boldsymbol{Z}_b(s)\boldsymbol{M}^T\boldsymbol{I}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)] = 0\\
&[\boldsymbol{M}\boldsymbol{Z}_b(s)\boldsymbol{M}^T]\boldsymbol{I}(s) = \boldsymbol{M}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{M}\boldsymbol{V}_s(s)\\
&\boldsymbol{Z}_m(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s), \quad \boldsymbol{Z}_m(s) = \boldsymbol{M}\boldsymbol{Z}_b(s)\boldsymbol{M}^T, \boldsymbol{E}_s(s) = \boldsymbol{M}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{M}\boldsymbol{V}_s(s)
\end{aligned}
$$

- Steps:

$$
\begin{aligned}
&\boldsymbol{I}(s) = \boldsymbol{Z}_m^{-1}(s)\boldsymbol{E}_s(s)\\
&\boldsymbol{J}(s) = \boldsymbol{M}^T\boldsymbol{I}(s)\\
&\boldsymbol{V}(s) = \boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)
\end{aligned}
$$

- Conditions:
  - $\det[\boldsymbol{Z}_m(s)] \neq 0$
  - No alone current sources as a branch.
  - Planar network topology

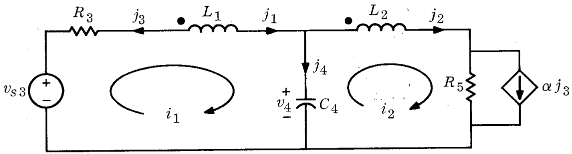
Shortcut Approach

- LTI network without coupling, dependent sources, alone current source, and initial conditions
  - Mesh impedance matrix $\boldsymbol{Z}_m(s) = \boldsymbol{M}\boldsymbol{Z}_b(s)\boldsymbol{M}^T$
    - Diagonal elements: Sum of the impedances in mesh i, i.e., $[\boldsymbol{Z}_m(s)]_{ii} = \sum_{k=1}^{b}(m_{ik})^2 Z_k(s)$
    - Off-diagonal elements: Negative sum of the common impedances in meshes i and j, i.e., $[\boldsymbol{Z}_m(s)]_{ij} = \sum_{k=1}^{b} m_{ik} Z_k(s) m_{jk}$
  - Voltage source vector: $\boldsymbol{E}_s(s) = \boldsymbol{M}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{M}\boldsymbol{V}_s(s)$
    - Vector elements: Sum of the voltages aligned to mesh i, i.e., $[\boldsymbol{E_s}(s)]_i = \sum_{k=1}^{b} m_{ik}[Z_k(s)J_{s_k}(s) - V_{s_k}(s)]$
- Coupled elements: Add the corresponding off-diagonal elements to $\boldsymbol{Z}_m(s)$
- Dependent sources: Consider them as independent sources
- Alone current source: Use supermesh or source transformation
- Initial conditions: Model them using series independent sources

Mesh Analysis

Example (Shortcut mesh analysis for an LTI circuit)

Shortcut mesh analysis can be used for the LTI circuit below.

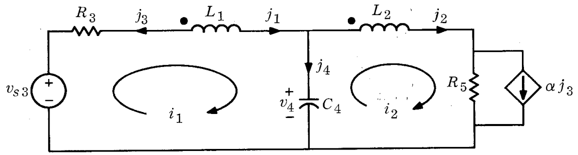


$$\begin{bmatrix} R_3 + L_1 s + \frac{1}{C_4 s} & -\frac{1}{C_4 s} \\ -\frac{1}{C_4 s} & L_2 s + R_5 + \frac{1}{C_4 s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_{s3}(s) - MsI_2(s) + L_1 i_{L1}(0) + Mi_{L2}(0) - \frac{v_{C4}(0)}{s} \\ -MsI_1(s) + L_2 i_{L2}(0) + Mi_{L1}(0) + \frac{v_{C4}(0)}{s} + \alpha R_5(-I_1(s)) \end{bmatrix}$$

$$\begin{bmatrix} R_3 + L_1 s + \frac{1}{C_4 s} & -\frac{1}{C_4 s} + Ms \\ -\frac{1}{C_4 s} + \alpha R_5 + Ms & L_2 s + R_5 + \frac{1}{C_4 s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_{s3}(s) + L_1 i_{L1}(0) + Mi_{L2}(0) - \frac{v_{C4}(0)}{s} \\ L_2 i_{L2}(0) + Mi_{L1}(0) + \frac{v_{C4}(0)}{s} \end{bmatrix}$$

Example (Shortcut mesh analysis for an LTI circuit)

Shortcut mesh analysis can be used for the LTI circuit below.



$$\begin{bmatrix} R_3 + L_1 j\omega + \frac{1}{C_4 j\omega} & -\frac{1}{C_4 j\omega} \\ -\frac{1}{C_4 j\omega} & L_2 j\omega + R_5 + \frac{1}{C_4 j\omega} \end{bmatrix} \begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} V_{s3}(j\omega) - MsI_2(j\omega) \\ -MsI_1(j\omega) + \alpha R_5(-I_1(j\omega)) \end{bmatrix}$$

$$\begin{bmatrix} R_3 + L_1 j\omega + \frac{1}{C_4 j\omega} & -\frac{1}{C_4 j\omega} + Mj\omega \\ -\frac{1}{C_4 j\omega} + \alpha R_5 + Mj\omega & L_2 j\omega + R_5 + \frac{1}{C_4 j\omega} \end{bmatrix} \begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} V_{s3}(j\omega) \\ 0 \end{bmatrix}$$

Mesh Analysis

- From time domain:
  - To phasor domain: Make the initial conditions zero and replace D with $j\omega$.
  - To Laplace domain: Specify the capacitive initial conditions in Laplace domain, add the inductive initial conditions, and replace D with s.
- From phsor domain:
  - To time domain: Add the initial conditions and replace $j\omega$ with D.
  - To Laplace domain: Add the initial conditions and replace $j\omega$ with s.
- From Laplace domain:
  - To phasor domain: Make the initial conditions zero and replace s with $j\omega$.
  - To time domain: Specify the capacitive initial conditions in time domain, make the inductive initial conditions zero, and replace s with D.

Cut-set Analysis

Systematic Approach

- Method:

$$\boldsymbol{J}(s) = \boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)$$
$$\boldsymbol{Q}\boldsymbol{J}(s) = 0 \Rightarrow \boldsymbol{Q}[\boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)] = 0$$
$$\boldsymbol{V}(s) = \boldsymbol{Q}^T\boldsymbol{E}(s) \Rightarrow \boldsymbol{Q}[\boldsymbol{Y}_b(s)\boldsymbol{Q}^T\boldsymbol{E}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)] = 0$$
$$[\boldsymbol{Q}\boldsymbol{Y}_b(s)\boldsymbol{Q}^T]\boldsymbol{E}(s) = \boldsymbol{Q}\boldsymbol{Y}_b(s)\boldsymbol{V}_s(s) - \boldsymbol{Q}\boldsymbol{J}_s(s)$$
$$\boldsymbol{Y}_q(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s), \quad \boldsymbol{Y}_q(s) = \boldsymbol{Q}\boldsymbol{Y}_b(s)\boldsymbol{Q}^T, \boldsymbol{I}_s(s) = \boldsymbol{Q}\boldsymbol{Y}_b(s)\boldsymbol{V}_s(s) - \boldsymbol{Q}\boldsymbol{J}_s(s)$$

- Steps:

$$\boldsymbol{E}(s) = \boldsymbol{Y}_q^{-1}(s)\boldsymbol{I}_s(s)$$
$$\boldsymbol{V}(s) = \boldsymbol{Q}^T\boldsymbol{E}(s)$$
$$\boldsymbol{J}(s) = \boldsymbol{Y}_b(s)\boldsymbol{V}(s) + \boldsymbol{J}_s(s) - \boldsymbol{Y}_b(s)\boldsymbol{V}_s(s)$$

- Conditions:
  - $\det[\boldsymbol{Y}_q(s)] \neq 0$
  - No alone voltage sources between two cut-sets.

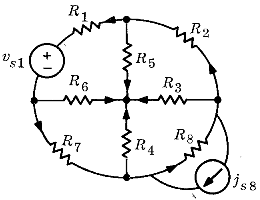
Shortcut Approach

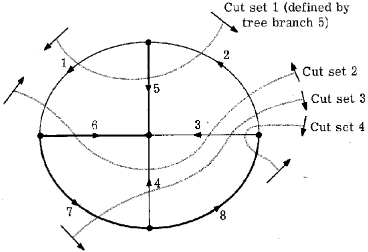
- LTI network without coupling, dependent sources, alone voltage source, and initial conditions
  - Cut-set admittance matrix $\boldsymbol{Y}_q(s) = \boldsymbol{Q}\boldsymbol{Y}_b(s)\boldsymbol{Q}^T$
    - Diagonal elements: Sum of the addimtances connected to the cut-set i
    - Off-diagonal elements: Positive sum of the addimtances connected to aligned cut-sets i and j, or negative sum of the addimtances connected to misaligned cut-sets i and j.
  - Current source vector: $\boldsymbol{I}_s(s) = \boldsymbol{Q}\boldsymbol{Y}_b(s)\boldsymbol{V}_s(s) - \boldsymbol{Q}\boldsymbol{J}_s(s)$
    - Vector elements: Sum of the currents flowing to cut-set i, positive sign for the current sources misaligned with the cut-set, and negative sign for the current sources aligned with the cut-set
- Coupled elements: Add the corresponding off-diagonal elements to $\boldsymbol{Y}_q(s)$
- Dependent sources: Consider them as independent sources
- Alone voltage source: Use source transformation
- Initial conditions: Model them using parallel independent sources

Cut-set Analysis

Example (Shortcut cut-set analysis for a resistive circuit)

Shortcut cut-set analysis can be used for the resistive circuit below.





$$
\begin{bmatrix}
G_1+G_2+G_5 & -G_1-G_2 & G_2 & G_2 \\
-G_1-G_2 & G_1+G_2+G_3+G_4+G_6 & -G_2-G_3-G_4 & -G_2-G_3 \\
G_2 & -G_2-G_3-G_4 & G_2+G_3+G_4+G_7 & G_2+G_3 \\
G_2 & -G_2-G_3 & G_2+G_3 & G_2+G_3+G_8
\end{bmatrix}
\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}
=
\begin{bmatrix} G_1 v_{s1} \\ -G_1 v_{s1} \\ 0 \\ j_{s8} \end{bmatrix}
$$

Loop Analysis

Systematic Approach

- Method:

$$\boldsymbol{V}(s) = \boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)$$
$$\boldsymbol{B}\boldsymbol{V}(s) = 0 \Rightarrow \boldsymbol{B}[\boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)] = 0$$
$$\boldsymbol{J}(s) = \boldsymbol{B}^T\boldsymbol{I}(s) \Rightarrow \boldsymbol{B}[\boldsymbol{Z}_b(s)\boldsymbol{B}^T\boldsymbol{I}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)] = 0$$
$$[\boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{B}^T]\boldsymbol{I}(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{B}\boldsymbol{V}_s(s)$$
$$\boldsymbol{Z}_l(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s), \quad \boldsymbol{Z}_l(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{B}^T, \boldsymbol{E}_s(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{B}\boldsymbol{V}_s(s)$$

- Steps:

$$\boldsymbol{I}(s) = \boldsymbol{Z}_l^{-1}(s)\boldsymbol{E}_s(s)$$
$$\boldsymbol{J}(s) = \boldsymbol{B}^T\boldsymbol{I}(s)$$
$$\boldsymbol{V}(s) = \boldsymbol{Z}_b(s)\boldsymbol{J}(s) + \boldsymbol{V}_s(s) - \boldsymbol{Z}_b(s)\boldsymbol{J}_s(s)$$

- Conditions:
  - $\det[\boldsymbol{Z}_l^{-1}(s)] \neq 0$
  - No alone current sources as a branch.
  - Planar network topology

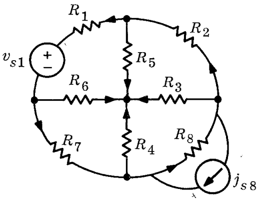
Shortcut Approach

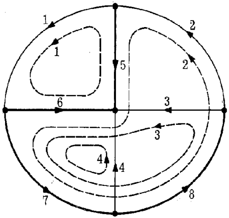
- LTI network without coupling, dependent sources, alone current source, and initial conditions
  - Loop impedance matrix $\boldsymbol{Z}_l(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{B}^T$
    - Diagonal elements: Sum of the impedances in the loop i
    - Off-diagonal elements: Positive sum of the common impedances in the aligned loops i and j, or negative sum of the common impedances in the misaligned loops i and j.
  - Voltage source vector: $\boldsymbol{E}_s(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{B}\boldsymbol{V}_s(s)$
    - Vector elements: Sum of the voltage sources in the loop i, positive sign for the voltage sources whose generated currents are aligned with the loop, and negative sign for the voltage sources whose generated currents are misaligned with the loop
- Coupled elements: Add the corresponding off-diagonal elements to $\boldsymbol{Z}_l(s)$
- Dependent sources: Consider them as independent sources
- Alone current source: Use source transformation
- Initial conditions: Model them using series independent sources

Loop Analysis

Example (Systematic loop analysis for a resistive circuit)

Systematic loop analysis can be used for the resistive circuit below.





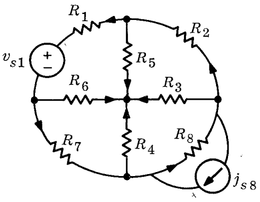
Links 1, 2, 3, 4

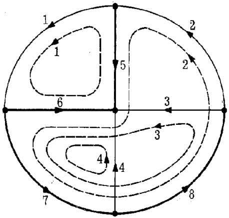
Tree branches 5, 6, 7, 8

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Example (Systematic loop analysis for a resistive circuit (cont.))

Systematic loop analysis can be used for the resistive circuit below.





Links 1, 2, 3, 4

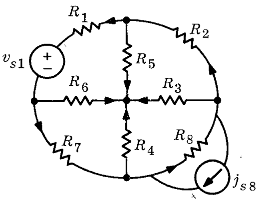
Tree branches 5, 6, 7, 8

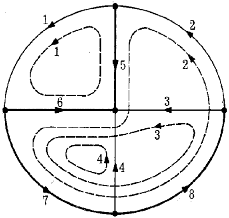
$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_8 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} + \begin{bmatrix} v_{s1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \boldsymbol{Z}_b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -j_{s8} \end{bmatrix}$$

$$\boldsymbol{Z}_l(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{B}^T, \quad \boldsymbol{E}_s(s) = \boldsymbol{B}\boldsymbol{Z}_b(s)\boldsymbol{J}_s(s) - \boldsymbol{B}\boldsymbol{V}_s(s), \quad \boldsymbol{Z}_l(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s)$$

Example (Shortcut loop analysis for a resistive circuit)

Shortcut loop analysis can be used for the resistive circuit below.





Links 1, 2, 3, 4

Tree branches 5, 6, 7, 8

$$\begin{bmatrix} R_1+R_5+R_6 & -R_5-R_6 & -R_6 & -R_6 \\ -R_5-R_6 & R_2+R_5+R_6+R_7+R_8 & R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7+R_8 & R_3+R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7 & R_6+R_7 & R_4+R_6+R_7 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} -v_{s1} \\ -R_8 j_{s8} \\ -R_8 j_{s8} \\ 0 \end{bmatrix}$$

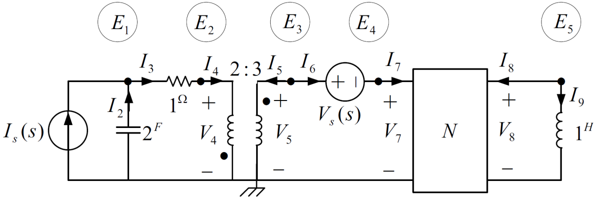
Modified Node Analysis

Modified Node Analysis

- Problematic branches
  - Independent voltage sources
  - Dependent voltage sources
  - Ideal transformer
  - Unit coupling factor
  - Operational amplifiers
  - Nonlinear elements whose current is not an explicit function of the voltage
  - Inductor current if the differential equations are intended
- Modified node analysis
  - Consider the current of the problematic branches unknown
  - Add an extra equation for each problematic branch
  - Solve the augmented system of equations
- Features
  - A system of differential equations is obtained.
  - Only initial conditions of the capacitors and inductors are required.
  - The equations can be simply specified in other domains.
  - More unknowns and equations should be handled.

Example (Modified node analysis for a sample circuit)

Modified node analysis can be used for the circuit below.



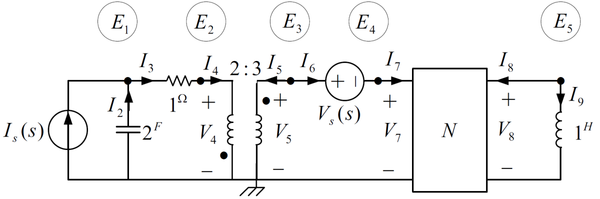
$$V_7(s) = (s+1)I_7(s)$$

$$V_8(s) = \frac{-1}{s}I_7(s) + \frac{1}{s}I_8(s)$$

$$\begin{cases} 2sE_1(s) + \frac{E_1(s)-E_2(s)}{1} = I_s(s) + 2v_2(0) \\ \frac{E_2(s)-E_1(s)}{1} + I_4(s) = 0 \\ I_5(s) + I_6(s) = 0 \\ -I_6(s) + I_7(s) = 0 \\ I_8(s) + I_9(s) = sE_5(s) + I_7(s) + I_9(s) = 0 \end{cases}, \quad \begin{cases} \frac{E_2(s)}{E_3(s)} = -\frac{2}{3} \Rightarrow 3E_2(s) + 2E_3(s) = 0 \\ \frac{I_4(s)}{I_5(s)} = \frac{3}{2} \Rightarrow 2I_4(s) + 3I_5(s) = 0 \\ E_3(s) - E_4(s) = V_s(s) \\ E_4(s) = (s+1)I_7(s) \Rightarrow E_4(s) - (s+1)I_7(s) = 0 \\ E_5(s) = sI_9(s) - i_9(0) \Rightarrow E_5(s) - sI_9(s) = -i_9(0) \end{cases}$$

Example (Modified node analysis for a sample circuit (cont.))

Modified node analysis can be used for the circuit below.



$$V_7(s) = (s+1)I_7(s)$$

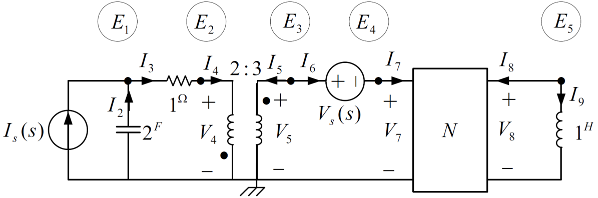
$$V_8(s) = \frac{-1}{s}I_7(s) + \frac{1}{s}I_8(s)$$

$$\begin{bmatrix} 2s+1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & s+1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -s \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \\ E_4(s) \\ E_5(s) \\ I_4(s) \\ I_5(s) \\ I_6(s) \\ I_7(s) \\ I_9(s) \end{bmatrix} = \begin{bmatrix} I_s(s) + 2v_2(0) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_s(s) \\ 0 \\ -i_9(0) \end{bmatrix}$$

Modified Node Analysis

Example (Modified node analysis for a sample circuit (cont.))

Modified node analysis can be used for the circuit below.



$$V_7(s) = (s+1)I_7(s)$$

$$V_8(s) = \frac{-1}{s} I_7(s) + \frac{1}{s} I_8(s)$$

$$\begin{bmatrix} 2D+1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & D & 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & D+1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -D \end{bmatrix} \begin{bmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \\ e_4(s) \\ e_5(s) \\ i_4(s) \\ i_5(s) \\ i_6(s) \\ i_7(s) \\ i_9(s) \end{bmatrix} = \begin{bmatrix} i_s(t)) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_s(t) \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{P}(D) \begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{i}(t) \end{bmatrix} = \boldsymbol{u}(t)$$

Solution Uniqueness

Solution Uniqueness

- Branch admittance & branch impedance matrices:
  - RLC circuit: Diagonal
  - RLCM circuit: Symmetric
  - RLCM+dependent source circuit: Asymmetric, in general
- Node/cut-set admittance & mesh/loop impedance Matrices
  - RLC circuit: Symmetric
  - RLCM circuit: Symmetric
  - RLCM+dependent source circuit: Asymmetric, in general

Solution Uniqueness

- Node analysis:
  $\boldsymbol{Y}_n(D)\boldsymbol{e}(t) = \boldsymbol{i}_s(t) \Rightarrow \boldsymbol{Y}_n(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s) \Rightarrow \Delta_n(s) = \det[\boldsymbol{Y}_n(s)]$
- Mesh analysis:
  $\boldsymbol{Z}_m(D)\boldsymbol{i}(t) = \boldsymbol{e}_s(t) \Rightarrow \boldsymbol{Z}_m(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s) \Rightarrow \Delta_m(s) = \det[\boldsymbol{Z}_m(s)]$
- Cut-set analysis:
  $\boldsymbol{Y}_q(D)\boldsymbol{e}(t) = \boldsymbol{i}_s(t) \Rightarrow \boldsymbol{Y}_q(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s) \Rightarrow \Delta_q(s) = \det[\boldsymbol{Y}_q(s)]$
- Loop analysis:
  $\boldsymbol{Z}_l(D)\boldsymbol{i}(t) = \boldsymbol{e}_s(t) \Rightarrow \boldsymbol{Z}_l(s)\boldsymbol{I}(s) = \boldsymbol{E}_s(s) \Rightarrow \Delta_l(s) = \det[\boldsymbol{Z}_l(s)]$
- Modified analysis:
  $\boldsymbol{P}(D)\begin{bmatrix}\boldsymbol{e}(t)\\ \boldsymbol{i}(t)\end{bmatrix} = \boldsymbol{u}(t) \Rightarrow \boldsymbol{P}(s)\begin{bmatrix}\boldsymbol{E}(s)\\ \boldsymbol{I}(s)\end{bmatrix} = \boldsymbol{U}(s) \Rightarrow \Delta_p(s) = \det[\boldsymbol{P}(s)]$
- State analysis: $D\boldsymbol{X}(t) = \boldsymbol{AX}(t) + \boldsymbol{BW}(t) \Rightarrow (s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{BX}(s) + \boldsymbol{X}_0 \Rightarrow \Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}]$

Solution Uniqueness

Theorem (Natural Frequencies of a Network)

The nonzero natural frequencies of any linear time-invariant network are identical to the nonzero roots of the equation $\Delta(s) = det[\boldsymbol{P}(s)] = 0$*, where* $\boldsymbol{P}(s)$ *is the matrix of any system of differential equations which describe the network.*

- Node admittance matrix: $\Delta_n(s) = \det[\boldsymbol{Y}_n(s)] = 0$
- Mesh impedance matrix: $\Delta_m(s) = \det[\boldsymbol{Z}_m(s)] = 0$
- Cut-set admittance matrix: $\Delta_q(s) = \det[\boldsymbol{Y}_q(s)] = 0$
- Loop impedance matrix: $\Delta_l(s) = \det[\boldsymbol{Z}_l(s)] = 0$
- Modified node matrix: $\Delta_p(s) = \det[\boldsymbol{P}(s)] = 0$
- State matrix: $\Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}] = 0$

Solution Uniqueness

Theorem (Sufficient Condition for Unique Solution)

Suppose that N is a strictly passive LTI RLCMT network, such that all its resistors have positive resistances, all its capacitors have positive capacitances, all its inductors have positive inductances. Suppose further that every set of coupled inductors has a positive definite inductance matrix. Under these conditions, given any initial state and any set of inputs, the network N has a unique solution.

- Proof for strictly passive RLC networks: Contradiction

$$\det[\boldsymbol{Y}_n(s)] = 0 \Rightarrow \boldsymbol{Y}_n(s)\boldsymbol{x} = 0, \boldsymbol{x} \neq 0$$

$$\Rightarrow 0 = \boldsymbol{x}^T\boldsymbol{Y}_n(s)\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{Y}_b(s)\boldsymbol{A}^T\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{x})^T\boldsymbol{Y}_b(s)(\boldsymbol{A}^T\boldsymbol{x}) = \boldsymbol{z}^T\boldsymbol{Y}_b(s)\boldsymbol{z}$$

$$\boldsymbol{x} \neq 0 \;\&\; \boldsymbol{A}^T \text{ full rank} \Rightarrow \boldsymbol{z} \neq 0$$

$$\Rightarrow \sum_{i=1}^{b} y_{ii}(s)z_i^2 = 0 \Rightarrow \sum_{i=1}^{b} y_{ii}(s_1)z_i^2 = 0, s_1 \in [0,\infty), y_{ii}(s_1) > 0$$

- No unique solution: No solution, several solutions, infinite number of solutions

The End