Basic Circuit Elements

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Overview

1. Signals
2. Resistor
3. Capacitor
4. Inductor
5. Memristor
6. Power and Energy
7. Elements Interconnections

Signals

Elementary Signals

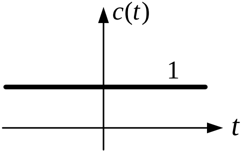


Figure: Constant signal
$c(t) = 1, \forall t$.

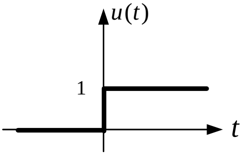


Figure: Step signal
$u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$.

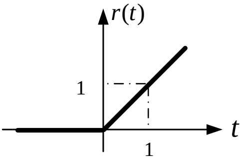


Figure: Step signal
$r(t) = \begin{cases} t, t \geq 0 \\ 0, t < 0 \end{cases} = tu(t) = \int_{-\infty}^{t} u(\lambda) d\lambda$.

Elementary Signals

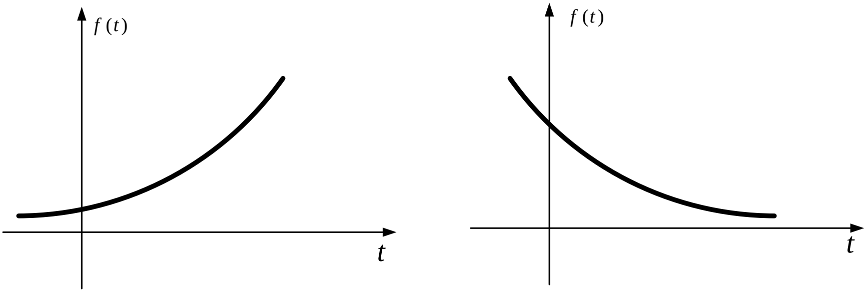
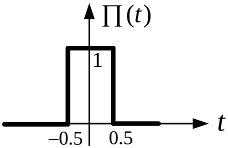


Figure: Exponential signal $f(t) = Ae^{at}$.

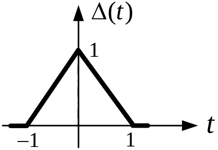
Example (Rectangular signal)

$\sqcap(t) = u(t + 0.5) - u(t - 0.5).$



Example (Triangle signal)

$\Lambda(t) = r(t+1) - 2r(t) + r(t-1).$



$$\Lambda(t) = (t+1)[u(t+1) - u(t)] + (1-t)[u(t) - u(t-1)] = r(t+1) - 2r(t) + r(t-1)$$

Singular Signals

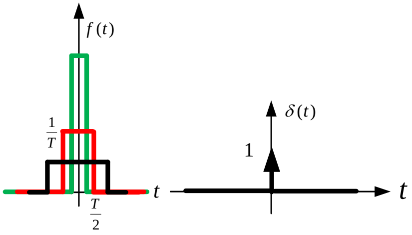


Figure: Impulse signal $\delta(t)$.

1. Definition: $\delta(t) = \lim_{T\to 0} \frac{1}{T} \sqcap\left(\frac{t}{T}\right) = \lim_{T\to 0} \frac{u(t+0.5T)-u(t-0.5T)}{T} = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0 \end{cases}$
2. Surface: $\int_{-\infty}^{+\infty} \delta(t)dt = \int_{0-}^{0^+} \delta(t)dt = 1$
3. Sampling: $\int_{-\infty}^{+\infty} f(t)\delta(t)dt = \int_{t_1}^{t_2} f(t)\delta(t)dt = f(0), 0 \in (t_1, t_2), \quad f(t)\delta(t) = f(0)\delta(t)$
4. Scaling: $\delta(at) = \frac{1}{|a|}\delta(t)$
5. Integral: $u(t) = \int_{-\infty}^{t} \delta(\lambda)d\lambda$
6. Derivative: $\delta'(t) = \frac{d\delta(t)}{dt}$

Singular Signals

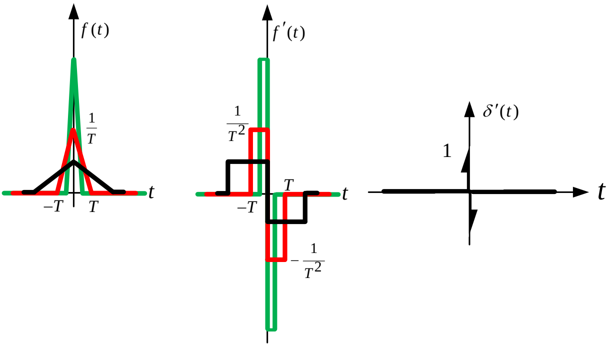


Figure: Doublet signal $\delta'(t)$.

1. Definition: $\delta(t) = \lim_{T\to 0} \frac{1}{T}\Lambda(\frac{t}{T}) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$, $\delta'(t) = \frac{d\delta(t)}{dt}$
2. Surface: $\int_{-\infty}^{+\infty} \delta'(t)dt = \int_{0^-}^{0^+} \delta'(t)dt = 0$
3. Sampling: $\int_{t_1}^{t_2} f(t)\delta'(t)dt = -f'(0), 0 \in (t_1, t_2)$, $f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$

Example (Sampling property of $\delta'(t)$)

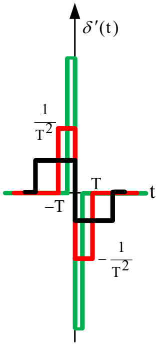
The sampling property of $\delta'(t)$ can be roughly verified as the limit of $\frac{1}{T}\Lambda(\frac{t}{T})$.

$$0 \in (t_1, t_2)$$

$$\int_{t_1}^{t_2} f(t)\delta'(t)dt = \lim_{T\to 0}[f(-0.5T)\frac{1}{T^2}T - f(0.5T)\frac{1}{T^2}T]$$

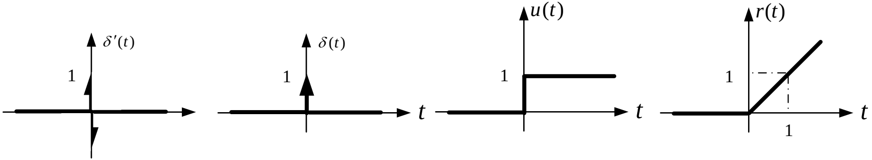
$$= \lim_{T\to 0} \frac{f(-0.5T) - f(+0.5T)}{T}$$

$$= -\lim_{T\to 0} \frac{f(0.5T) - f(-0.5T)}{0.5T - (-0.5T)} = -f'(0)$$



Example (Relations of singular functions)

Singular functions relate to each other using derivative and integral operations.



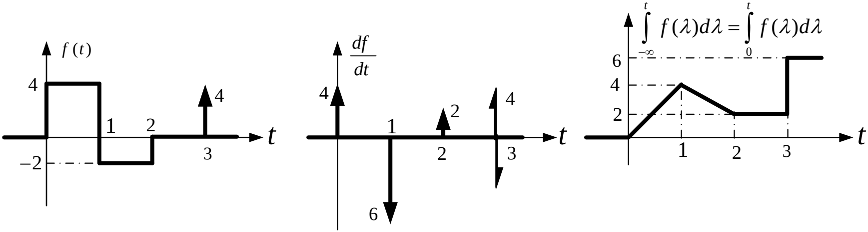
$$\cdots, \quad \delta'(t) = \frac{d\delta(t)}{dt}, \quad \delta(t) = \frac{du(t)}{dt}, \quad u(t) = \frac{dr(t)}{dt}, \quad \cdots$$

$$\cdots, \quad \delta(t) = \int_{-\infty}^{t} \delta'(\lambda) d\lambda, \quad u(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda, \quad r(t) = \int_{-\infty}^{t} u(\lambda) d\lambda, \quad \cdots$$

Singular Signals

Example (Derivative and integral of discontinuous function)

Singular functions can be used in derivative and integral calculations.



$$f(t) = 4u(t) - 6u(t-1) + 2u(t-2) + 4\delta(t-3)$$

$$\frac{df(t)}{dt} = 4\delta(t) - 6\delta(t-1) + 2\delta(t-2) + 4\delta'(t-3)$$

$$\int_{-\infty}^{t} f(\lambda)d\lambda = 4tu(t) - 6(t-1)u(t-1) + 2(t-2)u(t-2) + 4u(t-3)$$

Periodic Signals

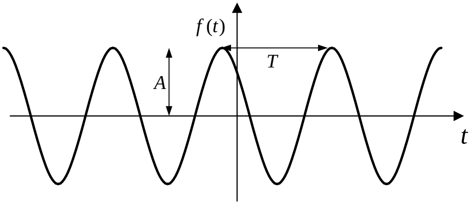


Figure: Sinusoidal periodic signals with period T.

1. Expression: $f(t) = A\cos(\omega t + \theta) \equiv A\sin(\omega t + \theta)$
2. Period: $T = \frac{2\pi}{\omega} = \frac{1}{f}$
3. Frequency: $f = \frac{\omega}{2\pi} = \frac{1}{T}$
4. Phase: θ
5. Amplitude: A
6. Peak to peak amplitude: $2A$
7. Average: $f_{av} = \frac{1}{T}\int_T f(t)dt = \frac{1}{T}\int_T A\cos(\omega t + \theta)dt = 0$
8. RMS: $f_{rms} = \sqrt{\frac{1}{T}\int_T |f(t)|^2 dt} = \sqrt{\frac{1}{T}\int_T A^2\cos^2(\omega t + \theta)dt} = \frac{A}{\sqrt{2}}$

Periodic Signals

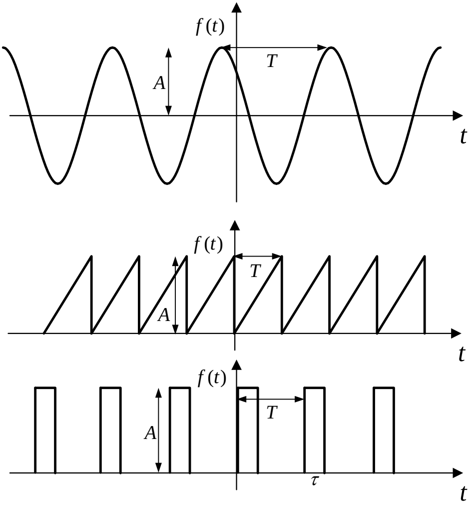
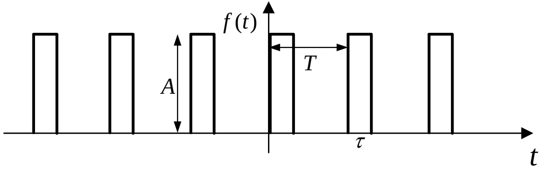


Figure: Sinusoidal, sawtooth, and pulse train periodic signals with period T.

Periodic Signals

Example (Pulse train)

A pulse train can be characterized in terms of its average, rms, and duty cycle.



$$f(t) = A \sqcap (\frac{t - 0.5\tau}{\tau}), 0 \leq t < T, f(t \pm T) = f(t)$$

$$D = \frac{\tau}{T}$$

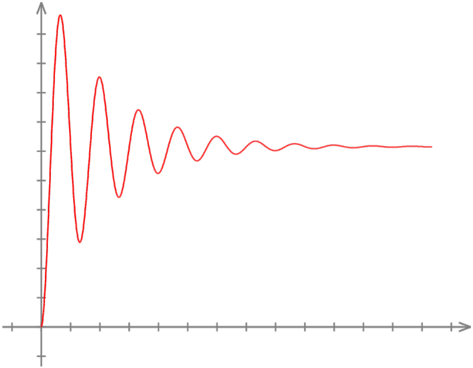
$$f_{av} = \frac{1}{T} \int_T f(t)dt = A\frac{\tau}{T} = AD$$

$$f_{rms} = \sqrt{\frac{1}{T} \int_T |f(t)|^2 dt} = A\sqrt{\frac{\tau}{T}} = A\sqrt{D}$$

Other Signals

Example (Underdampled signal)

An underdamped signal can be expressed as the multiplication of sinusoidal and exponential signals.



$$f(t) = A + Be^{-\alpha t}\cos(\omega t + \phi)$$

Other Signals

Example (Complex exponential signal)

A complex signal can be described using its polar or Cartesian presentations.

$$f(t) = Ae^{-\alpha t}e^{j(\omega t+\phi)} = \Re\{f(t)\} + j\Im\{f(t)\} = |f(t)|e^{j\angle f(t)}$$

$$\Re\{f(t)\} = Ae^{-\alpha t}\cos(\omega t + \phi)$$

$$\Im\{f(t)\} = Ae^{-\alpha t}\sin(\omega t + \phi)$$

$$|f(t)| = |A|e^{-\alpha t}$$

$$\angle f(t) = \omega t + \phi + \pi u(-A)$$

Resistor

Resistor

Statement (Linear Function)

The function $f(x)$ is (map-) linear if it is homogeneous, i.e., $f(\alpha x) = \alpha f(x)$, and additive, i.e., $f(x_1 + x_2) = f(x_1) + f(x_2)$.

Statement (Continuous Function)

The function $f(x)$ is continuous if $\lim_{x \to x_0} f(x) = f(x_0), \forall x_0$.

Statement (Bounded Function)

The function $f(x)$ is bounded if $|f(x_0)| < M, \forall x_0$.

1. $f(x) = ax$ is a linear function.
2. $f(x) = ax + b, b \neq 0$ is not a linear function.
3. $f(x(t)) = \frac{dx(t)}{dt}$ is a linear function.
4. $f(x) = u(x)$ is not continuous but is bounded.
5. $f(x) = \delta(x)$ is not continuous and is not bounded.

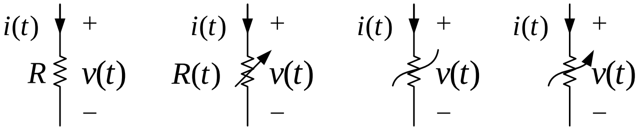
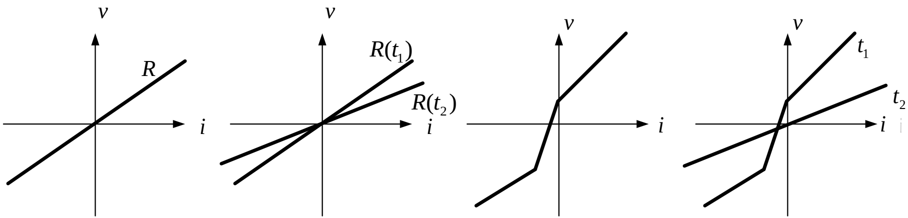


Figure: LTI, LTV, NTI, NTV resistors. The units of voltage, current, resistance, and conductance are V, A, Ω, $\mho$.

1. Linear time-invariant resistor: $v(t) = Ri(t) \equiv i(t) = Gv(t)$
2. Linear time-variant resistor: $v(t) = R(t)i(t) \equiv i(t) = G(t)v(t)$
3. Nonlinear time-invariant resistor: $f(v(t), i(t)) = 0$
4. Nonlinear time-variant resistor: $f(v(t), i(t), t) = 0$
5. Voltage-controlled resistor: $i(t) = f(v(t), t)$
6. Current-controlled resistor: $v(t) = f(i(t), t)$
7. Bilateral resistor: $f(v(t), i(t)) = f(-v(t), -i(t))$



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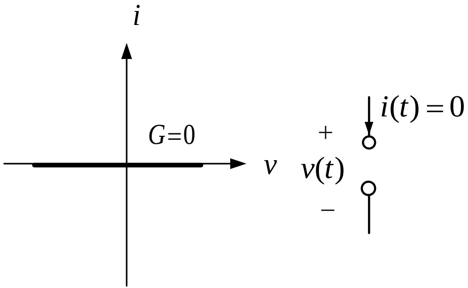
Resistor

Example (Open circuit)

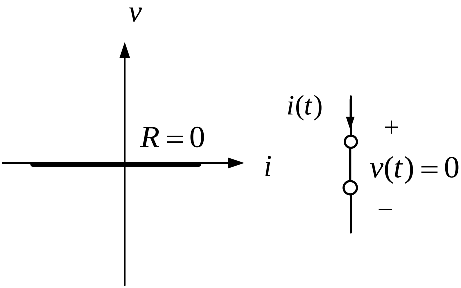
Open circuit is a voltage-controlled bilateral LTI resistor with $G = 0$.



Resistor

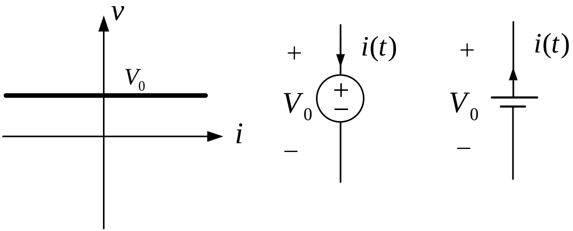
Example (Short circuit)

Short circuit is a current-controlled bilateral LTI resistor with $R = 0$.



Example (DC voltage source)

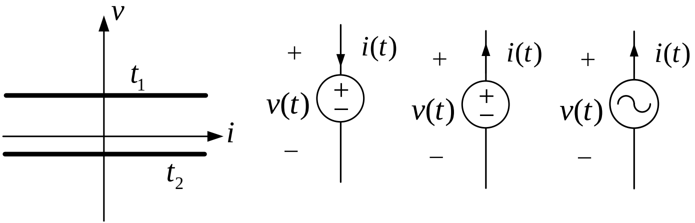
DC voltage source is a current-controlled NTI resistor.



Resistor

Example (AC voltage source)

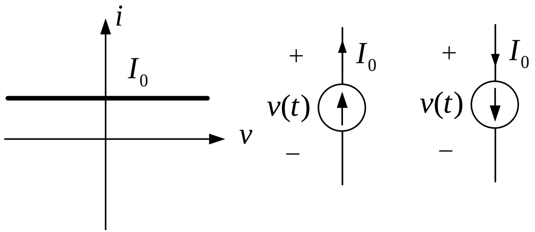
AC voltage source is a current-controlled NTV resistor.



Resistor

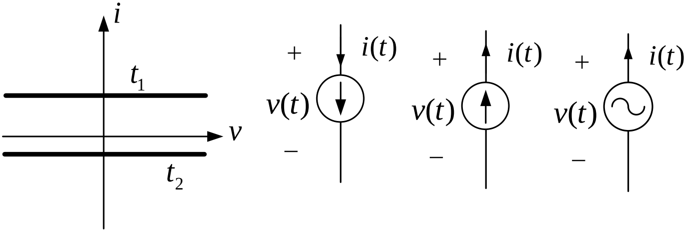
Example (DC current source)

DC current source is a voltage-controlled NTI resistor.



Example (AC current source)

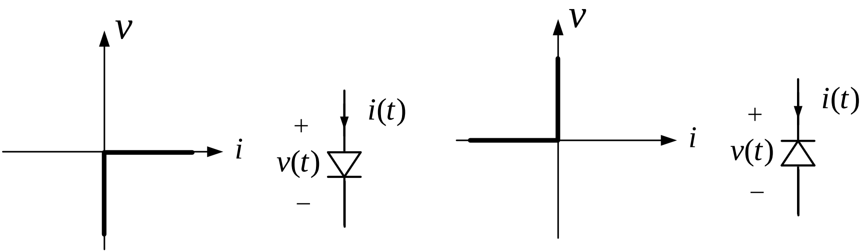
AC current source is a voltage-controlled NTV resistor.



Resistor

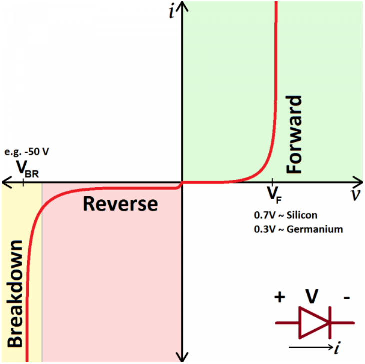
Example (Ideal diode)

An ideal diode is an NTI resistor.



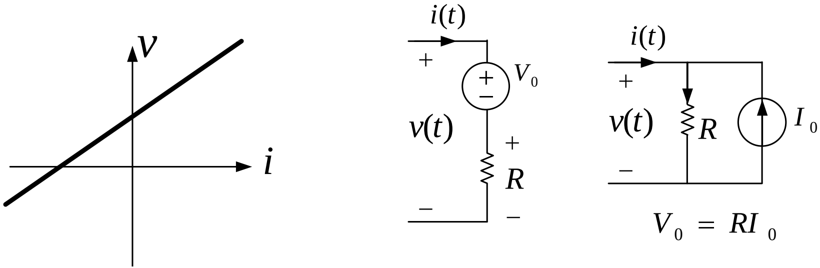
Example (Ideal diode)

A real diode with the characteristic curve $i = I_s(e^{\frac{qv}{kT}} - 1) = I_s(e^{\frac{v}{V_T}} - 1)$ is an NTI resistor, where the thermal voltage equals $V_T = kT/q \approx 26$ mV in room temperature.



Example (Battery)

A battery can be modeled as a series connection of a resistor and a voltage source.



$$v(t) = V_0 + V_R(t) = V_0 + Ri(t)$$

$$i(t) = -I_0 + i_R(t) = -\frac{V_0}{R} + \frac{v(t)}{R}$$

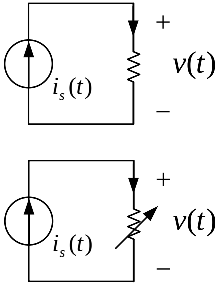
Example (Time-variant resistor)

A time-variant resistor can create new frequencies from an input single-frequency tone signal.

$i_s(t) = I\sin(2\pi f_1 t)$

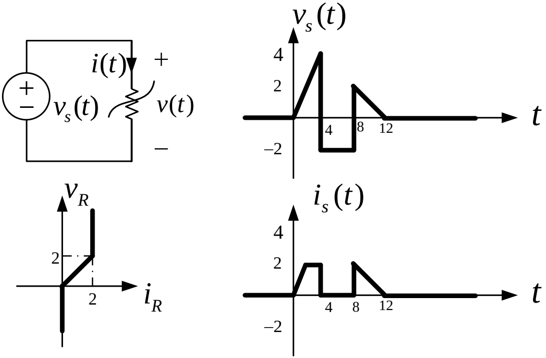
$R = 1 \Rightarrow v(t) = I\sin(2\pi f_1 t)$

$R(t) = 1 + 2\cos(2\pi f_2 t) \Rightarrow$
$v(t) = I\sin(2\pi f_1 t) + I\sin(2\pi(f_1 + f_2)t) + I\sin(2\pi(f_1 - f_2)t)$



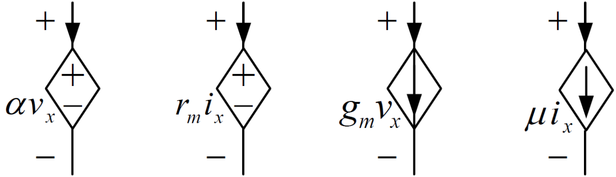
Example (Nonlinear resistor)

The characteristic curve of a nonlinear resistor can be used to draw its voltage or current.



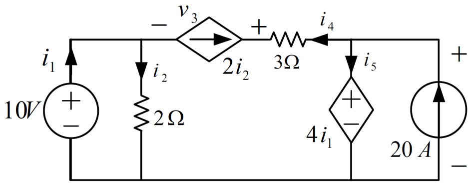
Example (Dependent sources)

Linear dependent sources can be usually considered as NTV resistors.



Example (Circuit with dependent sources)

Tellegen's theorem can be verified for the circuit below.



$$i_6 = 20,\ i_2 = \frac{10}{2} = 5,\ i_3 = 2i_2 = 10,\ i_4 = -i_3 = -10,\ i_1 = i_2 - i_4 = 15,\ i_5 = 20 - i_4 = 30$$

$$v_1 = 10,\ v_2 = 10,\ v_4 = 3i_4 = -30,\ v_5 = 4i_1 = 60,\ v_6 = v_5 = 60,\ v_3 = -v_4 + v_5 - v_2 = 80$$

$$p_1 = -10i_1 = -150,\ p_2 = v_2 i_2 = 50,\ p_3 = -v_3 i_3 = -800$$

$$p_4 = v_4 i_4 = 300,\ p_5 = v_5 i_5 = 1800,\ p_6 = -v_6 i_6 = -1200$$

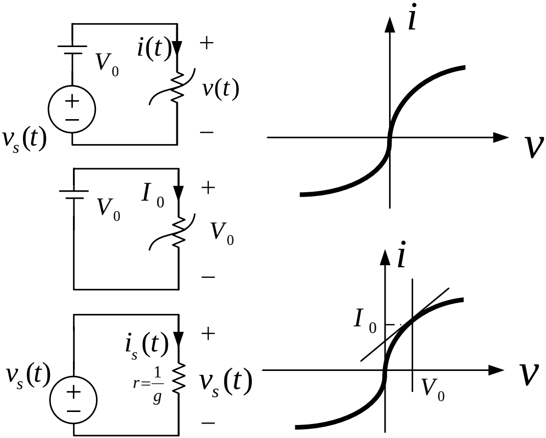
$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 0$$

Resistor

Example (Small-signal analysis)

Circuits with nonlinear resistors can be investigated using small-signal analysis.

$$i = f(v)$$
$$i(t) = f(V_0 + v_s(t)), |v_s(t)| \ll |V_0|$$
$$i(t) \approx f(V_0) + \frac{df}{dv}|_{v=V_0} v_s(t)$$
$$i(t) \approx I_0 + g v_s(t)$$



Capacitor

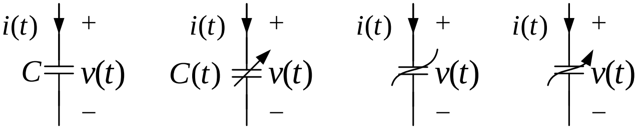
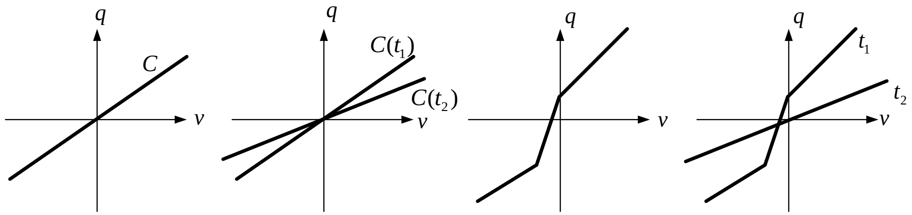


Figure: LTI, LTV, NTI, NTV capacitors. The units of charge, voltage, capacitance, and elastance are C, V, F, F^{-1}.

1. Linear time-invariant capacitor: $q(t) = Cv(t) \equiv v(t) = Sq(t)$
2. Linear time-variant capacitor: $q(t) = C(t)v(t) \equiv v(t) = S(t)q(t)$
3. Nonlinear time-invariant capacitor: $f(q(t), v(t)) = 0$
4. Nonlinear time-variant capacitor: $f(q(t), v(t), t) = 0$
5. Voltage-controlled capacitor: $q(t) = f(v(t), t)$
6. Charge-controlled capacitor: $v(t) = f(q(t), t)$
7. Bilateral capacitor: $f(q(t), v(t)) = f(-q(t), -v(t))$

Capacitor



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Capacitor

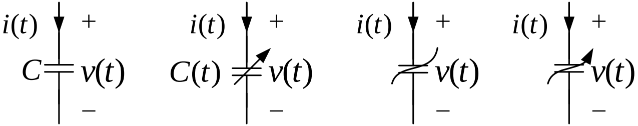
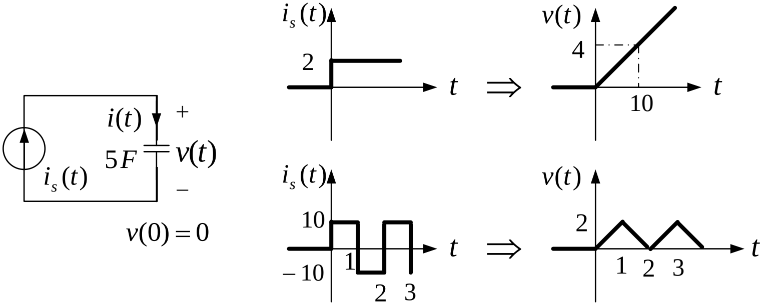


Figure: LTI, LTV, NTI, NTV capacitors. The units of charge, voltage, capacitance, and elastance are C, V, F, F^{-1}.

1. Linear time-invariant capacitor:
   - Current equation: $i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt}, \quad v(t_0)$
   - Voltage equation: $v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\lambda)d\lambda$
   - Full description by capacitance C and initial voltage $v(t_0)$
   - Memory element
   - Linearity of current in terms of voltage
   - Continuity of voltage for bounded current
2. Linear time-variant capacitor: $i(t) = C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt}, \quad v(t_0), C(t_0)$
3. Voltage-controlled capacitor: $i(t) = \frac{\partial f}{\partial v}\frac{dv(t)}{dt} + \frac{\partial f}{\partial t}$

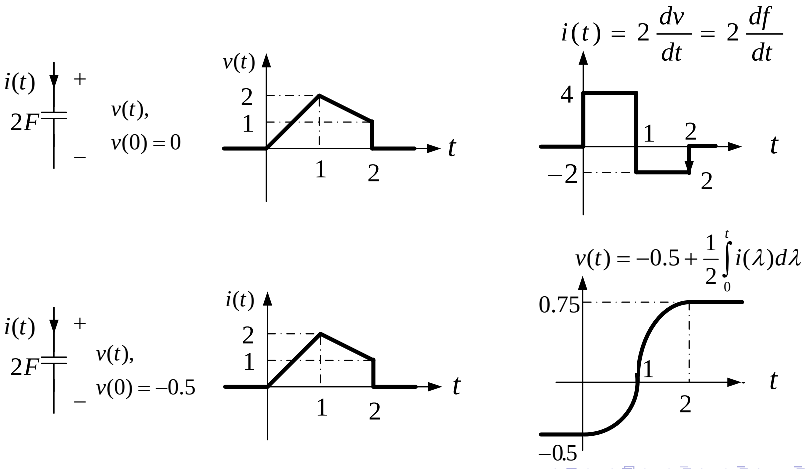
Example (LTI capacitor)

A capacitor integrates its flowing current.



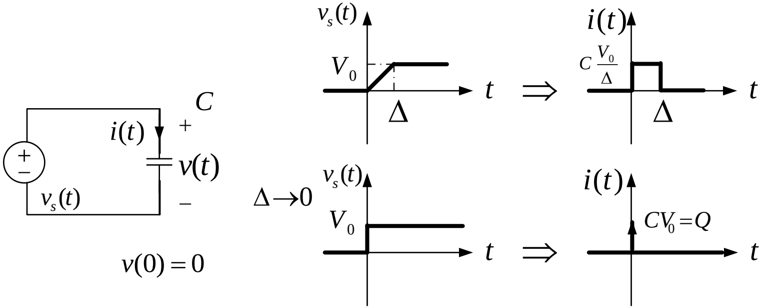
Example (LTI capacitor)

The capacitor voltage remains continuous for the bounded flowing current.



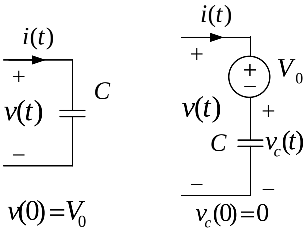
Example (LTI capacitor)

The capacitor voltage experiences discontinuity for the unbounded flowing current.



Example (Initial condition modeling)

The initial voltage can be modeled using an independent voltage source.

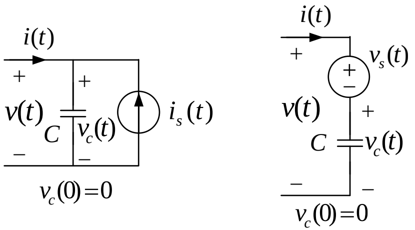


$$v(t) = v(0) + \frac{1}{C}\int_0^t i(\lambda)d\lambda = V_0 + \frac{1}{C}\int_0^t i(\lambda)d\lambda$$

Capacitor

Example (Thevenin-Norton Equivalency)

The two circuits below are equivalent if $i_s(t) = C\frac{dv_s(t)}{dt} \equiv v_s(t) = \frac{1}{C}\int_0^t i_s(\lambda)d\lambda$ and $v_C(0) = 0$



$$v(t) = v_s(t) + \frac{1}{C}\int_0^t i(\lambda)d\lambda$$

$$i(t) = -i_s(t) + C\frac{dv(t)}{dt}$$

Inductor

Inductor

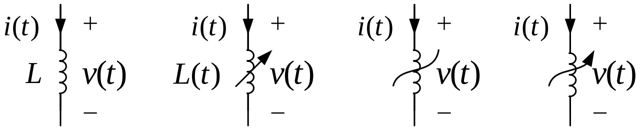


Figure: LTI, LTV, NTI, NTV inductors. The units of flux, current, inductance, and reciprocal inductance are Wb, A, H, H^{-1}.

1. Linear time-invariant inductor: $\phi(t) = Li(t) \equiv i(t) = \Gamma\phi(t)$
2. Linear time-variant inductor: $\phi(t) = L(t)i(t) \equiv i(t) = \Gamma(t)\phi(t)$
3. Nonlinear time-invariant inductor: $f(\phi(t), i(t)) = 0$
4. Nonlinear time-variant inductor: $f(\phi(t), i(t), t) = 0$
5. Current-controlled inductor: $\phi(t) = f(i(t), t)$
6. Flux-controlled inductor: $i(t) = f(\phi(t), t)$
7. Bilateral inductor: $f(\phi(t), i(t)) = f(-\phi(t), -i(t))$

Inductor

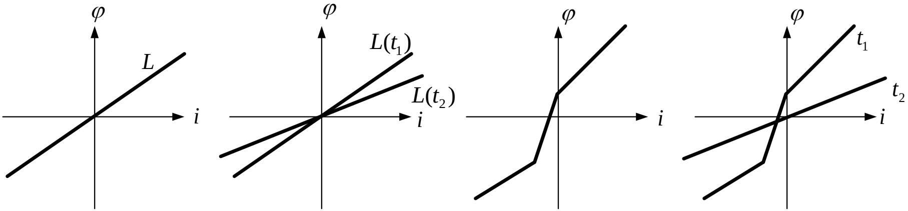


Figure: LTI, LTV, NTI, NTV inductors. The units of flux, current, inductance, and reciprocal inductance are Wb, A, H, H^{-1}.

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Inductor

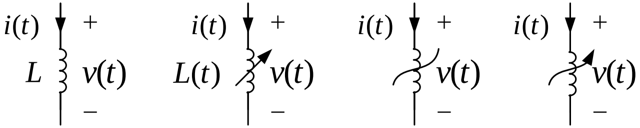
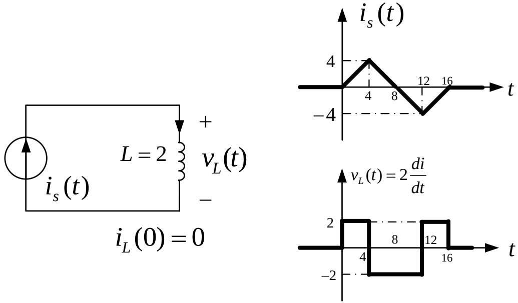


Figure: LTI, LTV, NTI, NTV inductors. The units of flux, current, inductance, and reciprocal inductance are Wb, A, H, H^{-1}.

1. Linear time-invariant inductor:
   - Voltage equation: $v(t) = \frac{d\phi(t)}{dt} = L\frac{di(t)}{dt}, \quad i(t_0)$
   - Current equation: $i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\lambda)d\lambda$
   - Full description by inductance L and initial current $i(t_0)$
   - Memory element
   - Linearity of voltage in terms of current
   - Continuity of current for bounded voltage
2. Linear time-variant inductor: $v(t) = L(t)\frac{di(t)}{dt} + i(t)\frac{dL(t)}{dt}, \quad i(t_0), L(t_0)$
3. Current-controlled inductor: $i(t) = \frac{\partial f}{\partial i}\frac{di(t)}{dt} + \frac{\partial f}{\partial t}$

Example (LTI inductor)

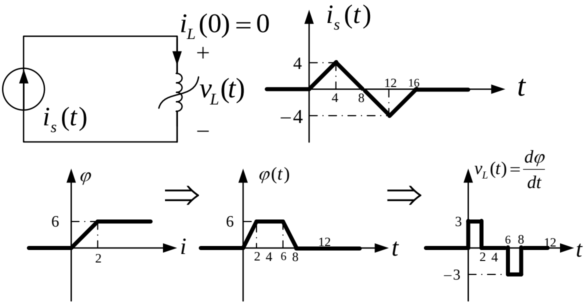
An inductor differentiates its flowing current.



Inductor

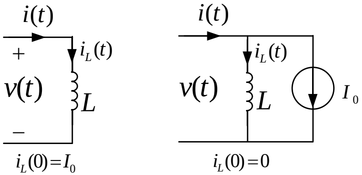
Example (NTI inductor)

An NTI inductor can be described by its characteristic curve.



Example (Initial condition modeling)

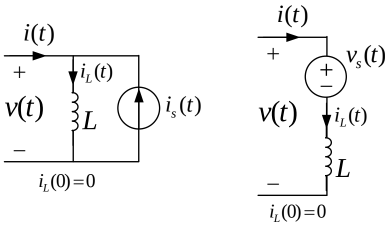
The initial current can be modeled using an independent current source.



$$i(t) = i(0) + \frac{1}{L}\int_0^t v(\lambda)d\lambda = I_0 + \frac{1}{L}\int_0^t v(\lambda)d\lambda$$

Example (Thevenin-Norton Equivalency)

The two circuits below are equivalent if $v_s(t) = L\frac{di_s(t)}{dt} \equiv i_s(t) = \frac{1}{L}\int_0^t v_s(\lambda)d\lambda$ and $i_L(0) = 0$

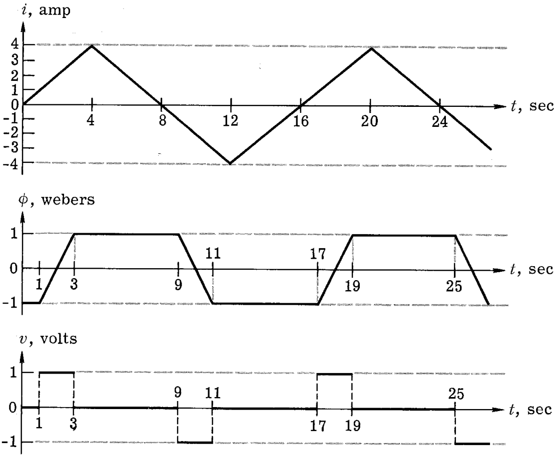


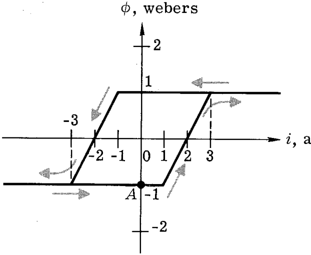
$$i(t) = -i_s(t) + \frac{1}{L}\int_0^t v(\lambda)d\lambda$$

$$v(t) = v_s(t) + L\frac{di(t)}{dt}$$

Example (Hysteresis)

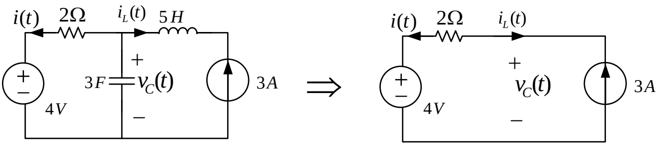
An inductor with hysteresis characteristic is an NTI inductor.





Example (DC steady state)

If a DC driven inductor (capacitor) reaches its steady state situation, it acts like a short (open) circuit.



Memristor

Memristor

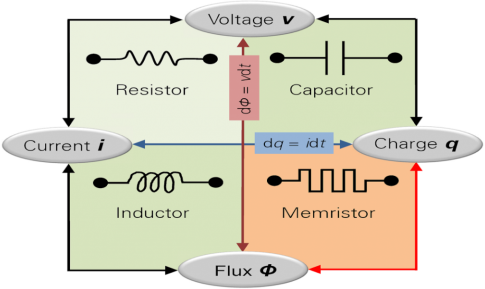


Figure: Basic one-port circuit elements.

- Nonlinear time-variant memristor: $f(q(t), \phi(t), t) = 0$

Power and Energy

Power and Energy

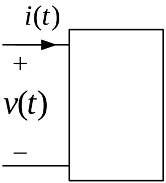


Figure: A general one-port element with passive sign convention.

1. Absorbed power: $p(t) = v(t)i(t) = \frac{d\epsilon(t)}{dt}$
2. Absorbed energy: $w(t_0, t) = \epsilon(t) - \epsilon(t_0) = \int_{t_0}^{t} p(\lambda)d\lambda$
3. Absolute energy: $\epsilon(t) = \epsilon(t_0) + w(t_0, t)$

Resistor

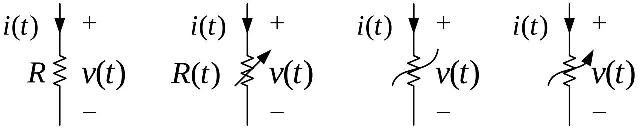


Figure: LTI, LTV, NTI, NTV resistors. Resistors dissipate power.

1. LTI resistor absorbed energy: $w(t_0,t)=\int_{t_0}^{t}v(\lambda)i(\lambda)d\lambda=R\int_{t_0}^{t}i^2(\lambda)d\lambda$
2. LTI resistor passivity condition: $w(t_0,t)=R\int_{t_0}^{t}i^2(\lambda)d\lambda\geq 0\Rightarrow R\geq 0$
3. LTV resistor passivity condition: $R(t)\geq 0,\forall t$
4. NTV resistor passivity condition:
$w(t_0,t)=\int_{t_0}^{t}p(\lambda)d\lambda\geq 0\Rightarrow p(t)=v(t)i(t)\geq 0,\forall t$

Capacitor

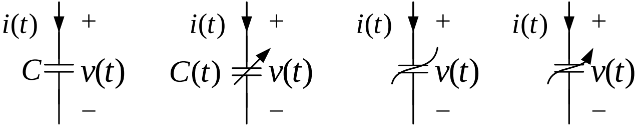


Figure: LTI, LTV, NTI, NTV capacitors. Capacitors store electrical energy.

1. LTI capacitor absorbed energy: $w(t_0, t) = \int_{t_0}^{t} v(\lambda)i(\lambda)d\lambda = \int_{t_0}^{t} v(\lambda)C\frac{dv(\lambda)}{d\lambda}d\lambda = C\int_{v(t_0)}^{v(t)} udu = \frac{C}{2}(v^2(t) - v^2(t_0))$
2. LTI capacitor absolute energy: $\epsilon_E(t) = \frac{C}{2}v^2(t) = \frac{1}{2C}q^2(t)$
3. LTI capacitor passivity condition: $\epsilon_E(t) = \frac{C}{2}v^2(t) \geq 0 \Rightarrow C \geq 0$
4. LTV capacitor passivity condition: $C(t), C'(t) \geq 0, \forall t$
5. NTI capacitor passivity condition: $q(t)v(t) \geq 0, \forall t$

Inductor

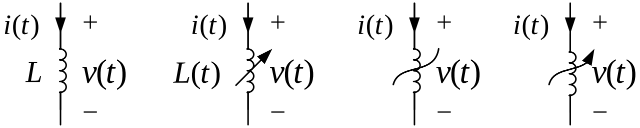


Figure: LTI, LTV, NTI, NTV inductors. Inductors store magnetic energy.

1. LTI inductor absorbed energy:
   $w(t_0, t) = \int_{t_0}^{t} v(\lambda)i(\lambda)d\lambda = \int_{t_0}^{t} L\frac{di(\lambda)}{d\lambda}i(\lambda)d\lambda = L\int_{i(t_0)}^{i(t)} u du = \frac{L}{2}(i^2(t) - i^2(t_0))$
2. LTI inductor absolute energy: $\epsilon_M(t) = \frac{L}{2}i^2(t) = \frac{1}{2L}\phi^2(t)$
3. LTI inductor passivity condition: $\epsilon_M(t) = \frac{L}{2}i^2(t) \geq 0 \Rightarrow L \geq 0$
4. LTV inductor passivity condition: $L(t), L'(t) \geq 0, \forall t$
5. NTI inductor passivity condition: $\phi(t)i(t) \geq 0, \forall t$

Example (Activity)

A DC voltage source with the voltage V_0 is active since $p(t) = v(t)i(t) = V_0(-V_0) = -V_0^2 < 0$.

Example (Passivity)

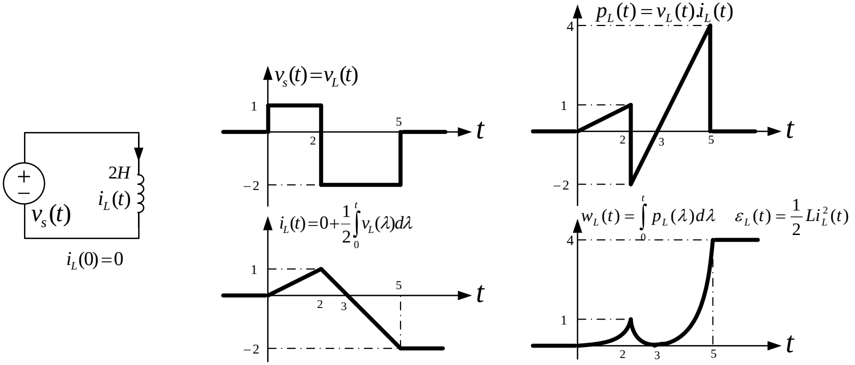
The NTI resistor with the characteristic curve $i(t) = 2(v(t))^3$ is passive since $p(t) = v(t)i(t) = 2(v(t))^4 \geq 0$.

Example (Activity)

The LTV resistor with the resistance $R(t) = -(2t+1)$ is active since $R(0) = -1 < 0$.

Example (Power and Energy)

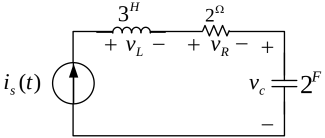
The energy and power curves for the shown inductor are plotted as below.



Power and Energy

Example (Power and Energy)

For the circuit below, $i(t) = 3t, t > 0, v_C(0) = 3$, and $i_L(0) = 0$. $w_R(0,1) = 6$, $p_L(2) = 54$, and $\epsilon_C(4) = 225$.



$$w_R(0,1) = 2\int_0^1 (3\lambda)^2 d\lambda = 6$$

$$i_L(2) = 6, v_L(2) = 3i_L'(2) = 9 \Rightarrow p_L(2) = v_L(2)i_L(2) = 54$$

$$v_c(4) = 3 + \frac{1}{2}\int_0^4 3\lambda d\lambda = 15 \Rightarrow \epsilon_C(4) = \frac{1}{2}(2)v_C^2(4) = 225$$

Elements Interconnections

Equivalent One-ports

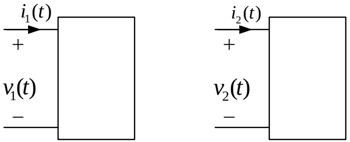


Figure: A same equation governs ports of two equivalent one-ports.

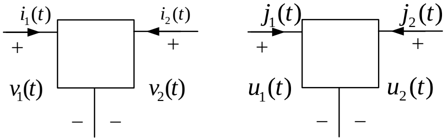


Figure: Same equations govern ports of two equivalent two-ports.

Resistors

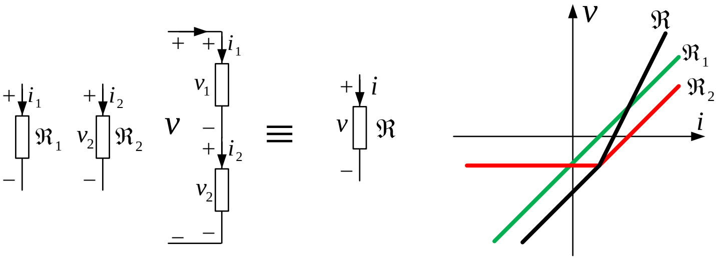


Figure: Two series resistors with $i = i_1 = i_2$ and $v = v_1 + v_2$. Series connection of two current-controlled resistors has the characteristic curve $v = v_1 + v_2 = f_1(i_1) + f_2(i_2) = f(i)$.

Resistors

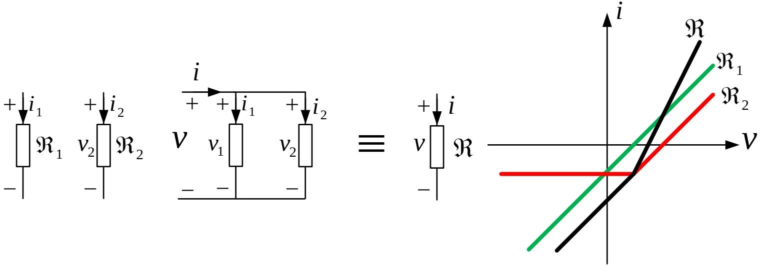
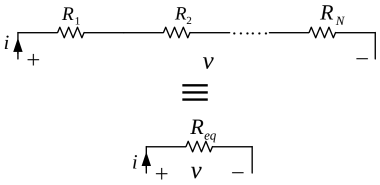


Figure: Two parallel resistors with $v = v_1 = v_2$ and $i = i_1 + i_2$. Parallel connection of two voltage-controlled resistors has the characteristic curve $i = i_1 + i_2 = f_1(v_1) + f_2(v_2) = f(v)$.

Example (Series connection of LTI resistors)

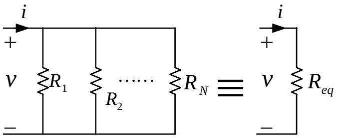
If the LTI resistors R_1, R_2, ..., R_N are connected in series, they can be replaced with the equivalent LTI resistor $R_{eq} = \sum_{k=1}^{N} R_k$.



$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} R_k i_k = i \sum_{k=1}^{N} R_k = R_{eq} i$$

Example (Parallel connection of LTI resistors)

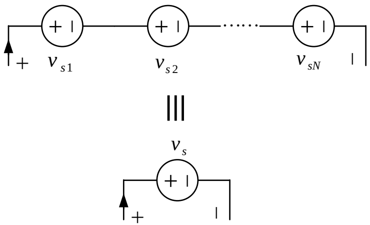
If the LTI resistors G_1, G_2, ..., G_N are connected in parallel, they can be replaced with the equivalent LTI resistor $G_{eq} = \sum_{k=1}^{N} G_k$.



$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} G_k v_k = v \sum_{k=1}^{N} G_k = G_{eq} i$$

Example (Series connection of voltage sources)

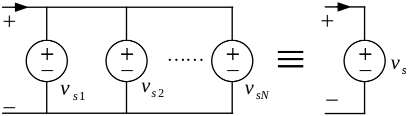
If the voltage sources v_{s1}, v_{s2}, ..., v_{sN} are connected in series, they can be replaced with the equivalent voltage source $v_s = \sum_{k=1}^{N} v_{sk}$.



$$v_s = \sum_{k=1}^{N} v_{sk}$$

Example (Parallel connection of voltage sources)

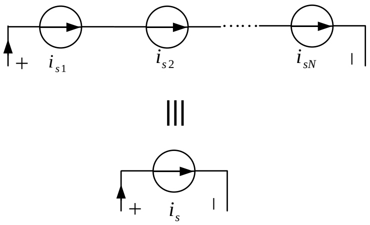
The parallel connection of the voltage sources v_{s1}, v_{s2}, ..., v_{sN} is possible if $v_{s1} = v_{s2} = \cdots = v_{sN}$.



$$v_s = v_{s1} = v_{s2} = \cdots = v_{sN}$$

Example (Series connection of current sources)

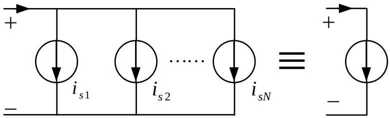
The series connection of the current sources i_{s1}, i_{s2}, ..., i_{sN} is possible if $i_{s1} = i_{s2} = \cdots = i_{sN}$.



$$i_s = i_{s1} = i_{s2} = \cdots = i_{sN}$$

Example (Parallel connection of current sources)

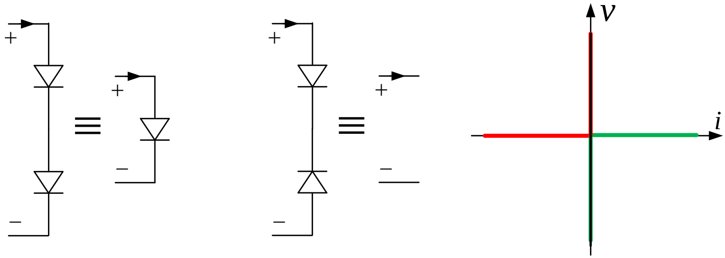
If the current sources i_{s1}, i_{s2}, ..., i_{sN} are connected in parallel, they can be replaced with the equivalent current source $i_s = \sum_{k=1}^{N} i_{sk}$.



$$i_s = \sum_{k=1}^{N} i_{sk}$$

Example (Series connection of diodes)

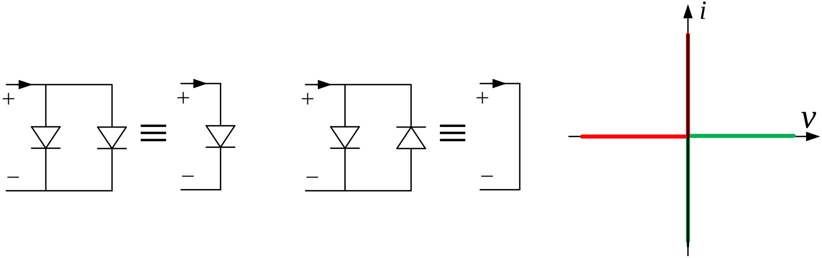
Series connection of two ideal diodes results in an equivalent ideal diode or open circuit.



Resistors

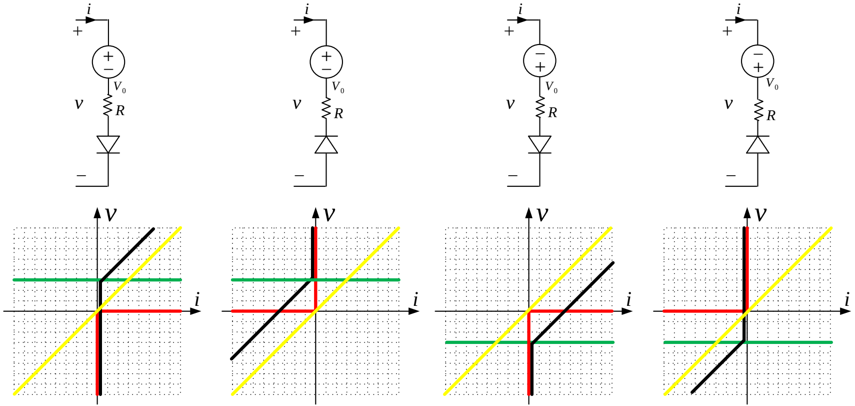
Example (Parallel connection of diodes)

Parallel connection of two ideal diodes results in an equivalent ideal diode or short circuit.



Example (Series connection of several elements)

The direction of elements is important in elements interconnection.



Example (Parallel connection of several elements)

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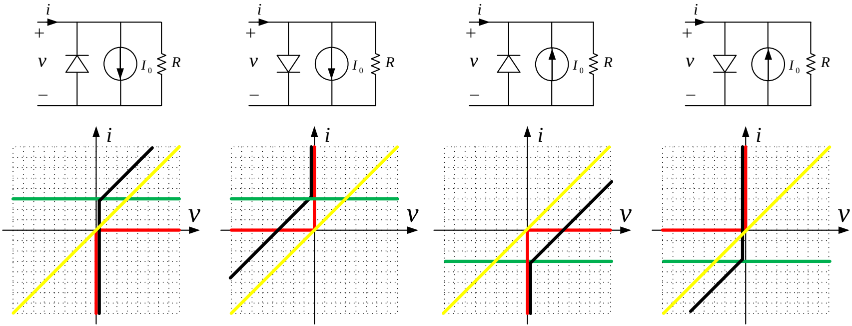
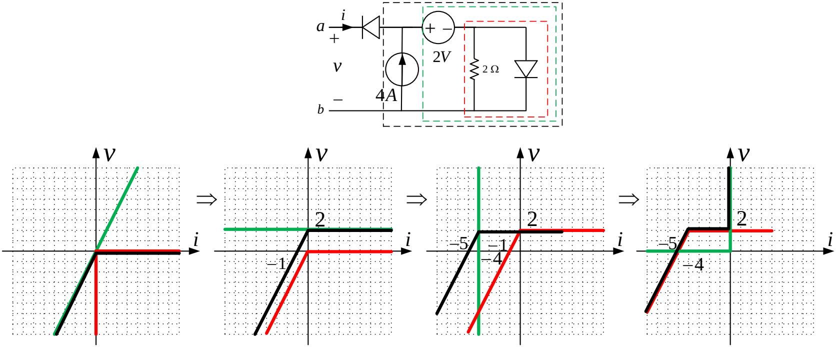


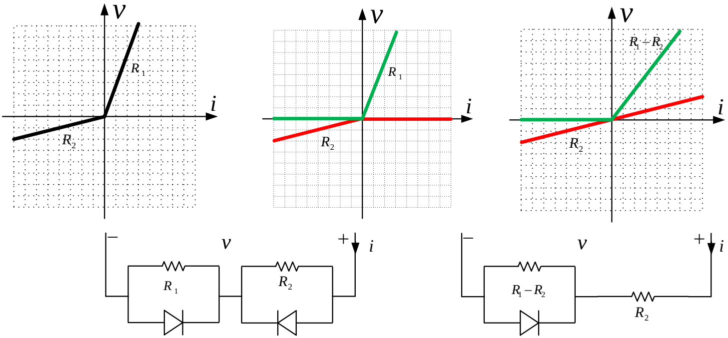
Example (Interconnection of several elements)

Interconnection of various elements leads to interesting characteristic curves.



Example (Circuit Synthesis)

A desired circuit can be synthesized in different ways.



Resistors

Example (Circuit Synthesis)

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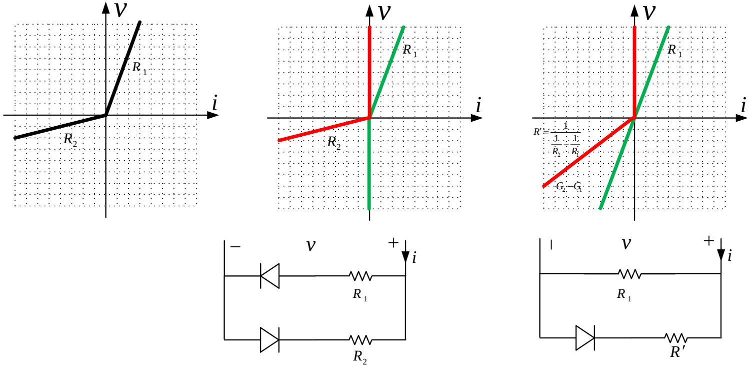
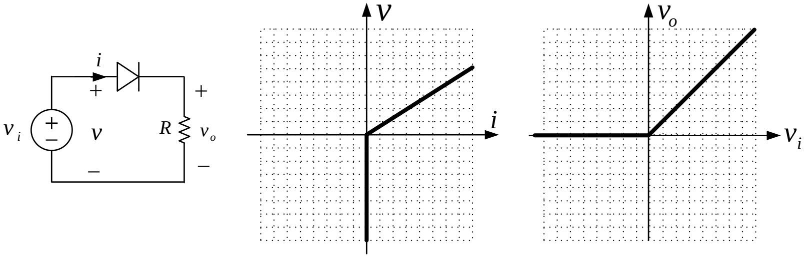


Resistors

Example (Rectifier)

Diodes can be used for rectification.



$$v_o = v_i u(v_i)$$

Resistors

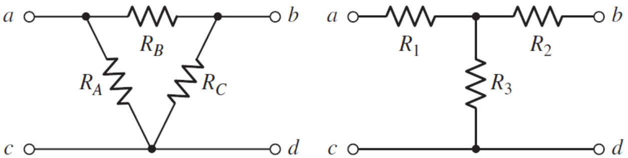


Figure: Resistive Δ (triangle, $\prod$) and Y (star, T) networks. If the two networks are equivalent, then the port voltages and currents must be equal.

$$R_A = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2} \qquad R_1 = \frac{R_AR_B}{R_A + R_B + R_C}$$

$$R_B = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_3} \qquad R_2 = \frac{R_BR_C}{R_A + R_B + R_C}$$

$$R_C = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1} \qquad R_3 = \frac{R_CR_A}{R_A + R_B + R_C}$$

Capacitors

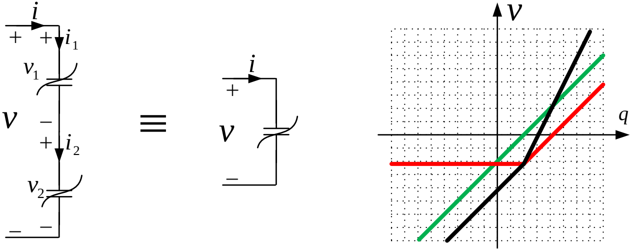


Figure: Two series NTI capacitors with $i = i_1 = i_2$ and $v = v_1 + v_2$. Series connection of two charge-controlled capacitors has the characteristic curve $v = v_1 + v_2 = f_1(q_1) + f_2(q_2) = f(q)$ provided that $q_1(0) = q_2(0)$.

$$i = i_1 = i_2 \Rightarrow \frac{dq}{dt} = \frac{dq_1}{dt} = \frac{dq_2}{dt} \Rightarrow q(t) - q(0) = q_1(t) - q_1(0) = q_2(t) - q_2(0)$$

$$q(0) = q_1(0) = q_2(0) \Rightarrow q(t) = q_1(t) = q_2(t)$$

Capacitors

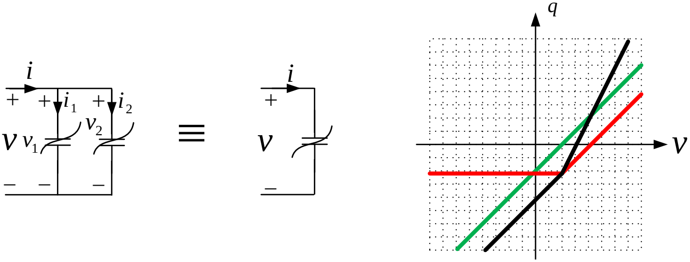


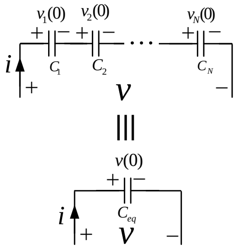
Figure: Two parallel NTI capacitors with $v = v_1 = v_2$ and $i = i_1 + i_2$. Parallel connection of two voltage-controlled capacitors has the characteristic curve $q = q_1 + q_2 = f_1(v_1) + f_2(v_2) = f(v)$.

$$i = i_1 + i_2 \Rightarrow \frac{dq}{dt} = \frac{dq_1}{dt} + \frac{dq_2}{dt} \Rightarrow q(t) - q(0) = q_1(t) - q_1(0) + q_2(t) - q_2(0)$$

$$q(0) = q_1(0) + q_2(0) \Rightarrow q(t) = q_1(t) + q_2(t)$$

Example (Series connection of LTI capacitors)

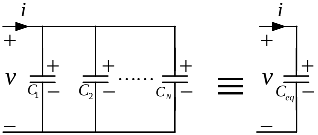
If the LTI capacitors S_1, S_2, ..., S_N with the initial voltages $v_1(0)$, $v_2(0)$, ..., $v_N(0)$ are connected in series, they can be replaced with the equivalent LTI capacitor $S_{eq} = \sum_{k=1}^{N} S_k$ with the initial voltage $v(0) = \sum_{k=1}^{N} v_k(0)$.



$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} \left[v_k(0) + S_k \int_0^t i_k(\lambda) d\lambda \right] = \sum_{k=1}^{N} v_k(0) + \left(\sum_{k=1}^{N} S_k \right) \int_0^t i(\lambda) d\lambda$$

Example (Parallel connection of LTI capacitors)

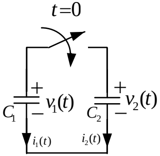
If the LTI capacitors C_1, C_2, ..., C_N with the initial voltages $v_1(0^-)$, $v_2(0^-)$, ..., $v_N(0^-)$ are connected in parallel, they can be replaced with the equivalent LTI capacitor $C_{eq} = \sum_{k=1}^{N} C_k$ with a suitable initial voltage $v(0^+) = v_1(0^+) = \cdots = v_N(0^+)$.



$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} C_k \frac{dv_k}{dt} = \left(\sum_{k=1}^{N} C_k\right) \frac{dv}{dt}$$

Example (Initial voltage of two parallel LTI capacitors)

If the LTI capacitors C_1 and C_2 with the initial voltages $v_1(0^-)$ and $v_2(0^-)$ are connected in parallel, they can be replaced with the equivalent LTI capacitor $C_{eq} = C_1 + C_2$ with having the initial voltage $v(0^+) = \frac{C_1 v_1(0^-) + C_2 v_2(0^-)}{C_1 + C_2}$.

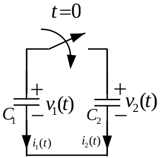


$$q(0^-) = q(0^+) \Rightarrow C_1 v_1(0^-) + C_2 v_2(0^-) = C_1 v_1(0^+) + C_2 v_2(0^+) = (C_1 + C_2) v(0^+)$$

Capacitors

Example (Initial voltage of two parallel LTI capacitors)

If the LTI capacitors C_1 and C_2 with the initial voltages $v_1(0^-)$ and $v_2(0^-)$ are connected in parallel, they can be replaced with the equivalent LTI capacitor $C_{eq} = C_1 + C_2$ with having the initial voltage $v(0^+) = \frac{C_1 v_1(0^-)+C_2 v_2(0^-)}{C_1+C_2}$.



$$i_1(t) + i_2(t) = C_1\frac{dv_1}{dt} + C_2\frac{dv_2}{dt} = 0 \Rightarrow \int_{0^-}^{0^+} [C_1\frac{dv_1}{dt} + C_2\frac{dv_2}{dt}]dt = 0 \Rightarrow C_1 \int_{v_1(0^-)}^{v_1(0^+)} dv_1 + C_2 \int_{v_2(0^-)}^{v_2(0^+)} dv_2 = 0$$

$$C_1[v_1(0^+) - v_1(0^-)] + C_2[v_2(0^+) - v_2(0^-)] = 0 \Rightarrow C_1 v_1(0^-) + C_2 v_2(0^-) = (C_1 + C_2)v(0^+)$$

Inductors

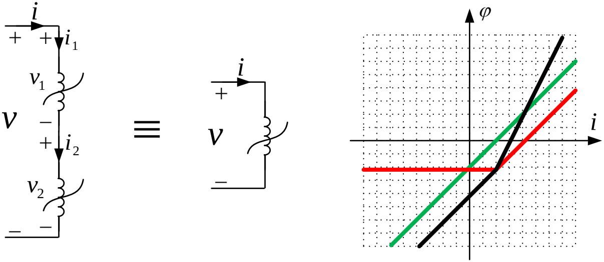
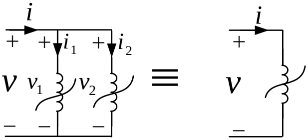


Figure: Two series NTI inductors with $i = i_1 = i_2$ and $v = v_1 + v_2$. Series connection of two current-controlled inductors has the characteristic curve $\phi = \phi_1 + \phi_2 = f_1(i_1) + f_2(i_2) = f(i)$.

$$v = v_1 + v_2 \Rightarrow \frac{d\phi}{dt} = \frac{d\phi_1}{dt} + \frac{d\phi_2}{dt} \Rightarrow \phi(t) - \phi(0) = \phi_1(t) - \phi_1(0) + \phi_2(t) - \phi_2(0)$$

$$\phi(0) = \phi_1(0) + \phi_2(0) \Rightarrow \phi(t) = \phi_1(t) + \phi_2(t)$$

Inductors



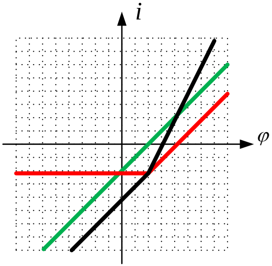


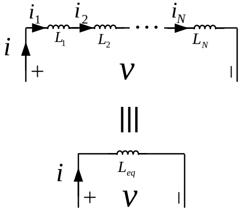
Figure: Two parallel NTI inductors with $v = v_1 = v_2$ and $i = i_1 + i_2$. Parallel connection of two flux-controlled inductors has the characteristic curve $i = i_1 + i_2 = f_1(\phi_1) + f_2(\phi_2) = f(\phi)$ provided that $\phi_1(0) = \phi_2(0)$.

$$v = v_1 = v_2 \Rightarrow \frac{d\phi}{dt} = \frac{d\phi_1}{dt} = \frac{d\phi_2}{dt} \Rightarrow \phi(t) - \phi(0) = \phi_1(t) - \phi_1(0) = \phi_2(t) - \phi_2(0)$$

$$\phi(0) = \phi_1(0) = \phi_2(0) \Rightarrow \phi(t) = \phi_1(t) = \phi_2(t)$$

Example (Series connection of LTI inductors)

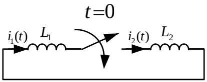
If the LTI inductors L_1, L_2, ..., L_N with the initial currents $i_1(0^-)$, $i_2(0^-)$, ..., $i_N(0^-)$ are connected in series, they can be replaced with the equivalent LTI inductor $L_{eq} = \sum_{k=1}^{N} L_k$ with a suitable initial current $i(0^+) = i_1(0^+) = \cdots = i_N(0^+)$.



$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} L_k \frac{di_k}{dt} = \Big(\sum_{k=1}^{N} L_k\Big) \frac{di}{dt}$$

Example (Initial current of two series LTI inductors)

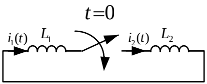
If the LTI inductors L_1 and L_2 with the initial currents $i_1(0^-)$ and $i_2(0^-)$ are connected in series, they can be replaced with the equivalent LTI inductor $L_{eq} = L_1 + L_2$ with having the initial current $i(0^+) = \frac{L_1 i_1(0^-) + L_2 i_2(0^-)}{L_1 + L_2}$.



$$\phi(0^-) = \phi(0^+) \Rightarrow L_1 i_1(0^-) + L_2 i_2(0^-) = L_1 i_1(0^+) + L_2 i_2(0^+) = (L_1 + L_2) i(0^+)$$

Example (Initial current of two series LTI inductors)

If the LTI inductors L_1 and L_2 with the initial currents $i_1(0^-)$ and $i_2(0^-)$ are connected in series, they can be replaced with the equivalent LTI inductor $L_{eq} = L_1 + L_2$ with having the initial current $i(0^+) = \frac{L_1 i_1(0^-) + L_2 i_2(0^-)}{L_1 + L_2}$.

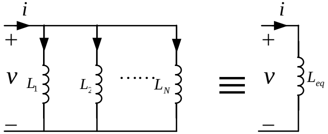


$$v_1(t) + v_2(t) = L_1 \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = 0 \Rightarrow \int_{0^-}^{0^+} [L_1 \frac{di_1}{dt} + L_2 \frac{di_2}{dt}] dt = 0 \Rightarrow L_1 \int_{i_1(0^-)}^{i_1(0^+)} di_1 + L_2 \int_{i_2(0^-)}^{i_2(0^+)} di_2 = 0$$

$$L_1[i_1(0^+) - i_1(0^-)] + L_2[i_2(0^+) - i_2(0^-)] = 0 \Rightarrow L_1 i_1(0^-) + L_2 i_2(0^-) = (L_1 + L_2) i(0^+)$$

Example (Parallel connection of LTI inductors)

If the LTI inductors Γ_1, Γ_2, ..., Γ_N with the initial currents $i_1(0)$, $i_2(0)$, ..., $i_N(0)$ are connected in parallel, they can be replaced with the equivalent LTI inductor $\Gamma_{eq} = \sum_{k=1}^{N} \Gamma_k$ with the initial current $i(0) = \sum_{k=1}^{N} i_k(0)$.



$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} \left[i_k(0) + \Gamma_k \int_0^t v_k(\lambda) d\lambda \right] = \sum_{k=1}^{N} i_k(0) + \left(\sum_{k=1}^{N} \Gamma_k \right) \int_0^t v(\lambda) d\lambda$$

The End