# Circuit Laws

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# Overview

# Lumped Circuits

# Lumped Condition

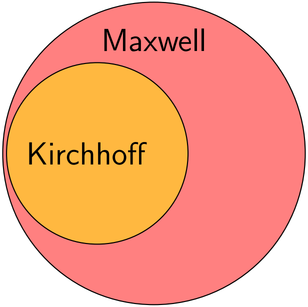



Figure: Maxwell equations describe how electric and magnetic fields are generated by charges, currents, and changes of the fields. Maxwell equations demonstrate that fluctuating electric and magnetic fields propagate at the speed $v = f\lambda$ no greater than the light speed $c$. Maxwell equations are simplified to Kirchhoff equations for lumped circuits.

- Maxwell's equation: Sophisticated vector quantities $\vec{E}, \vec{H}, \vec{D}, \vec{B}, \vec{J}, \rho$
- Distributed circuits: Space- and time-dependent description
- Kirchhoff's equations: Simplified scalar quantities $v, i, q, \phi$
- Lumped circuits: Time-dependent description
- Lumped condition: max{circuit dimension}≪min{circuit wavelength}

# Lumped Condition

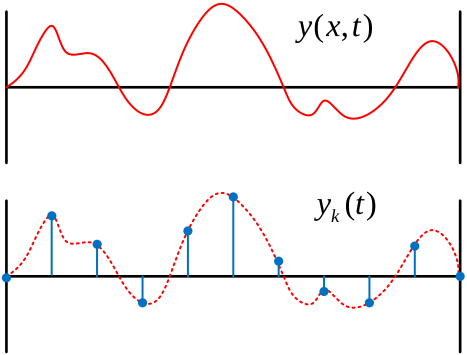



Figure: An illustration for the lumped approximation. A partial differential equation may govern $y(x, t)$ while an ordinary differential equation may characterize $y_k(t)$.

# Lumped Condition

## Example (Lump condition)

The 100-MHz dipole antenna below can be considered lumped if it is terminated at the point $B = 1.27$ cm while the lumped condition is violated at the point $C = 1.5$ m.

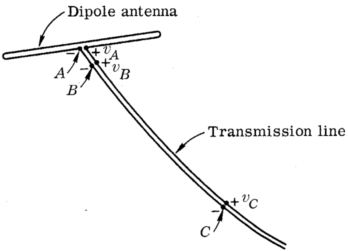
$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^8} = 3 \text{ m}$$

$$v(x,t) = V_0 \sin[2\pi f(t - \frac{x}{c})]$$

$$v(A,t) = V_0 \sin[2\pi \times 10^8 t]$$

$$v(B,t) = V_0 \sin[2\pi \times 10^8 (t - \frac{0.0127}{3 \times 10^8})] \approx v(A,t)$$

$$v(C,t) = V_0 \sin[2\pi \times 10^8 (t - \frac{1.5}{3 \times 10^8}] = -v(A,t)$$



# Lumped Condition

## Example (Lump condition)

Intel Core i7-4702HQ processor with the package size 37.5mm $\times$ 32mm $\times$ 1.6mm and the max turo frequency 3.2 GHz is not a lumped circuit since its maximum dimension $d \approx \sqrt{37.5^2 + 32^2 + 1.6^2} = 49.32$ mm is in the order of minimum operating wavelength $\lambda \approx 3 \times 10^{11}/(3.2 \times 10^9) = 93.72$ mm.

## Example (Lump condition)

The power transmission system is a lumped circuit over Tehran city since the maximum transmission distance $d \approx 50$ km is much less than the operating wavelength $\lambda \approx 3 \times 10^5/50 = 6000$ km.

# Circuit Elements

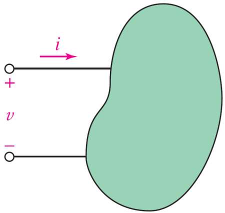



Figure: Passive sign convention for one-port circuit elements.

- One-port element: an electrical entity with one port (two terminals)
- Passive sign convention: the current flows to the plus terminal.
- Absorbed power: $p(t) = v(t)i(t)$.

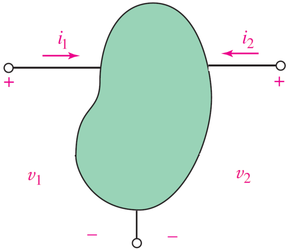



Figure: Passive sign convention for three-terminal circuit elements.

- Three-terminal element: an electrical entity with three terminals (two ports).
- Passive sign convention: the currents flow to the plus terminals.
- Absorbed power: $p(t) = v_1(t)i_1(t) + v_2(t)i_2(t)$.

# Lumped Circuits

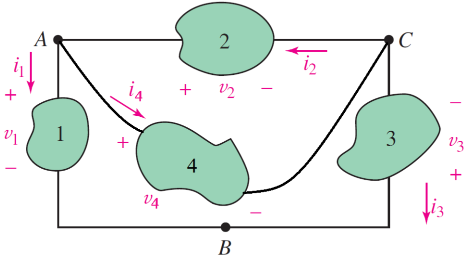



Figure: A sample lumped circuit, where the passive sign convention is not held for all the circuit elements.

- Circuit: an interconnection of elements under an arbitrary topology.
- Lumped circuit: a circuit for which the lumped condition is held.

# Circuit Laws

# Circuit Graphs

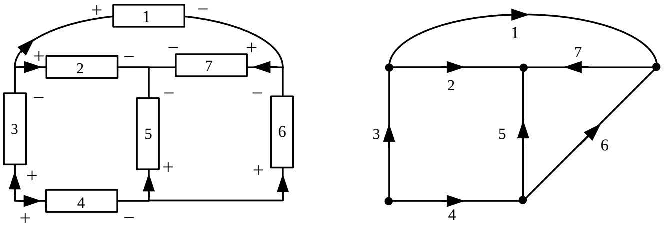


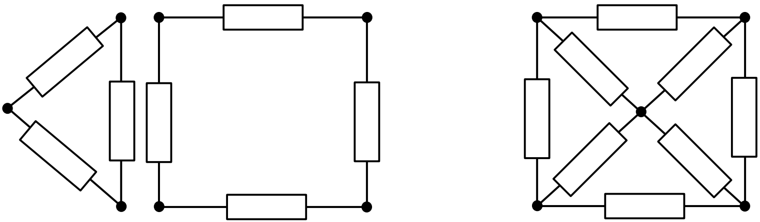

Figure: Each circuit can be represented by a directed graph if each element is replaced with a branch having two ending nodes. The branch directions coincide with the current arrows. If the passive sign convention is held, the voltage polarities are uniquely determined from the circuit graph. The nature of elements is discarded in the network graph.

1. $n$ : number of nodes (vertices)
2. $b$ : number of branches (edges)
3. $l = b - n + 1$ : number of meshes in a connected planar graph
4. Gaussian surface (super node)
5. Closed chain (closed path)

# Circuit Graphs

## Example (Connected circuit)

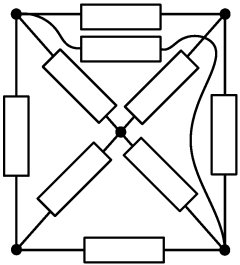
The left circuit is unconnected while the right one is connected.

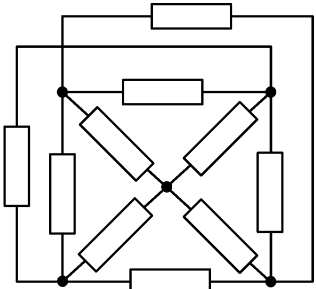


# Circuit Graphs

**Example (Planar circuit)**

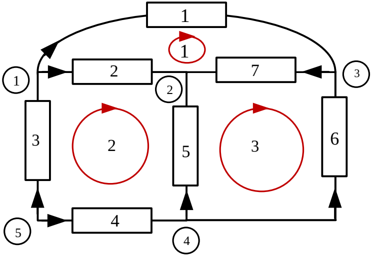
The left circuit is planar while the right one is not.





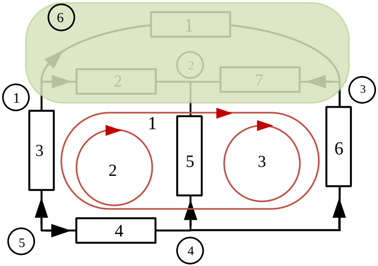
## Example (Nodes, branches, and meshes)

The planar circuit below has 7 branches, 5 nodes, and 3 meshes.



## Example (Gaussian surface closed chain)

For the planar circuit below, the Gaussian surface 6 crosses the branches 3, 5, and 6 while the closed path 1 includes the branches 2, 3, 4, 6, and 7.



# Circuit Laws

**Theorem (KCL)**

*For any lumped network and at any time, the algebraic sum of the branch currents entering (exiting) a node or a Gaussian surface branches is zero.*

**Theorem (KVL)**

*For any lumped network and at any time, the algebraic sum of the aligned branch voltages around a mesh or a closed chain is zero.*

**Corollary (Tellegen)**

*For any lumped network and at any time, the algebraic sum of the branch absorbed powers are zero.*

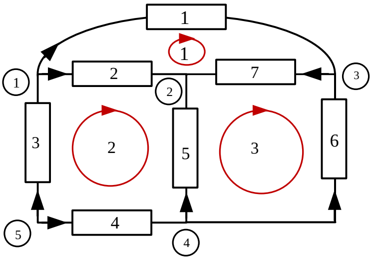
# Circuit Laws

## Example (KCL at nodes)

The circuit below has 4 independent KCLs.

$$\begin{cases} i_1(t) + i_2(t) - i_3(t) = 0 \\ -i_2(t) - i_5(t) - i_7(t) = 0 \\ -i_1(t) - i_6(t) + i_7(t) = 0 \\ i_3(t) + i_4(t) = 0 \end{cases}$$

$$\begin{aligned} &- [i_1(t) + i_2(t) - i_3(t)] \\ &- [-i_2(t) - i_5(t) - i_7(t)] \\ &- [-i_1(t) - i_6(t) + i_7(t)] \\ &- [i_3(t) + i_4(t)] \\ &= -i_4(t) + i_5(t) + i_6(t) = 0 \end{aligned}$$

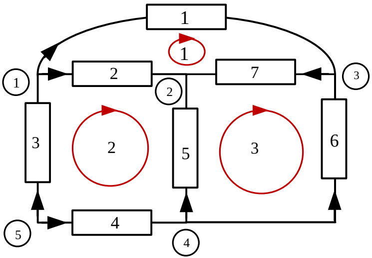


## Example (KVL at meshes)

The circuit below has 3 independent KVLs.

$$\begin{cases} v_1(t) + v_7(t) - v_2(t) = 0 \\ v_2(t) - v_5(t) - v_4(t) + v_3(t) = 0 \\ v_5(t) - v_7(t) - v_6(t) = 0 \end{cases}$$

$$\begin{aligned} &+ [v_1(t) + v_7(t) - v_2(t)] \\ &+ [v_2(t) - v_5(t) - v_4(t) + v_3(t)] \\ &+ [v_5(t) - v_7(t) - v_6(t)] \\ &= v_1(t) - v_4(t) + v_3(t) - v_6(t) = 0 \end{aligned}$$



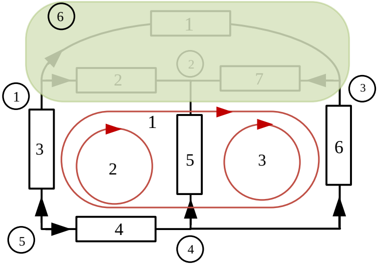
# Circuit Laws

## Example (KCL at super node)

The KCL at the Gaussian surface 6 below depends linearly on the KCLs at nodes 1, 2, and 3.

$$-i_3(t) - i_5(t) - i_6(t) = 0$$

$$\begin{aligned}
&+[i_1(t) + i_2(t) - i_3(t)] \\
&+[-i_2(t) - i_5(t) - i_7(t)] \\
&+[-i_1(t) - i_6(t) + i_7(t)] \\
&= -i_3(t) - i_5(t) - i_6(t) = 0
\end{aligned}$$

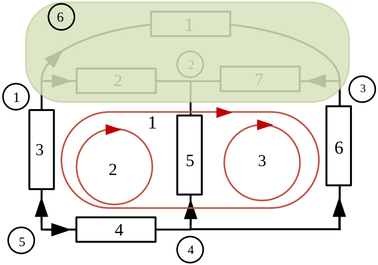


## Example (KVL at closed chain)

The KVL at the closed path 1 below depends linearly on the KVLs at meshes 2 and 3.

$$+v_2(t) - v_7(t) - v_6(t) - v_4(t) + v_3(t) = 0$$

$$\begin{aligned}&+[v_2(t) - v_5(t) - v_4(t) + v_3(t)]\\&+[v_5(t) - v_7(t) - v_6(t)]\\&= v_2(t) - v_7(t) - v_6(t) - v_4(t) + v_3(t) = 0\end{aligned}$$

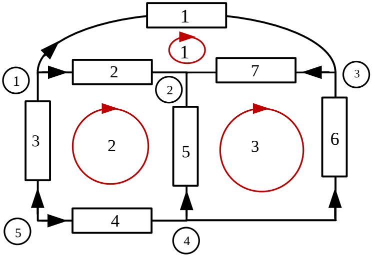


# Circuit Laws

## Example (Independent currents)

$i_1(t)$, $i_2(t)$, and $i_6(t)$ can be considered as the 3 independent currents of the circuit below.

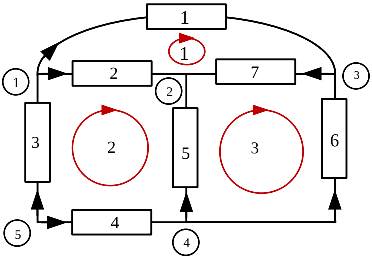
$$\begin{cases} i_1(t) = i_1(t) \\ i_2(t) = i_2(t) \\ i_3(t) = i_1(t) + i_2(t) \\ i_4(t) = -i_3(t) = -i_1(t) - i_2(t) \\ i_5(t) = i_4(t) - i_6(t) = -i_1(t) - i_2(t) - i_6(t) \\ i_6(t) = i_6(t) \\ i_7(t) = i_1(t) + i_6(t) \end{cases}$$



## Example (Independent voltages)

$v_1(t)$, $v_2(t)$, $v_3(t)$, and $v_4(t)$ can be considered as the 4 independent voltages of the circuit below.

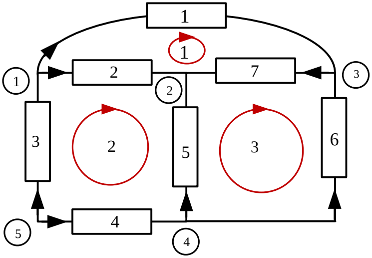
$$\begin{cases} v_1(t) = v_1(t) \\ v_2(t) = v_2(t) \\ v_3(t) = v_3(t) \\ v_4(t) = v_4(t) \\ v_5(t) = -v_4(t) + v_3(t) + v_2(t) \\ v_6(t) = -v_4(t) + v_3(t) + v_1(t) \\ v_7(t) = -v_1(t) + v_2(t) \end{cases}$$



## Example (Tellegen's corollary)

For the circuit below, the Tellegen's corollary can be verified using the governing KCLs and KVLs.

$$
\begin{aligned}
&v_1 i_1 + v_2 i_2 + v_3 i_3 + v_4 i_4 + v_5 i_5 + v_6 i_6 + v_7 i_7 \\
=&v_1 i_1 + v_2 i_2 + v_3(i_1 + i_2) + v_4(-i_1 - i_2) \\
+&v_5(-i_1 - i_2 - i_6) + v_6 i_6 + v_7(i_1 + i_6) \\
=&i_1(v_1 + v_3 - v_4 - v_5 + v_7) \\
+&i_2(v_2 + v_3 - v_4 - v_5) + i_6(-v_5 + v_6 + v_7) = 0
\end{aligned}
$$



# Circuit Laws

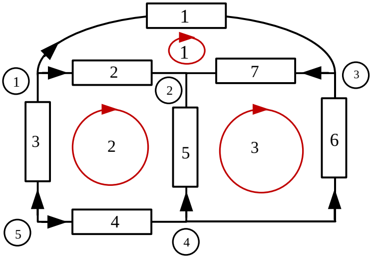
## Example (Tellegen's corollary)

For the circuit below, the branch absorbed powers are completely determined if $i_1(t) = i_2(t) = i_6(t) = 1$ A and $v_1(t) = v_2(t) = v_3(t) = v_4(t) = 1$ V.

$$\begin{cases} i_1(t) = i_2(t) = i_6(t) = 1 \\ i_3(t) = i_1(t) + i_2(t) = 2 \\ i_4(t) = -i_1(t) - i_2(t) = -2 \\ i_5(t) = -i_1(t) - i_2(t) - i_6(t) = -3 \\ i_7(t) = i_1(t) + i_6(t) = 2 \end{cases}$$

$$\begin{cases} v_1(t) = v_2(t) = v_3(t) = v_4(t) = 1 \\ v_5(t) = -v_4(t) + v_3(t) + v_2(t) = 1 \\ v_6(t) = -v_4(t) + v_3(t) + v_1(t) = 1 \\ v_7(t) = -v_1(t) + v_2(t) = 0 \end{cases}$$

$$\begin{cases} p_1(t) = v_1(t)i_1(t) = 1 \\ p_2(t) = v_2(t)i_2(t) = 1 \\ p_3(t) = v_3(t)i_3(t) = 2 \\ p_4(t) = v_4(t)i_4(t) = -2 \\ p_5(t) = v_5(t)i_5(t) = -3 \\ p_6(t) = v_6(t)i_6(t) = 1 \\ p_7(t) = v_7(t)i_7(t) = 0 \end{cases}$$



# KCL

- KCL equations
  1. originate from charge conservation.
  2. are independent of the nature of the elements.
  3. are linear homogeneous equations with real coefficient $-1, 0, 1$.
  4. are dependent equations.
  5. include $n-1$ independent equations.
  6. include $b-n+1$ independent currents.
- KVL equations
  1. originate from conservativity of electric field.
  2. are independent of the nature of the elements.
  3. are linear homogeneous equations with real constant coefficient $-1, 0, 1$.
  4. are dependent equations.
  5. include $b-n+1$ independent equations.
  6. include $n-1$ independent voltages.

# The End