## Question 1

**For the circuit shown in Fig. 1,**

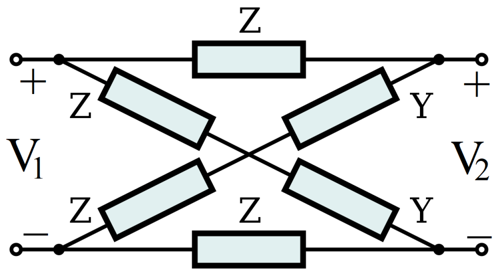


Figure 1: Lattice network.

*(a) Find the network function* $H(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)}$*, where* $Z(j\omega)$ *and* $Y(j\omega)$ *denote impedance and admittance of the diagonal single-port passive networks.*

$$V_2(j\omega) = \frac{Y^{-1}+Z}{Y^{-1}+Z+Z}V_1(j\omega) - \frac{Z}{Y^{-1}+Z+Z}V_1(j\omega) = \frac{Y^{-1}}{Y^{-1}+2Z}V_1(j\omega) = \frac{1}{1+2ZY}V_1(j\omega)$$

So,

$$H(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)} = \frac{1}{1+2Z(j\omega)Y(j\omega)}$$

*(b) Let* $Z(j\omega) = j\omega + \frac{1}{j\omega}$ *and* $Y(j\omega) = j\omega + \frac{1}{j\omega}$ *be a series and a parallel LC network, respectively. Find the simplified transfer function* $H(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)}$ *and the corresponding amplitude and phase responses.*

$$H(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)} = \frac{1}{1+2Z(j\omega)Y(j\omega)} = \frac{1}{1+2(j\omega+\frac{1}{j\omega})(j\omega+\frac{1}{j\omega})} = \frac{-\omega^2}{2\omega^4-5\omega^2+2}$$

$$|H(j\omega)| = \frac{\omega^2}{|2\omega^4-5\omega^2+2|} = \begin{cases} \frac{\omega^2}{2\omega^4-5\omega^2+2}, & \omega < \frac{1}{\sqrt{2}} \\ \frac{\omega^2}{-(2\omega^4-5\omega^2+2)}, & \frac{1}{\sqrt{2}} < \omega < \sqrt{2} \\ \frac{\omega^2}{2\omega^4-5\omega^2+2}, & \omega > \sqrt{2} \end{cases}$$

$$\angle H(j\omega) = \angle -\omega^2 - \angle 2\omega^4 - 5\omega^2 + 2 = \begin{cases} \pi, & \omega < \frac{1}{\sqrt{2}} \\ 0, & \frac{1}{\sqrt{2}} < \omega < \sqrt{2} \\ \pi, & \omega > \sqrt{2} \end{cases}$$

Question 1 continued on next page. . . 

(c) *Plot the amplitude and phase response of $H(j\omega)$ calculated in part (b).*

Magnitude and phase responses are shown in Fig. 2 and Fig. 3, respectively. The magnitude has two vertical asymptote lines at $\omega = \frac{1}{\sqrt{2}}$ and $\omega = \sqrt{2}$. The magnitude is zero at $\omega = 0$. The magnitude also approaches zero for $\omega \to \infty$. The phase experiences sudden jumps at the frequencies $\omega = \frac{1}{\sqrt{2}}$ and $\omega = \sqrt{2}$.

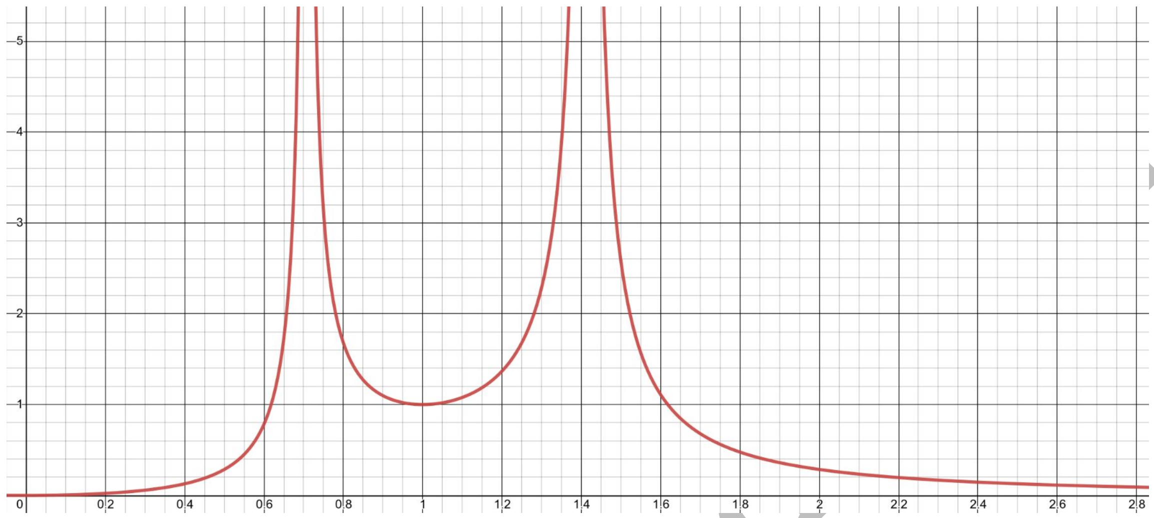


Figure 2: Magnitude response curve.

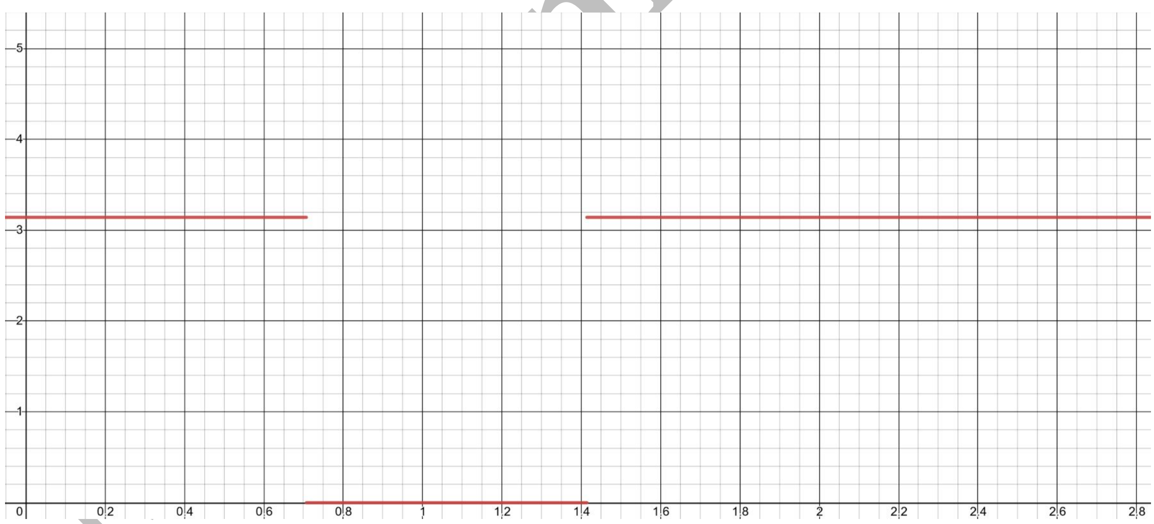


Figure 3: Phase response curve.

## Question 2

**Calculate the Thevenin and Norton equivalent circuits seen from port ab in Fig. 4,**

*Question 2 continued on next page...* 

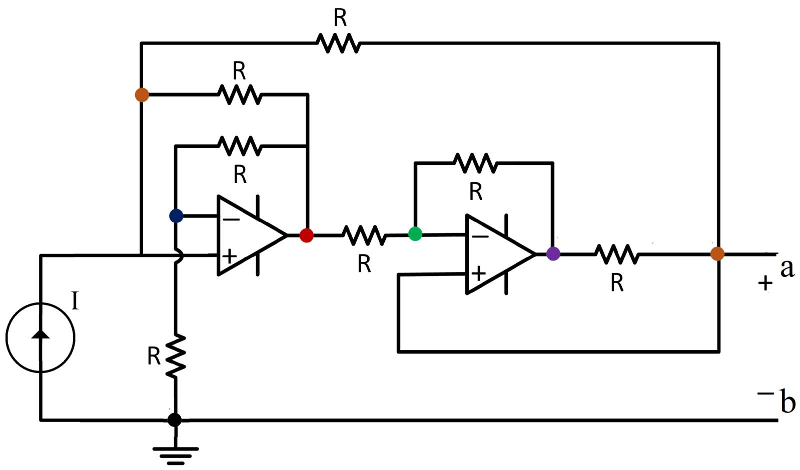


Figure 4: An op-amp circuit.

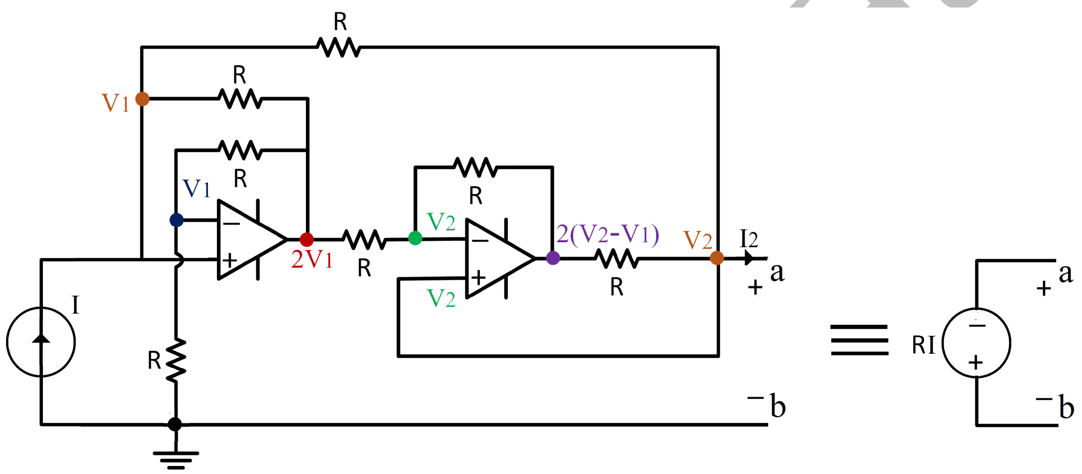


Figure 5: An op-amp circuit.

Consider the annotated circuit shown in Fig. 2. We have,

$$I = \frac{V_1 - 2V_1}{R} + \frac{V_1 - V_2}{R} = -\frac{V_2}{R} \Rightarrow V_2 = -RI = -RI + 0I_2 \Rightarrow V_{oc} = -RI, \quad R_{th} = 0$$

So, the Thevenin equivalent circuit is an alone voltage source as shown in Fig. 5. Consequently, there is no Norton equivalent circuit.

## Question 3

**The zero-state response of an LTI circuit to the input** $x(t) = \big(e^{-t} - \cos(t) + \sin(t)\big)u(t)$ **is** $y(t) = 2\sin(t)u(t)$**. Find the impulse response, step response, and zero-state response to the input** $x_1(t) = e^{-2t}\sin(3t)u(t)$**.**

*Question 3 continued on next page...* 

We use properties of LTI circuits to find the impulse response.

$$
\begin{aligned}
x(t) = \big(e^{-t} - \cos(t) + \sin(t)\big)u(t) &\to y(t) = 2\sin(t)u(t)\\
x'(t) = \big(-e^{-t} + \sin(t) + \cos(t)\big)u(t) &\to y'(t) = 2\cos(t)u(t)\\
x(t) + x'(t) = 2\sin(t)u(t) &\to y(t) + y'(t) = 2\big(\sin(t) + \cos(t)\big)u(t)\\
x'(t) + x''(t) = 2\cos(t)u(t) &\to y'(t) + y''(t) = 2\big(\cos(t) - \sin(t)\big)u(t) + 2\delta(t)\\
x''(t) + x'''(t) = -2\sin(t)u(t) + 2\delta(t) &\to y''(t) + y'''(t) = 2\big(-\sin(t) - \cos(t)\big)u(t)\\
&\qquad + 2\delta(t) + 2\delta'(t)\\
x(t) + x'(t) + x''(t) + x'''(t) = 2\delta(t) &\to y(t) + y'(t) + y''(t) + y'''(t) = 2\delta(t) + 2\delta'(t)\\
\delta(t) &\to h(t) = \delta(t) + \delta'(t)
\end{aligned}
$$

For the step response,

$$x(t) = u(t) \to s(t) = h(t) * u(t) = \big(\delta(t) + \delta'(t)\big) * u(t) = u(t) + \delta(t)$$

And finally, the output corresponding to the input $x_1(t) = e^{-2t}\sin(3t)u(t)$ is

$$y_1(t) = x_1(t) * h(t) = \big(\delta(t) + \delta'(t)\big) * x_1(t) = x_1(t) + x_1'(t) = \big(3e^{-2t}\cos(3t) - e^{-2t}\sin(3t)\big)u(t)$$

## Question 4

**The one ports $\mathcal{N}_1$ and $\mathcal{N}_2$ in Fig. 6 are in sinusoidal steady state. When the one-ports are connected as Fig. 6(a), $V_{aa'} = \sqrt{2}\underline{/-45^\circ}, I_{ab} = 1\underline{/0^\circ}$ while $V_{aa'} = \sqrt{2}\underline{/45^\circ}, I_{ab'} = 3\underline{/0^\circ}$ when the one-ports are connected as Fig. 6(b).**

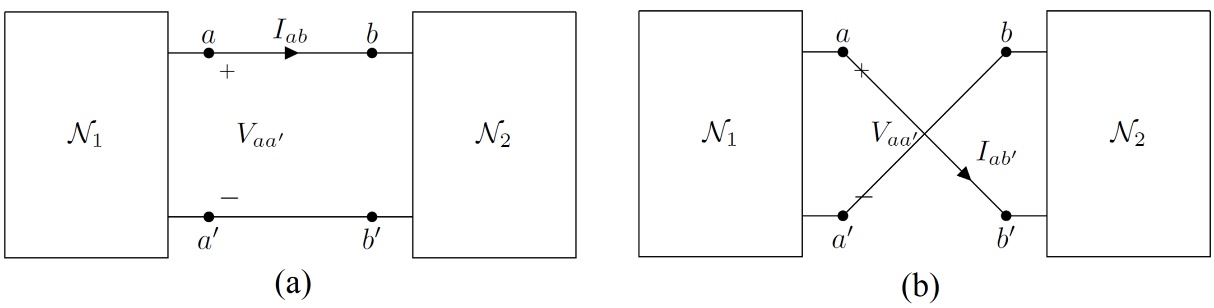


Figure 6: Two one-ports in sinusoidal steady state.

Question 4 continued on next page...

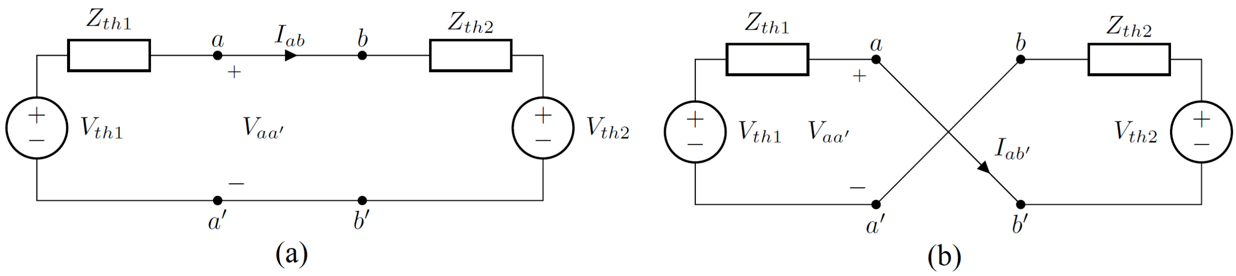


Figure 7: Thevenin equivalent circuits of the one-ports in Fig. 6.

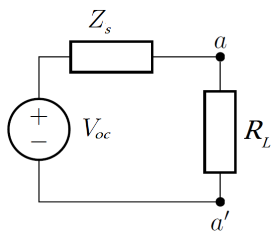


Figure 8: Thevenin equivalent circuit seen from port $aa'$ in Fig. 6(a).



*(a) Find the Thevenin equivalent circuits of the two one-ports.*

The one ports are replaced with their equivalent Thevenin circuits in Fig. 7. We have,

$$I_{ab1} = \frac{V_{th1} - V_{th2}}{Z_{th1} + Z_{th2}}, \quad V_{aa'1} = V_{th1} - I_{ab1}Z_{th1}$$

$$I_{ab'2} = \frac{V_{th1} + V_{th2}}{Z_{th1} + Z_{th2}}, \quad V_{aa'2} = V_{th1} - I_{ab'2}Z_{th1}$$

So,

$$V_{aa'1} - V_{aa'2} = Z_{th1}(I_{ab'2} - I_{ab1})$$

$$\Rightarrow Z_{th1} = \frac{V_{aa'1} - V_{aa'2}}{I_{ab'2} - I_{ab1}} = \frac{\sqrt{2}(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}) - \sqrt{2}(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})}{3 - 1} = -j$$

$$V_{th1} = V_{aa'1} + I_{ab1}Z_{th1} = \sqrt{2}(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}) - j = 1 - 2j$$

$$I_{ab1} + I_{ab'2} = \frac{2V_{th1}}{Z_{th1} + Z_{th2}} \Rightarrow Z_{th2} = \frac{2V_{th1}}{I_{ab1} + I_{ab'2}} - Z_{th1} = \frac{2 - 4j}{4} + j = \frac{1}{2}$$

$$V_{th2} = I_{ab'2}(Z_{th1} + Z_{th2}) - V_{th1} = 3(\frac{1}{2} - j) - 1 + 2j = \frac{1}{2} - j$$

*Question 4 continued on next page...* 

*(b) Find the resistance of the resistive load $R_L$ connected between $a$ and $a'$ in Fig. 6(a) that absorbs the maximum average power.*

Note that we have a pure resistive load here and we cannot use the simple form of the maximum power transfer theorem. Consider Fig. 8, where that the series combination of the voltage source $V_s$ and impedance $Z_s$ drives the load $Z_L = R_L$. We have,

$$I = \frac{V_s}{Z_s + Z_L} = \frac{V_s}{(R_s + R_L) + jX_s} \Rightarrow P_{avL} = \frac{\Re\{Z_L\}}{2}|I|^2 = \frac{1}{2}\frac{R_L}{(R_L + R_s)^2 + X_s^2}|V_s|^2$$
$$\frac{dP_{avL}}{dR_L} = 0 \Rightarrow (R_L + R_s)^2 + X_s^2 - 2(R_L + R_s)R_L = 0 \Rightarrow R_L = \sqrt{R_s^2 + X_s^2} = |Z_s|$$

The equivalent impedance seen from the port $aa'$ in Fig. 7(a) is shown in Fig. 8 and equal to

$$Z_s = Z_{th1}||Z_{th2} = \frac{-j\frac{1}{2}}{-j + \frac{1}{2}}$$

So, the desired load should have the resistance

$$R_L = |Z_s| = \left|\frac{-j\frac{1}{2}}{-j + \frac{1}{2}}\right| = \frac{\sqrt{5}}{5}$$

*(c) Find the maximum power absorbed by the resistive load $R_L$ calculated in the previous part.*

We know that

$$Z_s = Z_{th1}||Z_{th2} = \frac{-j\frac{1}{2}}{-j + \frac{1}{2}} = \frac{2}{5} - j\frac{1}{5} = R_s + jX_s$$

Further, using superposition in Fig. 7(a),

$$V_{oc} = \frac{Z_{th2}}{Z_{th1} + Z_{th2}}V_{th1} + \frac{Z_{th1}}{Z_{th1} + Z_{th2}}V_{th2} = \frac{1 + 3j}{-1 + 2j} = 1 - j$$

The maximum power is consumed for $R_L = \frac{\sqrt{5}}{5}$ and equals

$$P_{avL} = \frac{1}{2}\frac{R_L}{(R_L + R_s)^2 + X_s^2}|V_s|^2 = \frac{1}{2}\frac{\frac{\sqrt{5}}{5}}{(\frac{\sqrt{5}}{5} + \frac{2}{5})^2 + (-\frac{1}{5})^2}|1 - j|^2 = \frac{5}{2\sqrt{5} + 4}$$

## Question 5

**Let $R_1 = 3\Omega$, $R_2 = 6\Omega$, and $L = \frac{2}{3}$H in the first order RL circuit shown in Fig. 9.**

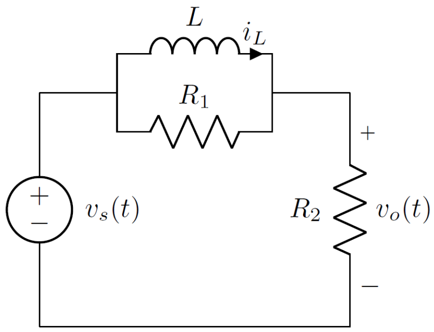


Figure 9: An RL circuit.

*(a) Assume that* $v_s(t) = e^{-\frac{9}{2}t}u(t)$. *Calculate the initial current* $I_0$ *of the inductor such that* $v_o(t) = 0, t > 0$.

$$v_L = L\frac{di_L}{dt}, \quad i_{R2} = i_L + i_{R1} = i_L + \frac{L}{R_1}\frac{di_L}{dt}, \quad v_o = R_2 i_{R_2} = R_2(i_L + \frac{L}{R_1}\frac{di_L}{dt})$$

$$v_s = v_o + v_L = R_2(i_L + \frac{L}{R_1}\frac{di_L}{dt}) + L\frac{di_L}{dt} = R_2 i_L + L(1 + \frac{R_2}{R_1})\frac{di_L}{dt}$$

$$2\frac{di_L}{dt} + 6i_L = e^{-\frac{9}{2}t}, t > 0 \Rightarrow i_L = Ke^{-3t} - \frac{1}{3}e^{-\frac{9}{2}t}, t > 0$$

$$v_o = 6i_L + \frac{4}{3}\frac{di_L}{dt} = 2Ke^{-3t} = 0, t > 0 \Rightarrow K = 0 \Rightarrow i_L(0) = I_0 = K - \frac{1}{3} = -\frac{1}{3}$$

*(b) Assume that* $v_s(t) = 8\cos(\sqrt{3}t + \phi)u(t)$ *V and let the inductor initial current* $i_L(0) = I_0 = 1$ *A. Calculate the phase* $\phi$ *such that no transient response appears in* $i_L(t), t > 0$.

We know that

$$v_s = R_2 i_L + L(1 + \frac{R_2}{R_1})\frac{di_L}{dt} \Rightarrow 2\frac{di_L}{dt} + 6i_L = 8\cos(\sqrt{3}t + \phi), t > 0$$

$$i_L = i_h + i_p = Ke^{-3t} + I_m\cos(\sqrt{3}t + \alpha)$$

Using phasor analysis,

$$2j\sqrt{3}I_p + 6I_p = 8e^{j\phi} \Rightarrow I_p = \frac{8e^{j\phi}}{2j\sqrt{3} + 6} = \frac{8e^{j\phi}}{\sqrt{48}e^{j\frac{\pi}{6}}} = \frac{2}{\sqrt{3}}e^{j(\phi - \frac{\pi}{6})} \Rightarrow i_p = \frac{2}{\sqrt{3}}\cos(\sqrt{3}t + \phi - \frac{\pi}{6})$$

Overall,

$$i_L = Ke^{-3t} + \frac{2}{\sqrt{3}}\cos(\sqrt{3}t + \phi - \frac{\pi}{6})$$

To remove the transient part of the response $K = 0$. Further,

$$i_L(0) = I_0 = 1 = K + \frac{2}{\sqrt{3}}\cos(\phi - \frac{\pi}{6}) = \frac{2}{\sqrt{3}}\cos(\phi - \frac{\pi}{6}) \Rightarrow \cos(\phi - \frac{\pi}{6}) = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$