First-order Circuits

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Overview

First-order Circuits

Circuit Types

Statement (Linear Circuit)

A linear circuit is a circuit that includes linear elements and/or independent sources.

Statement (LTI Circuit)

An LTI circuit is a circuit that includes LTI elements and/or independent sources.

Statement (First-order Circuit)

A first-order circuit is a circuit that has one independent energy-storage element.

Statement (First-order LTI Circuit)

A first-order LTI circuit is an LTI circuit that has one independent energy-storage element.

- Capacitors and inductors are energy-storage elements.

Definition (Circuit Inputs)

Independent sources are called circuit inputs.

Definition (Circuit Initial Conditions)

The initial voltage of the capacitors and initial currents of the inductors are referred to as circuit initial conditions.

Definition (Circuit Response)

The circuit response is the voltage or current of a desired element of the circuit.

Definition (Zero-input response)

Zero-input response is defined as the response of a circuit when its inputs are identically zero.

Definition (Zero-state response)

Zero-state response is defined as the the response of a circuit when its initial conditions are zero.

Definition (Complete response)

Complete response is defined as the response of a circuit to both inputs and initial states.

Circuit Responses

Definition (Impulse Response)

The zero-state response of the LTI circuit to the impulse input is called impulse response.

Definition (Step Response)

The zero-state response of the LTI circuit to the unit step input is called step response.

Circuit Responses

Definition (Transient Response)

Transient response is the part of the circuit response that damps as time proceeds.

Definition (Steady-state response)

Steady-state response is the part of the circuit response that remains as time proceeds.

Linear Circuits

Statement (Zero-input response)

In a linear circuit, the zero-input response is a linear function of the initial conditions.

Statement (Zero-state response)

In a linear circuit, the zero-state response is a linear function of the inputs.

Statement (Complete response)

In a linear circuit, the complete response is the sum of zero-input and zero-state responses.

- Homogeneity property of a linear function:
  $i_s(t) \to y(t) \Rightarrow Ki_s(t) \to Ky(t)$
- Additivity property of a linear function:
  $i_{s1}(t) \to y_1(t), i_{s2}(t) \to y_2(t) \Rightarrow i_{s1}(t) + i_{s2}(t) \to y_1(t) + y_2(t)$

LTI Circuits

Statement (Input Shift Property)

In an LTI circuit, the zero-state response to a shifted input experiences the same shift.

- Shift property: $i_s(t), y(0^-) = 0 \to y(t), t > 0 \Rightarrow i_s(t - t_0), y(t_0^-) = 0 \to y(t - t_0), t > t_0$

Statement (Input Derivative Property)

In an LTI circuit, the zero-state response to the derivative of an input equals the derivative of the zero-state response to the input.

- Derivative property:
$i_s(t), y(0^-) = 0 \to y(t), t > 0 \Rightarrow \frac{di_s(t)}{dt}, y(0^-) = 0 \to \frac{dy(t)}{dt}, t > 0$

First-order LTI RC Circuit

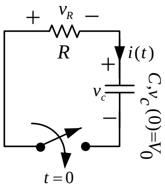
Example (Zero-input response for series RC circuit)

The zero-input response of capacitor voltage in the series RC circuit has exponential form.

$$v_c(0^-) = V_0 \Rightarrow v_c(0^+) = V_0 \Rightarrow v_c(0) = V_0$$

$$i_R(t) - i_c(t) = 0 \Rightarrow \frac{v_R(t)}{R} - C\frac{dv_c(t)}{dt} = 0$$

$$-\frac{v_c(t)}{R} - C\frac{dv_c(t)}{dt} = 0 \Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t) = 0, \tau = RC$$



$$v_c(t) = Ke^{-\frac{t}{\tau}}, v_c(0) = K = V_0$$

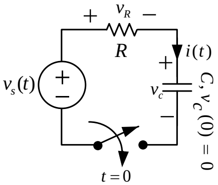
$$\Rightarrow v_c(t) = V_0 e^{-\frac{t}{\tau}}, t > 0$$

Example (Zero-state response for series RC circuit)

The DC-driven zero-state response of capacitor voltage in the series RC circuit has exponential form.

$$v_c(0^-) = 0 \Rightarrow v_c(0^+) = 0 \Rightarrow v_c(0) = 0$$

$$\begin{aligned}
&v_R(t) + v_c(t) - v_s(t) = 0\\
&\Rightarrow Ri_R(t) + v_c(t) = Ri_c(t) + v_c(t) = v_s(t)\\
&\Rightarrow RC\frac{dv_C(t)}{dt} + v_c(t) = v_s(t)\\
&\Rightarrow \frac{dv_C(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{v_s(t)}{\tau}, \quad \tau = RC\\
&\Rightarrow \frac{dv_C(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{V_s}{\tau}
\end{aligned}$$



$$\begin{aligned}
&v_c(t) = Ke^{-\frac{t}{\tau}} + V_s, v_c(0) = K + V_s = 0 \Rightarrow K = -V_s\\
&\Rightarrow v_c(t) = V_s(1 - e^{-\frac{t}{\tau}}), t > 0
\end{aligned}$$

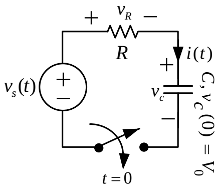
Series RC Circuit

Example (Complete response for series RC circuit)

The DC-driven complete response of capacitor voltage in the series RC circuit has exponential form.

$$v_c(0^-) = V_0 \Rightarrow v_c(0^+) = V_0 \Rightarrow v_c(0) = V_0$$

$$v_R(t) + v_c(t) - v_s(t) = 0$$
$$\Rightarrow Ri_R(t) + v_c(t) = Ri_c(t) + v_c(t) = v_s(t)$$
$$\Rightarrow RC\frac{dv_C(t)}{dt} + v_c(t) = v_s(t)$$
$$\Rightarrow \frac{dv_C(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{v_s(t)}{\tau}, \quad \tau = RC$$
$$\Rightarrow \frac{dv_C(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{V_s}{\tau}$$

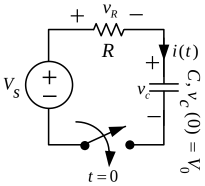


$$v_c(t) = Ke^{-\frac{t}{\tau}} + V_s, v_c(0) = K + V_s = V_0 \Rightarrow K = V_0 - V_s$$
$$\Rightarrow v_c(t) = (V_0 - V_s)e^{-\frac{t}{\tau}} + V_s, t > 0$$

Example (Complete response for series RC circuit)

The compete respone is the sum of the zero-input and zero-state responses in the DC-driven series RC circuit.

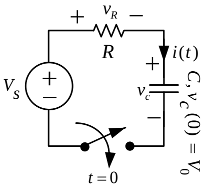
$$v_c(t) = (V_0 - V_s)e^{-\frac{t}{\tau}} + V_s, t > 0$$
$$v_c(t) = V_0 e^{-\frac{t}{\tau}} + V_s(1 - e^{-\frac{t}{\tau}}), t > 0$$



Example (Transient and steady state responses for series RC circuit)

The DC-driven series RC circuit with $\tau > 0$ has a damping exponential transient response and a constant steady state response.

$$v_c(t) = (V_0 - V_s)e^{-\frac{t}{\tau}} + V_s, t > 0$$



Series RC Circuit

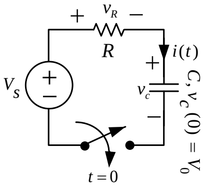
Example (Current response for series RC circuit)

The current response experiences a discontinuity at $t = 0$ in the series RC circuit.

$$i(0^-) = 0, i(0^+) = \frac{V_s - V_0}{R}$$

$$v_c(t) = (V_0 - V_s)e^{-\frac{t}{\tau}} + V_s, t > 0$$

$$i(t) = C\frac{dv_c(t)}{dt} = \frac{V_s - V_0}{R}e^{-\frac{t}{\tau}}$$



Series RC Circuit

Example (Stored energy for series RC circuit)

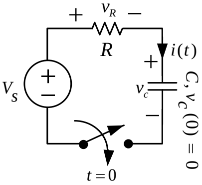
$\frac{1}{2}CV_s^2$ of electrical energy is stored in the zero-state DC-driven series RC circuit.

$$v_c(t) = V_s(1 - e^{-\frac{t}{\tau}}), t > 0$$

$$i(t) = \frac{V_s}{R} e^{-\frac{t}{\tau}}$$

$$p(t) = \frac{V_s^2}{R} e^{-\frac{t}{\tau}} (1 - e^{-\frac{t}{\tau}})$$

$$w(0, \infty) = \int_0^\infty p(t)dt = \frac{1}{2}CV_s^2$$



Example (Step response)

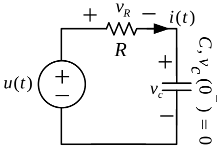
Step response of the series RC circuit has exponential form.

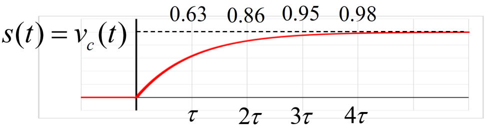
$$v_c(0^-) = 0 \Rightarrow v_c(0^+) = 0$$

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau} v_c(t) = \frac{1}{\tau}$$

$$\Rightarrow v_c(t) = 1 - e^{-\frac{t}{\tau}}, t > 0$$

$$v_c(t) = s(t) = \begin{cases} 0, & t \le 0 \\ 1 - e^{-\frac{t}{\tau}}, & t > 0 \end{cases} = (1 - e^{-\frac{t}{\tau}})u(t)$$





Example (Linearity and shift property of LTI circuits)

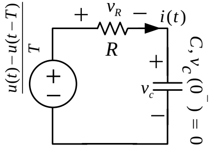
Zero-state response to a complex input can be found using linearity and shift properties of an LTI circuit.

$$v_c(0^-) = 0, v_s(t) = \frac{u(t) - u(t-T)}{T}$$

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau} v_c(t) = \frac{u(t) - u(t-T)}{T\tau}$$

$$\Rightarrow v_c(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1-e^{-\frac{t}{\tau}}}{T}, & 0 \leq t < T \\ \frac{e^{-\frac{t-T}{\tau}} - e^{-\frac{t}{\tau}}}{T}, & t \geq T \end{cases}$$

$$\Rightarrow v_c(t) = \frac{1}{T}(1 - e^{-\frac{t}{\tau}})u(t) - \frac{1}{T}(1 - e^{-\frac{t-T}{\tau}})u(t-T)$$



Series RC Circuit

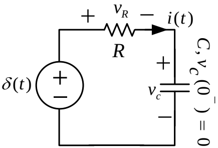
Example (Impulse response)

Impulse response of the series RC circuit is the derivative of the step response.

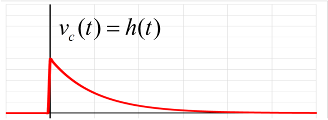
$$v_c(0^-) = 0, \delta(t) = \lim_{T \to 0} \frac{u(t) - u(t-T)}{T}$$

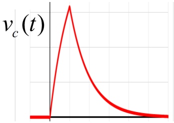
$$\frac{dv_C(t)}{dt} + \frac{1}{\tau} v_c(t) = \frac{u(t) - u(t-T)}{T\tau}$$

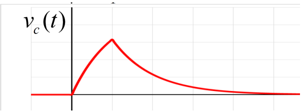
$$\Rightarrow v_c(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1 - e^{-\frac{t}{\tau}}}{T}, & 0 \leq t < T \\ \frac{e^{-\frac{t-T}{\tau}} - e^{-\frac{t}{\tau}}}{T}, & t \geq T \end{cases}$$



$$h(t) = \lim_{T \to 0} v_c(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{\tau} e^{-\frac{t}{\tau}}, & t \geq 0 \end{cases} = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) = \frac{ds(t)}{dt}$$







Example (Impulse response (cont.))

Impulse response of the series RC circuit is the derivative of the step response.

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{\delta(t)}{\tau}, v_c(0^-) = 0$$

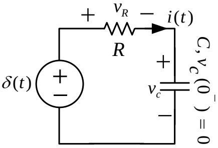
$$\frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t) = 0, t > 0, v_c(0^+) = ?$$

$$\int_{0^-}^{0^+} \left[\frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t)\right] dt = \int_{0^-}^{0^+} \frac{\delta(t)}{\tau} dt$$

$$(v_c(0^+) - v_c(0^-)) + 0 = \frac{1}{\tau} \Rightarrow v_c(0^+) = \frac{1}{\tau}$$

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t) = 0, v_c(0^+) = \frac{1}{\tau}$$

$$h(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t) = \frac{ds(t)}{dt}$$



Parallel RC Circuit

Example (Complete response for parallel RC circuit)

The DC-driven complete response of capacitor voltage in the parallel RC circuit has exponential form.

$v_c(0^-) = V_0 \Rightarrow v_c(0^+) = V_0 \Rightarrow v_c(0) = V_0$

$i_R(t) + i_c(t) - i_s(t) = 0$

$$\Rightarrow \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt} = i_s(t)$$

$$\Rightarrow RC\frac{dv_C(t)}{dt} + v_c(t) = Ri_s(t)$$

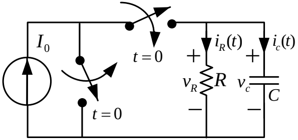
$$\Rightarrow \frac{dv_C(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{i_s(t)}{C}, \quad \tau = RC$$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{I_s}{C}, t > 0$$

$$\Rightarrow v_c(t) = Ke^{-\frac{t}{\tau}} + RI_s, v_c(0) = K + RI_s = V_0 \Rightarrow K = V_0 - RI_s$$

$$\Rightarrow v_c(t) = (V_0 - RI_s)e^{-\frac{t}{\tau}} + RI_s, t > 0$$

$$\Rightarrow v_c(t) = V_0 e^{-\frac{t}{\tau}} + RI_s(1 - e^{-\frac{t}{\tau}}), t > 0$$



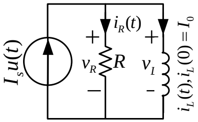
First-order LTI Circuit

DC-driven RL Circuit

Example (Complete response for an RL circuit)

The DC-driven complete response of inductor current in the RL circuit has exponential form.

$$
\begin{aligned}
& i_L(0^-) = I_0 \Rightarrow i_L(0^+) = I_0 \Rightarrow i_L(0) = I_0 \\
& i_R(t) + i_L(t) - i_s(t) = 0 \\
& \Rightarrow \frac{v_L(t)}{R} + i_L(t) = i_s(t) \\
& \Rightarrow \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t) \\
& \Rightarrow \frac{di_L(t)}{dt} + \frac{1}{\tau} i_L(t) = \frac{i_s(t)}{\tau}, \quad \tau = \frac{L}{R} \\
& \Rightarrow \frac{di_L(t)}{dt} + \frac{1}{\tau} i_L(t) = \frac{I_s}{\tau}, t > 0 \\
& \Rightarrow i_L(t) = Ke^{-\frac{t}{\tau}} + I_s, i_L(0) = K + I_s = I_0 \Rightarrow K = I_0 - I_s \\
& \Rightarrow i_L(t) = (I_0 - I_s)e^{-\frac{t}{\tau}} + I_s, t > 0 \\
& \Rightarrow i_L(t) = I_0 e^{-\frac{t}{\tau}} + I_s(1 - e^{-\frac{t}{\tau}}), t > 0
\end{aligned}
$$



DC-driven Complete Response

Statement (DC-driven First Order LTI Circuit Describing Equation)

A DC-driven first-order LTI circuit is described by the constant-coefficient linear differential equation $\frac{df(t)}{dt} + \frac{1}{\tau} f(t) = F_s, f(t_0^+) = F_0, t > t_0$ *, where* F_0 *is a suitably calculated initial value at* $t = t_0$*.*

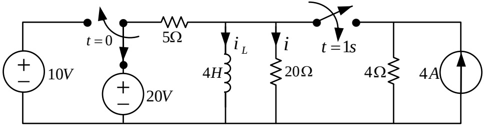
Statement (DC-driven First Order LTI Circuit Complete Response)

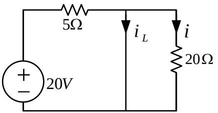
A DC-driven first-order LTI circuit achieving steady state situation has the complete response $f(t) = f(\infty) + [f(t_0^+) - f(\infty)]e^{-\frac{t-t_0}{\tau}}, t > t_0$ *, where* $\tau = R_{eq}C$ *or* $\tau = \frac{L}{R_{eq}}$ *and* R_{eq} *is the equivalent resistance seen from the energy-storage element in the in-rest circuit.* $f(t_0^+)$ *is a suitably calculated initial value at* $t = t_0$ *and* $f(\infty)$ *is the steady state value when the energy-storage element acts like short or open circuit.*

DC-driven LTI First-order Circuit

Example (LTI first-order circuit with two time constants)

Switching may change the time constant of an LTI first-order circuit.



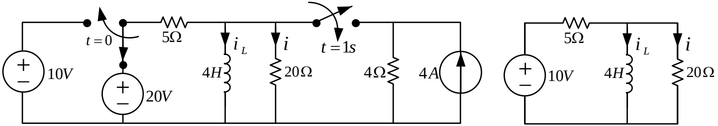


$$i(t) = 0, t < 0 \Rightarrow i_L(t) = \frac{20}{5} = 4, t < 0$$

$$i(0^-) = 0, i_L(0^-) = 4$$

Example (LTI first-order circuit with two time constants (cont.))

Switching may change the time constant of an LTI first-order circuit.



$$i_L(0^+) = i_L(0^-) = 4 \Rightarrow -10 + 5(i(0^+) + i_L(0^+)) + 20i(0^+) = 0 \Rightarrow i(0^+) = -0.4$$

$$i(\infty) = 0 \Rightarrow i_L(\infty) = \frac{10}{5} = 2$$

$$R_{eq} = 20||5 = 4 \Rightarrow \tau = \frac{L}{R_{eq}} = 1$$

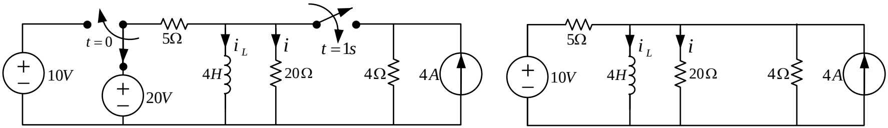
$$i_L(t) = [i_L(0^+) - i_L(\infty)]e^{-\frac{t}{\tau}} + i_L(\infty) = 2 + 2e^{-t}, 0 < t < 1$$

$$i(t) = [i(0^+) - i(\infty)]e^{-\frac{t}{\tau}} + i(\infty) = -0.4e^{-t} = \frac{v_L(t)}{20} = \frac{4}{20}\frac{di_L(t)}{dt}, 0 < t < 1$$

$$i_L(1^-) = 2 + 2e^{-1} \approx 2.74$$

Example (LTI first-order circuit with two time constants (cont.))

Switching may change the time constant of an LTI first-order circuit.



$$i_L(1^+) = i_L(1^-) = 2.74 \Rightarrow -10 + 5(i(1^+) + i_L(1^+) + \frac{20i(1^+)}{4} - 4) + 20i(1^+) = 0 \Rightarrow i(1^+) = -0.33$$

$$i(\infty) = 0 \Rightarrow i_L(\infty) = \frac{10}{5} + 4 = 6$$

$$R_{eq} = 20||5||4 = 2 \Rightarrow \tau = \frac{L}{R_{eq}} = 2$$

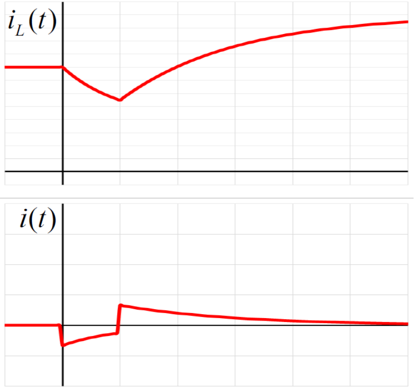
$$i_L(t) = [i_L(1^+) - i_L(\infty)]e^{-\frac{t-1}{\tau}} + i_L(\infty) = 6 - 3.2e^{-\frac{t-1}{2}}, t > 1$$

$$i(t) = [i(1^+) - i(\infty)]e^{-\frac{t-1}{\tau}} + i(\infty) = -0.33e^{-\frac{t-1}{2}} = \frac{v_L(t)}{20} = \frac{4}{20}\frac{di_L(t)}{dt}, t > 1$$

DC-driven LTI First-order Circuit

Example (LTI first-order circuit with two time constants (cont.))

Switching may change the time constant of an LTI first-order circuit.



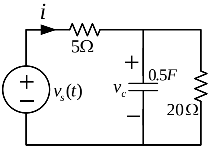
DC-driven LTI First-order Circuit

Example (Step response calculation)

Step response can be found using various methods.

$$-v_s(t) + 5i(t) + v_C(t) = 0, \quad i(t) = \frac{1}{2}\frac{v_C(t)}{dt} + \frac{v_C(t)}{20}$$

$$\frac{dv_C(t)}{dt} + \frac{1}{2}v_C(t) = \frac{2}{5}v_s(t)$$



$$\frac{dv_C(t)}{dt} + \frac{1}{2}v_C(t) = \frac{2}{5}u(t), v_C(0^-) = 0$$

$$\frac{dv_C(t)}{dt} + \frac{1}{2}v_C(t) = \frac{2}{5}, t > 0, v_C(0^+) = 0$$

$$v_C(t) = \frac{4}{5}(1 - e^{-\frac{t}{2}}), t > 0, \quad s_v(t) = \frac{4}{5}(1 - e^{-\frac{t}{2}})u(t)$$

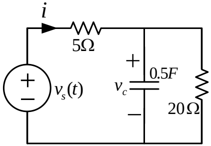
$$s_v(t) = \begin{cases} 0, & t < 0 \\ [v_C(0^+) - v_C(\infty)]e^{-\frac{t}{\tau}} + v_C(\infty), & t > 0 \end{cases} = \frac{4}{5}(1 - e^{-\frac{t}{2}})u(t)$$

Example (Step response calculation (cont.))

Step response can be found using various methods.

$$-v_s(t)+5i(t)+v_C(t)=0, \quad i(t)=\frac{1}{2}\frac{v_C(t)}{dt}+\frac{v_C(t)}{20}$$

$$\frac{di(t)}{dt}+\frac{1}{2}i(t)=\frac{1}{50}v_s(t)+\frac{1}{5}\frac{dv_s(t)}{dt}$$



$$\frac{di(t)}{dt}+\frac{1}{2}i(t)=\frac{1}{50}u(t)+\frac{1}{5}\delta(t), i(0^-)=0$$

$$\int_{0^-}^{0^+}\frac{di(t)}{dt}dt+\frac{1}{2}\int_{0^-}^{0^+} i(t)dt=\frac{1}{50}\int_{0^-}^{0^+} u(t)dt+\frac{1}{5}\int_{0^-}^{0^+} \delta(t)dt$$

$$(i(0^+)-i(0^-))+0=0+\frac{1}{5}\Rightarrow i(0^+)=\frac{1}{5}$$

$$\frac{di(t)}{dt}+\frac{1}{2}i(t)=\frac{1}{50}, i(0^+)=\frac{1}{5}$$

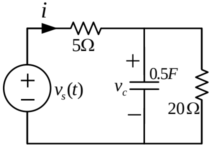
$$i(t)=\frac{4}{25}e^{-\frac{t}{2}}+\frac{1}{25}, t>0, \quad s_i(t)=[\frac{4}{25}e^{-\frac{t}{2}}+\frac{1}{25}]u(t)$$

Example (Step response calculation (cont.))

Step response can be found using various methods.

$$i(t) = \frac{v_s(t) - v_C(t)}{5} \Rightarrow s_i(t) = \frac{u(t) - s_v(t)}{5} = [\frac{4}{25}e^{-\frac{t}{2}} + \frac{1}{25}]u(t)$$

$$s_i(t) = \begin{cases} 0, & t < 0 \\ [i(0^+) - i(\infty)]e^{-\frac{t}{\tau}} + i(\infty), & t > 0 \end{cases} = [\frac{4}{25}e^{-\frac{t}{2}} + \frac{1}{25}]u(t)$$



Example (Impulse response calculation)

Impulse response can be found using various methods.

$$-v_s(t)+5i(t)+v_C(t)=0,\quad i(t)=\frac{1}{2}\frac{dv_C(t)}{dt}+\frac{v_C(t)}{20}$$

$$\frac{dv_C(t)}{dt}+\frac{1}{2}v_C(t)=\frac{2}{5}v_s(t)$$

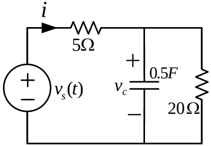
$$\frac{dv_C(t)}{dt}+\frac{1}{2}v_C(t)=\frac{2}{5}\delta(t), v_C(0^-)=0$$

$$\int_{0^-}^{0^+}\frac{dv_C(t)}{dt}dt+\frac{1}{2}\int_{0^-}^{0^+}v_C(t)dt=\frac{2}{5}\int_{0^-}^{0^+}\delta(t)dt$$

$$(v_C(0^+)-v_C(0^-))+0=0+\frac{2}{5}\Rightarrow v_C(0^+)=\frac{2}{5}$$

$$\frac{dv_C(t)}{dt}+\frac{1}{2}v_C(t)=0, v_C(0^+)=\frac{2}{5}$$

$$v_C(t)=\frac{2}{5}e^{-\frac{t}{2}}t>0,\quad h_v(t)=\frac{2}{5}e^{-\frac{t}{2}}u(t)$$



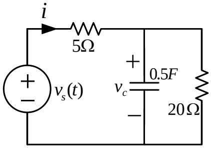
$$h_v(t)=\frac{ds_v(t)}{dt}=\frac{d}{dt}\left[(1-e^{-\frac{t}{2}})u(t)\right]=\frac{2}{5}e^{-\frac{t}{2}}u(t)$$

Example (Impulse response calculation (cont.))

Impulse response can be found using various methods.

$$h_i(t) = \frac{ds_i(t)}{dt} = \frac{d}{dt}\left[\left(\frac{4}{25}e^{-\frac{t}{2}} + \frac{1}{25}\right)u(t)\right] = \frac{-2}{25}e^{-\frac{t}{2}}u(t) + \frac{1}{5}\delta(t)$$

$$i(t) = \frac{v_s(t) - v_C(t)}{5} \Rightarrow h_i(t) = \frac{\delta(t) - h_v(t)}{5} = \frac{-2}{25}e^{-\frac{t}{2}}u(t) + \frac{1}{5}\delta(t)$$

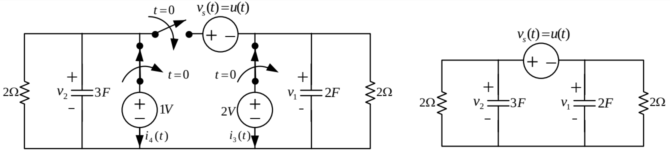


$$\frac{di(t)}{dt} + \frac{1}{2}i(t) = \frac{1}{50}\delta(t) + \frac{1}{5}\delta'(t), i(0^-) = 0$$

$$h_i(t) = \frac{-2}{25}e^{-\frac{t}{2}}u(t) + \frac{1}{5}\delta(t)$$

Example (LTI first-order circuit with two capacitors)

Each capacitive loop decrements the number of independent capacitors.



$v_1(0^-) = 2, \quad v_2(0^-) = 1$

$$\begin{cases} -v_2(t) + u(t) + v_1(t) = 0 \\ \frac{v_2(t)}{2} + 3\frac{dv_2(t)}{dt} + i_4(t) + i_3(t) + 2\frac{dv_1(t)}{dt} + \frac{v_1(t)}{2} = 0, \quad i_4(t) = i_3(t) = 0 \end{cases} , t > 0$$

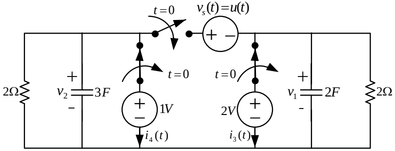
$$\begin{cases} -v_2(0^+) + u(0^+) + v_1(0^+) = 0 \\ 0 + 3(v_2(0^+) - v_2(0^-)) + 0 + 0 + 2(v_1(0^+) - v_1(0^-)) + 0 = 0 \end{cases} \Rightarrow \begin{cases} v_1(0^+) = 0.8 \\ v_2(0^+) = 1.8 \end{cases}$$

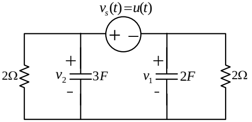
$$\frac{dv_2(t)}{dt} + \frac{1}{5}v_2(t) = 0.1u(t) + 0.4\delta(t), v_2(0^+) = 1.8$$

$$v_2(t) = 0.5 + 1.3e^{-\frac{t}{5}}, t > 0$$

Example (LTI first-order circuit with two capacitors (cont.))

Each capacitive loop decrements the number of independent capacitors.





$$v_1(0^-) = 2, \quad v_2(0^-) = 1 \Rightarrow v_1(0^+) = 0.8, \quad v_2(0^+) = 1.8$$

$$v_1(\infty) = -0.5, \quad v_2(\infty) = 0.5$$

$$R_{eq} = 2||2 = 1, \quad C_{eq} = 2 + 3 = 5 \Rightarrow \tau = R_{eq}C_{eq} = 5$$

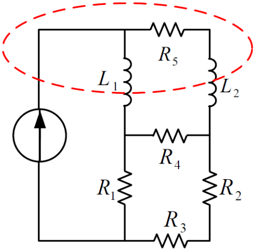
$$v_1(t) = (v_1(0^+) - v_1(\infty))e^{-\frac{t}{\tau}} + v_1(\infty) = -0.5 + 1.3e^{-\frac{t}{5}}$$

$$v_2(t) = (v_2(0^+) - v_2(\infty))e^{-\frac{t}{\tau}} + v_2(\infty) = 0.5 + 1.3e^{-\frac{t}{5}}$$

DC-driven LTI First-order Circuit

Example (LTI first-order circuit with two inductors)

Each inductive Gaussian surface decrements the number of independent inductors.



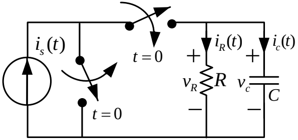
$$R_{eq} = R_5 + (R_4 || (R_1 + R_2 + R_3)), \quad L_{eq} = L_1 + L_2 \Rightarrow \tau = \frac{L_{eq}}{R_{eq}}$$

Exponentially-driven LTI First-order Circuit

Example (Exponential input)

In the circuit below with $R = 1\ \Omega$, $c = 0.5$ F, $v_C(0^+) = 2$ V, and $i_s(t) = 5e^{-t}$, the capacitor voltage has no steady state response.

$$\frac{dv_C(t)}{dt} + 2v_C(t) = 2i_s(t), v_C(0^+) = 2$$
$$\frac{dv_C(t)}{dt} + 2v_C(t) = 10e^{-t}, v_C(0^+) = 2$$
$$v_C(t) = Ke^{-2t} + Ae^{-t}$$
$$\frac{d[Ae^{-t}]}{dt} + 2Ae^{-t} = 10e^{-t} \Rightarrow A = 10$$
$$v_C(0^+) = K + 10 = 2 \Rightarrow K = -8$$
$$v_C(t) = -8e^{-2t} + 10e^{-t}, t > 0$$
$$v_C(t \to \infty) = 0$$



Exponentially-driven LTI First-order Circuit

Example (Sinusoidal input)

In the circuit below with $R = 1\ \Omega$, $c = 0.5$ F, $v_C(0^+) = 2$ V, and $i_s(t) = 5\cos(t)$, the capacitor voltage has sinusoidal steady state response.

$$\frac{dv_C(t)}{dt} + 2v_C(t) = 2i_s(t), v_C(0^+) = 2$$

$$\frac{dv_C(t)}{dt} + 2v_C(t) = 10\cos(t), v_C(0^+) = 2$$

$$v_C(t) = Ke^{-2t} + A\cos(t) + B\sin(t)$$

$$\frac{d[A\cos(t) + B\sin(t)]}{dt} + 2[A\cos(t) + B\sin(t)] = 10\cos(t)$$

$$(2A + B)\cos(t) + (2B - A)\sin(t) = 10\cos(t)$$

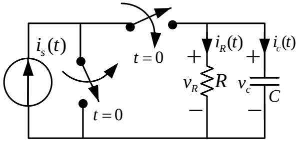
$$\begin{cases} 2A + B = 10 \\ 2B - A = 0 \end{cases} \Rightarrow A = 4, B = 2$$

$$v_C(0^+) = K + 4 = 2 \Rightarrow K = -2$$

$$v_C(t) = -2e^{-2t} + 4\cos(t) + 2\sin(t), t > 0$$

$$v_C(t) = -2e^{-2t} + 2\sqrt{5}\cos(t - \tan^{-1}(0.5)), t > 0$$

$$v_C(t \to \infty) = 2\sqrt{5}\cos(t - \tan^{-1}(0.5))$$



First-order NTV Circuit

Example (First-order LTV circuit)

Generally, a first-order LTV circuit is not described by constant coefficient linear differential equation.

$$R(t) = \frac{1}{1 + 0.5\cos(t)}, v_C(0) = 2$$

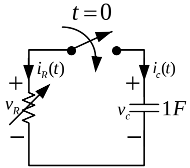
$$i_C(t) + i_R(t) = 0 \Rightarrow \frac{dv_C(t)}{dt} + (1 + 0.5\cos(t))v_C(t) = 0, t > 0$$

$$\frac{dv_C}{v_C} = -(1 + 0.5\cos(t))dt, t > 0$$

$$\int_{v_C(0)}^{v_C(t)} \frac{dv_C}{v_C} = -\int_0^t (1 + 0.5\cos(t))dt$$

$$\ln(v_C(t)) - \ln(v_C(0)) = -t - 0.5\sin(t), t > 0$$

$$v_C(t) = 2e^{-t-0.5\sin(t)}, t > 0$$



First-order NTV Circuit

Example (First-order NTI circuit)

Generally, a first-order NTI circuit is not described by constant coefficient linear differential equation.

$$i_R(t) = v_R^2(t), v_C(0) = 2$$

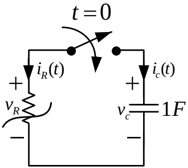
$$i_C(t) + i_R(t) = 0 \Rightarrow \frac{dv_C(t)}{dt} + v_C^2(t) = 0, t > 0$$

$$\frac{dv_C}{v_C^2} = -dt, t > 0$$

$$\int_{v_C(0)}^{v_C(t)} \frac{dv_C}{v_C^2} = -\int_0^t dt$$

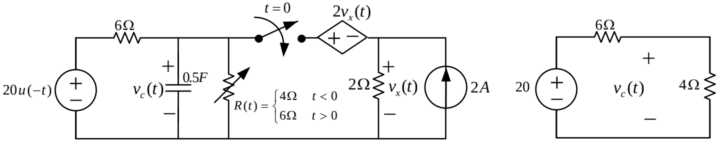
$$-\frac{1}{v_C(t)} + \frac{1}{v_C(0)} = -t, t > 0$$

$$v_C(t) = \frac{1}{t+0.5}, t > 0$$



Example (First-order LTV circuit)

Aa first-order LTV circuit may be treated as an LTI circuit in different time intervals.

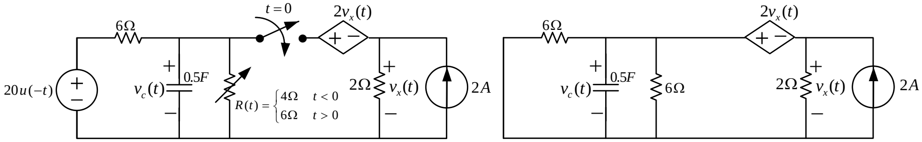


$$v_C(t) = \frac{4}{4+6}20 = 8, t < 0 \Rightarrow v_C(0^-) = 8$$

First-order NTV Circuit

Example (First-order LTV circuit (cont.))

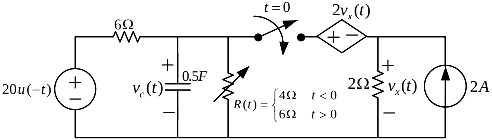
Generally, a first-order LTV circuit may be treated as an LTI circuit in different time intervals.

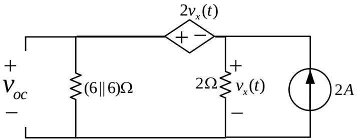


$$v_C(t) = [v_C(0^+) - v_C(\infty)]e^{-\frac{t}{\tau}} + v_C(\infty), t > 0$$
$$v_C(0^+) = v_C(0^-) = 8$$

Example (First-order LTV circuit (cont.))

Generally, a first-order LTV circuit may be treated as an LTI circuit in different time intervals.

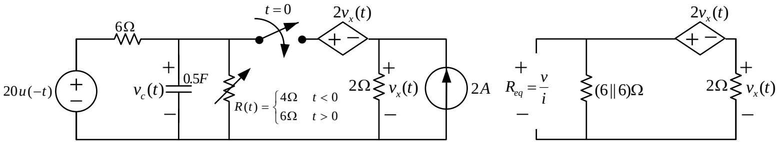




$$v_C(t) = [v_C(0^+) - v_C(\infty)]e^{-\frac{t}{\tau}} + v_C(\infty), t > 0$$
$$v_C(\infty) = v_{oc} = 4$$

Example (First-order LTV circuit (cont.))

Generally, a first-order LTV circuit may be treated as an LTI circuit in different time intervals.



$$v_C(t) = [v_C(0^+) - v_C(\infty)]e^{-\frac{t}{\tau}} + v_C(\infty) = 4 + 4e^{-t}, t > 0$$
$$R_{eq} = 2 \Rightarrow \tau = R_{eq}C = 1$$

The End