LTI Circuits

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Overview

LTI Circuits

Circuit Types

Statement (Linear Circuit)

A linear circuit is a circuit that includes linear elements and/or independent sources.

Statement (LTI Circuit)

An LTI circuit is a circuit that includes LTI elements and/or independent sources.

Circuit Analysis

Definition (Circuit Inputs)

Independent sources are called circuit inputs.

Definition (Single-input Circuit)

A single-input circuit is a circuit with only one input.

Definition (Circuit Initial Conditions)

The initial voltage of the capacitors and initial currents of the inductors are referred to as circuit initial conditions.

Definition (Circuit Response)

The circuit response is the voltage or current of a desired element of the circuit.

Circuit Responses

Definition (Zero-input response)

Zero-input response is defined as the response of a circuit when its inputs are identically zero.

Definition (Zero-state response)

Zero-state response is defined as the the response of a circuit when its initial conditions are zero.

Definition (Complete response)

Complete response is defined as the response of a circuit to both inputs and initial states.

Circuit Responses

Definition (Transient Response)

Transient response is the part of the circuit response that damps as time proceeds.

Definition (Steady-state response)

Steady-state response is the part of the circuit response that remains as time proceeds.

Statement (Zero-input response)

In a linear circuit, the zero-input response is a linear function of the initial conditions.

Statement (Zero-state response)

In a linear circuit, the zero-state response is a linear function of the inputs.

Statement (Complete response)

In a linear circuit, the complete response is the sum of zero-input and zero-state responses.

- Homogeneity property of a linear function:
  $i_s(t) \to y(t) \Rightarrow Ki_s(t) \to Ky(t)$
- Additivity property of a linear function:
  $i_{s1}(t) \to y_1(t), i_{s2}(t) \to y_2(t) \Rightarrow i_{s1}(t) + i_{s2}(t) \to y_1(t) + y_2(t)$

Linear Circuits

Statement (Describing Differential Equation)

A linear circuit is described by a linear differential equation.

- Single-input linear circuit: $\sum_{i=0}^{n} a_i(t)y^{(i)}(t) = \sum_{j=0}^{m} b_j(t)w^{(j)}(t)$

LTI Responses

Definition (Impulse Response)

The zero-state response of the LTI circuit to the impulse input is called impulse response.

Definition (Step Response)

The zero-state response of the LTI circuit to the unit step input is called step response.

LTI Circuits

Statement (Input Shift Property)

In an LTI circuit, the zero-state response to a shifted input experiences the same shift.

- Shift property: $i_s(t), y(0^-) = 0 \to y(t), t > 0 \Rightarrow i_s(t - t_0), y(t_0^-) = 0 \to y(t - t_0), t > t_0$

Statement (Input Derivative Property)

In an LTI circuit, the zero-state response to the derivative of an input equals the derivative of the zero-state response to the input.

- Derivative property:
$i_s(t), y(0^-) = 0 \to y(t), t > 0 \Rightarrow \frac{di_s(t)}{dt}, y(0^-) = 0 \to \frac{dy(t)}{dt}, t > 0$

LTI Circuits

Statement (Describing Differential Equation)

An LTI circuit is described by a constant-coefficient linear differential equation.

- Single-input LTI circuit: $\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j w^{(j)}(t)$

Statement (Zero-state Response of Arbitrary Input)

The zero-sate response of a single-input LTI circuit to an arbitrary input can be calculated using convolution integral of the input and impulse response.

- Convolution integral: $y(t) = w(t) * h(t) = \int_{-\infty}^{\infty} w(\lambda)h(t-\lambda)d\lambda$

Impulse Response

Single-input LTI Circuit Impulse Response

$$\begin{cases} \sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j \delta^{(j)}(t), t > 0^- \\ y^{(0)}(0^-) = 0, y^{(1)}(0^-) = 0, \cdots, y^{(n-1)}(0^-) = 0 \end{cases} \Rightarrow \begin{cases} \sum_{i=0}^{n} a_i y^{(i)}(t) = 0, t > 0^+ \\ y^{(0)}(0^+), y^{(1)}(0^+), \cdots, y^{(n-1)}(0^+) \end{cases}$$

- Impulse response:
  $h(t) = y_h(t)u(t) + k_{n+1}\delta(t) + k_{n+2}\delta'(t) + \cdots + k_{m+1}\delta^{(m-n)}(t)$
  1. $m < n$: $h(t) = y_h(t)u(t)$
  2. $m = n$: $h(t) = y_h(t)u(t) + k_{n+1}\delta(t)$
  3. $m = n+1$: $h(t) = y_h(t)u(t) + k_{n+1}\delta(t) + k_{n+2}\delta'(t)$
  4. $m = n+2$: $h(t) = y_h(t)u(t) + k_{n+1}\delta(t) + k_{n+2}\delta'(t) + k_{n+3}\delta''(t)$
- Distinguished real roots for the characteristic equation:
  $h(t) = u(t)\sum_{i=1}^{n} k_i e^{s_i t} + \sum_{i=n+1}^{m+1} k_i \delta^{(i-n-1)}(t)$

Single-input LTI Circuit Impulse Response

Example (Impulse response calculation)

The response $y(t)$ of a single-input circuit is described by $\frac{dy}{dt} + 2y(t) = 5w(t) + 8\frac{dw}{dt} + \frac{d^2w}{dt^2}$. The corresponding impulse response is $h(t) = -7e^{-2t}u(t) + 6\delta(t) + \delta'(t)$.

$$\frac{dy}{dt} + 2y(t) = 5w(t) + 8\frac{dw}{dt} + \frac{d^2w}{dt^2}$$

$$\frac{dy}{dt} + 2y(t) = 5\delta(t) + 8\delta'(t) + \delta''(t)$$

$$\frac{dy}{dt} + 2y(t) = 0, t > 0, y(0^+) \Rightarrow y_h(t) = k_1e^{-2t}u(t)$$

$$h(t) = k_1e^{-2t}u(t) + k_2\delta(t) + k_3\delta'(t)$$

$$h'(t) = -2k_1e^{-2t}u(t) + k_1\delta(t) + k_2\delta'(t) + k_3\delta''(t)$$

$$(2k_2 + k_1)\delta(t) + (2k_3 + k_2)\delta'(t) + k_3\delta''(t) = 5\delta(t) + 8\delta'(t) + \delta''(t) \Rightarrow \begin{cases} 2k_2 + k_1 = 5 \\ 2k_3 + k_2 = 8 \\ k_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} k_1 = -7 \\ k_2 = 6 \\ k_3 = 1 \end{cases} \Rightarrow h(t) = -7e^{-2t}u(t) + 6\delta(t) + \delta'(t)$$

Convolution Integral

Convolution Integral

Statement (Zero-state Response of Arbitrary Input)

The zero-sate response of a single-input LTI circuit to an arbitrary input can be calculated using convolution integral of the input and impulse response.

$$w(t) = \int_{-\infty}^{\infty} w(\lambda)\delta(t-\lambda)d\lambda$$

$$y(t) = \int_{-\infty}^{\infty} w(\lambda)h_{\lambda}(t)d\lambda$$

$$y(t) = \int_{-\infty}^{\infty} w(\lambda)h(t-\lambda)d\lambda = w(t) * h(t)$$

$$h(t) = 0, t < 0 \Rightarrow y(t) = \int_{-\infty}^{t} w(\lambda)h(t-\lambda)d\lambda = w(t) * h(t)$$

$$h(t) = 0, w(t) = 0t < 0 \Rightarrow y(t) = \int_{0}^{t} w(\lambda)h(t-\lambda)d\lambda = w(t) * h(t)$$

Convolution Integral

- Commutativity: $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
- Associativity:
  $x_1(t) * (x_2(t) * x_3(t)) = (x_1(t) * x_2(t)) * x_3(t) = x_1(t) * x_2(t) * x_3(t)$
- Distributivity: $x_1(t) * (x_2(t) + x_3(t)) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$
- Scalar multiplication:
  $k(x_1(t) * x_2(t)) = (kx_1(t)) * x_2(t) = x_1(t) * (kx_2(t))$
- Multiplicative identity: $x(t) * \delta(t) = \delta(t) * x(t) = x(t)$
- Shift: $x(t) * \delta(t - t_0) = \delta(t - t_0) * x(t) = x(t - t_0)$
- Differentiation: $\frac{d}{dt}(x_1(t) * x_2(t)) = \frac{x_1(t)}{dt} * x_2(t) = x_1(t) * \frac{x_2(t)}{dt}$
- Doublet: $x(t) * \delta'(t) = \delta'(t) * x(t) = \frac{x(t)}{dt}$
- Ramp: $u(t) * u(t) = r(t)$

Example (Convolution calculation)

Convolution can be calculated graphically.

$t < 1 \Rightarrow y(t) = 0$

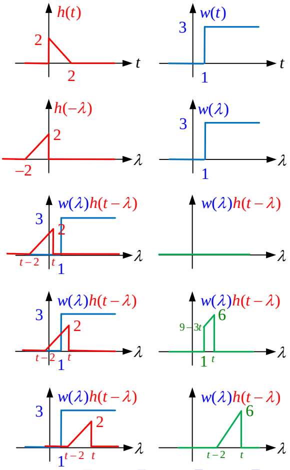
$\begin{cases} t \geq 1 \\ t-2 < 1 \end{cases} \Rightarrow y(t) = (t-1)\dfrac{6+9-3t}{2}$

$1 \leq t < 3 \Rightarrow y(t) = -1.5t^2 + 9t - 7.5$

$t - 2 \geq 1 \Rightarrow y(t) = \dfrac{6 \times 2}{2}$

$t \geq 3 \Rightarrow y = 6$

$$y(t) = w(t) * h(t) = \begin{cases} 0, & t < 1 \\ -1.5t^2 + 9t - 7.5, & 1 \leq t < 3 \\ 6, & t \geq 3 \end{cases}$$



Convolution Integral

Example (Convolution calculation)

Convolution can be calculated graphically.

$t-1<0 \Rightarrow y(t)=0$

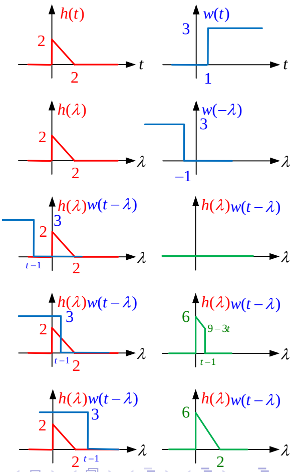
$$\begin{cases} t-1 \geq 0 \\ t-1 < 2 \end{cases} \Rightarrow y(t) = (t-1)\frac{6+9-3t}{2}$$

$1 \leq t < 3 \Rightarrow y(t) = -1.5t^2+9t-7.5$

$$t-1 \geq 2 \Rightarrow y(t) = \frac{6\times 2}{2}$$

$t \geq 3 \Rightarrow y = 6$

$$y(t) = h(t) * w(t) = \begin{cases} 0, & t<1 \\ -1.5t^2+9t-7.5, & 1 \leq t < 3 \\ 6, & t \geq 3 \end{cases}$$



Convolution Integral

Example (Convolution calculation)

Convolution can be calculated graphically.

$$t-1<1 \Rightarrow y(t)=0$$

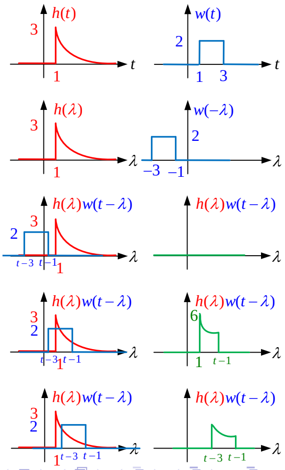
$$\begin{cases} t-1 \geq 1 \\ t-3<1 \end{cases} \Rightarrow y(t)=\int_{1}^{t-1} 2\times 3e^{-(\lambda-1)}d\lambda$$

$$2 \leq t < 4 \Rightarrow y(t)=6(1-e^{-(t-2)})$$

$$t-3 \geq 1 \Rightarrow y(t)=\int_{t-3}^{t-1} 2\times 3e^{-(\lambda-1)}d\lambda$$

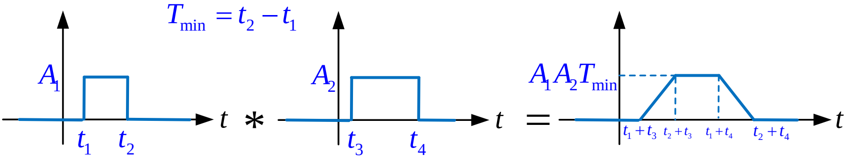
$$t \geq 4 \Rightarrow y(t)=6(e^{-(t-4)}-e^{-(t-2)})$$

$$y(t)=h(t)*w(t)=\begin{cases} 0, & t<2 \\ 6(1-e^{-(t-2)}), & 2\leq t<4 \\ 6(e^{-(t-4)}-e^{-(t-2)}), & t\geq 4 \end{cases}$$



Example (Graphical properties of the convolution)

Convolution softens and broadens the signals.



Convolution Integral

Example (Convolution calculation)

Convolution can be calculated analytically.

$$h(t) = 3e^{-2t}u(t), \quad w(t) = 2e^{-(t+2)}u(t+2) + 4\delta(t-4)$$

$$y(t) = w(t) * h(t) = 3e^{-2t}u(t) * \left[2e^{-(t+2)}u(t+2) + 4\delta(t-4)\right]$$

$$y(t) = 6(e^{-2t}u(t)) * (e^{-(t+2)}u(t+2)) + 12e^{-2(t-4)}u(t-4) = y_1(t) + 12e^{-2(t-4)}u(t-4)$$

$$y_1(t) = 6\int_{-\infty}^{\infty} e^{-(\lambda+2)}u(\lambda+2)e^{-2(t-\lambda)}u(t-\lambda)d\lambda = 6\int_{-2}^{\infty} e^{-(\lambda+2)}e^{-2(t-\lambda)}u(t-\lambda)d\lambda$$

$$y_1(t) = 6e^{-2(t+1)}\int_{-2}^{\infty} e^{\lambda}u(t-\lambda)d\lambda$$

$$t < -2 \Rightarrow y_1(t) = 6e^{-2(t+1)}\int_{-2}^{\infty} e^{\lambda}0d\lambda = 0$$

$$t > -2 \Rightarrow y_1(t) = 6e^{-2(t+1)}\int_{-2}^{t} e^{\lambda}d\lambda = 6e^{-2(t+1)}(e^{t} - e^{-2}) = 6(e^{-(t+2)} - e^{-2(t+2)})$$

$$y(t) = 6(e^{-(t+2)} - e^{-2(t+2)})u(t+2) + 12e^{-2(t-4)}u(t-4)$$

$$y(t) = \begin{cases} 0, & t < -2 \\ 6(e^{-(t+2)} - e^{-2(t+2)}), & -2 \le t < 4 \\ 6(e^{-(t+2)} - e^{-2(t+2)}) + 12e^{-2(t-4)}, & t \ge 4 \end{cases}$$

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