Question 1

Consider the series RC circuit shown in Fig. 1.

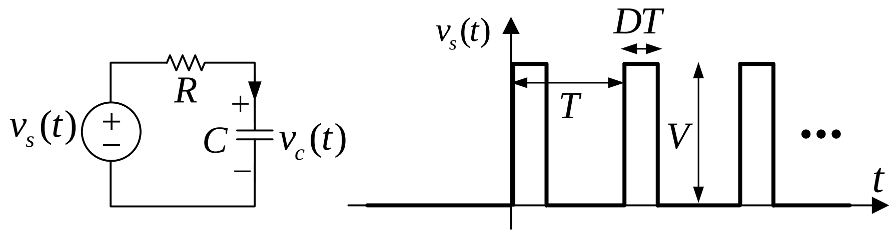

Figure 1: A series RC circuit.

(a) Express the signal $v_s(t)$ in terms of elementary signals. Note that D denotes the duty cycle and $v_s(t) = 0$ for $t < 0$.

$$v_s(t) = \sum_{k=0}^{\infty} V(u(t-kT) - u(t-DT-kT)) = \sum_{k=0}^{\infty} V \sqcap (\frac{t - \frac{DT}{2} - kT}{DT})$$

(b) Find the zero-state response of the capacitor voltage $v_c(t)$ for all time.

At first, we find the circuit response $v_{c1}(t)$ to $v_{s1}(t) = V(u(t)-tu(t-DT))$. For $0 < t < DT$,

$$v_{c1}(t) = v_c(\infty) + (v_c(0^+) - v_c(\infty))e^{-\frac{t}{\tau}} = V(1-e^{-\frac{t}{\tau}}), \tau = RC$$

Note that $v_c(0^+) = v_C(0^-) = 0$ due to continuity of the capacitor voltage and zero-state condition. Further, $v_c(\infty) = V$ since the DC-driven capacitor becomes open circuit at steady state situation. Similarly, for $t > DT$,

$$\begin{cases} v_c(DT^+) = v_c(DT^-) = V(1-e^{-\frac{DT}{\tau}}) \\ v_c(\infty) = 0 \\ v_{c1}(t) = v_c(\infty) + (v_c(0^+) - v_c(\infty))e^{-\frac{t-DT}{\tau}} = V(1-e^{-\frac{DT}{\tau}})e^{-\frac{t-DT}{\tau}}, \tau = RC \end{cases}$$

Overall,

$$v_{c1}(t) = \begin{cases} V(1-e^{-\frac{t}{\tau}}), & 0 < t < DT \\ V(1-e^{-\frac{DT}{\tau}})e^{-\frac{t-DT}{\tau}}, & t > DT \end{cases}$$

$$v_{c1}(t) = V(1-e^{-\frac{t}{\tau}})(u(t) - u(t-DT)) + V(1-e^{-\frac{DT}{\tau}})e^{-\frac{t-DT}{\tau}}u(t-DT)$$

$$v_{c1}(t) = V(1-e^{-\frac{t}{\tau}})u(t) - V(1-e^{-\frac{t-DT}{\tau}})u(t-DT)$$

Since the circuit is LTI and

$$v_s(t) = \sum_{k=0}^{\infty} V(u(t-kT) - tu(t-DT-kT)) = \sum_{k=0}^{\infty} v_{s1}(t-kT)$$

$$v_c(t) = \sum_{k=0}^{\infty} v_{c1}(t-kT) = \sum_{k=0}^{\infty} v_{c1}(t-kT)$$

Question 1 continued on next page...

$$v_c(t) = \sum_{k=0}^{\infty} \left[V(1 - e^{-\frac{t-kT}{\tau}})u(t - kT) - V(1 - e^{-\frac{t-kT-DT}{\tau}})u(t - kT - DT)\right]$$

(c) Find the complete response of the capacitor voltage for $t > 0$ if the capacitor initial voltage is $v_c(0^-) = V_0$.

Since the circuit is linear, the compelete response is the sum of zero-state and zero-input responses. Thus,

$$v_c(t) = V_0 e^{-\frac{t}{\tau}} + \sum_{k=0}^{\infty} \left[V(1 - e^{-\frac{t-kT}{\tau}})u(t - kT) - V(1 - e^{-\frac{t-kT-DT}{\tau}})u(t - kT - DT)\right], t > 0$$

Question 2

Find an expression for $v_c(t)$ in Fig. 1 valid for $t > 0$. Further, calculate the steady state voltage of the capacitor.

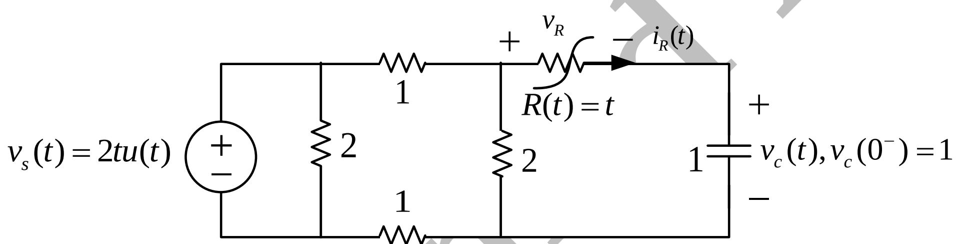

Figure 2: A first-order circuit.

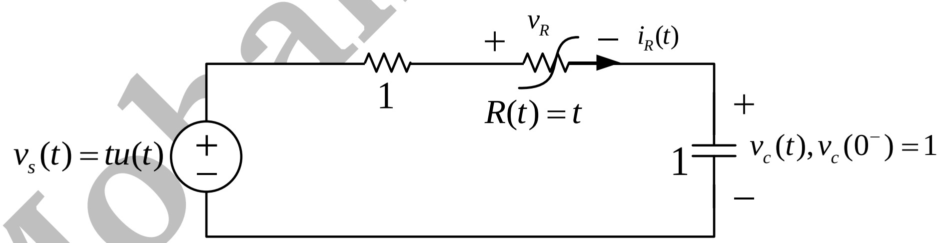

Figure 3: The equivalent circuit for Fig. 2.

The left part of the circuit is linear and can be replaced with its Thevenin equivalent as shown in Fig. 3. Now,

$$i_c(t) = \frac{dv_c}{dt} \Rightarrow i_c(t) + ti_c(t) + v_c(t) = (1 + t)\frac{dv_c}{dt} + v_c(t) = v_s(t) = tu(t), v_c(0^-) = 1$$

Question 2 continued on next page. . .

The capacitor voltage remains continuous at $t = 0$ since if the capacitor voltage jumps, its current experiences an impulse and there is no voltage source to supply the voltage impulse created on the 1-ohm resistor. So,

$$\frac{dv_c}{dt} + \frac{1}{1+t}v_c(t) = \frac{t}{t+1}, t > 0, v_c(0^+) = 1$$

This is a first-order differential equation with the solution

$$v_c(t) = \frac{t^2}{2\,(t+1)} + \frac{K}{t+1}, t > 0, v_c(0^+) = 1$$

Applying the initial condition,

$$v_c(t) = \frac{t^2}{2\,(t+1)} + \frac{1}{t+1}, t > 0$$

Since $v_c(t)$ grows unbounded as time proceeds, there is no steady state response for the capacitor voltage.

Question 3

Find the differential equation governing the response $v_3(t)$ in the circuit below if the initial conditions are $i_L(0^-)$, $v_1(0^-)$, $v_2(0^-)$, and $v_3(0^-)$.

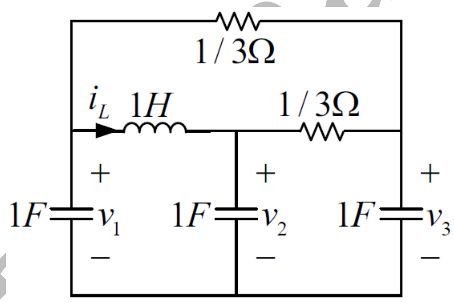

Figure 4: An LTI circuit.

Question 3 continued on next page...

Using the D operator,

$$\begin{cases} Dv_1 + D^{-1}(v_1 - v_2) + i_L(0^-) + 3(v_1 - v_3) = 0 \\ Dv_2 + D^{-1}(v_2 - v_1) - i_L(0^-) + 3(v_2 - v_3) = 0 \\ Dv_3 + 3(v_3 - v_1) + 3(v_3 - v_2) = 0 \end{cases}$$

$$\begin{bmatrix} D+3+D^{-1} & -D^{-1} & -3 \\ -D^{-1} & D+3+D^{-1} & -3 \\ -3 & -3 & D+6 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = \begin{bmatrix} -i_L(0^-) \\ i_L(0^-) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} D^2+3D+1 & -1 & -3 \\ -1 & D^2+3D+1 & -3 \\ -3D & -3D & D+6 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ v_3(t) \end{bmatrix} = \begin{bmatrix} -i_L(0^-) \\ i_L(0^-) \\ 0 \end{bmatrix}$$

Note that $v_1(t) = D\phi_1(t)$ and $v_2(t) = D\phi_2(t)$. Now, we should solve the matrix for $v_s(t)$. For example, we can use the elementary operations to make the matrix equation upper-triangle.

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ D^2+3D+1 & -1 & -3 & -i_L(0^-) \\ -3D & -3D & D+6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -1+(D^2+3D+1)(D^2+3D+1) & -3-3(D^2+3D+1) & \substack{-i_L(0^-)+ \\ (D^2+3D+1)i_L(0^-)} \\ 0 & -3D-3D(D^2+3D+1) & D+6-3D(-3) & 0-3Di_L(0^-) \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & D^4+6D^3+11D^2+6D & -3D^2-9D-6 & 0 \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \\ 0 & 0 & -3D^2-9D-6+(\frac{1}{3}D+1)(10D+6) & 0+(\frac{1}{3}D+1)0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \\ 0 & 0 & \frac{1}{3}D^2+3D & 0 \end{bmatrix}$$

$$(\frac{1}{3}D^2+3D)v_3(t) = 0 \Rightarrow v_3''(t) + 9v_3'(t) = 0$$

Note the the differential equation $(\frac{1}{3}D^3+3D^2)v_3(t) = 0$ or equivalently, $\frac{d^3v_3(t)}{dt^3} + 9\frac{d^2v_3(t)}{dt^2} = 0$ also describes $v_3(t)$ although its order is unnecessarily high by one.

Question 4

Calculate the current through the $2\ \Omega$ resistor in Fig. 5.

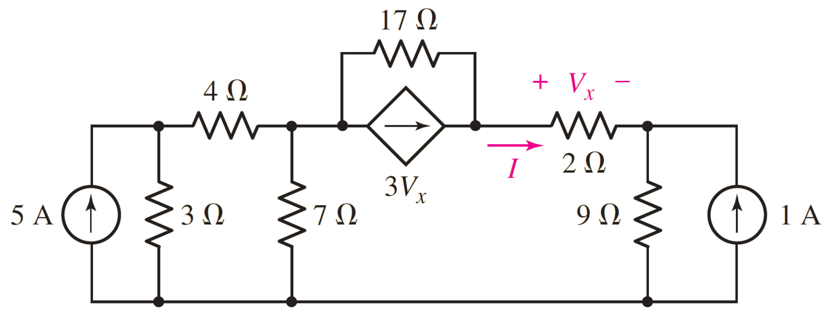

Figure 5: A resistive circuit.

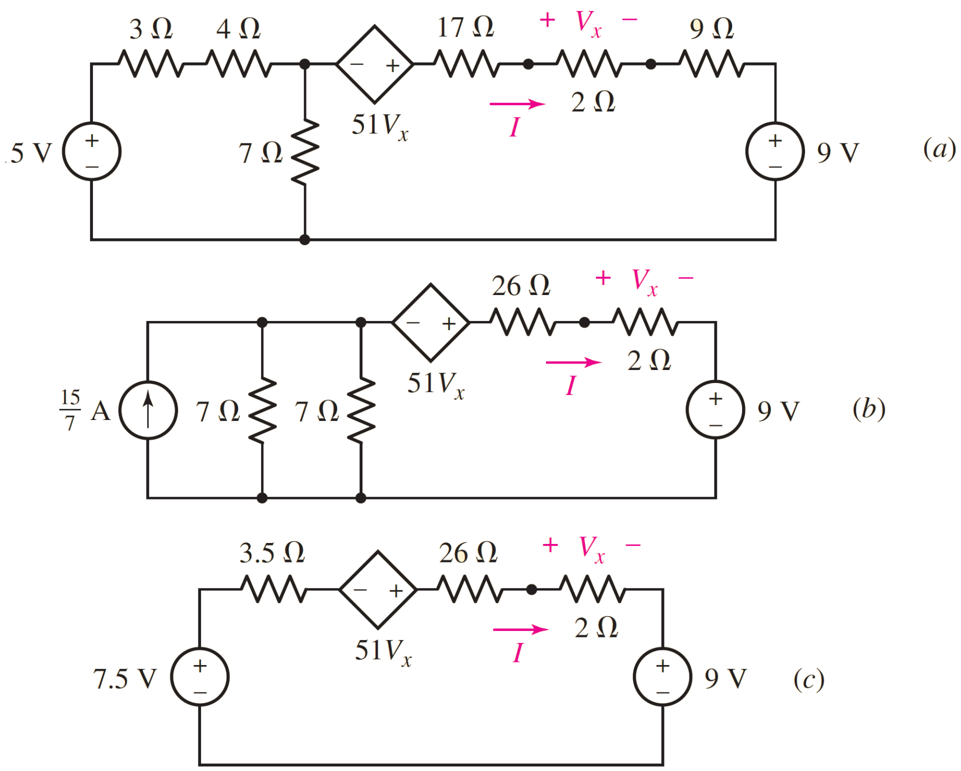

Figure 6: Source transformation for the resistive circuit of Fig. 5.

We begin by transforming each current source into a voltage source as Fig. 6(a), the strategy being to convert the circuit into a simple loop. We must be careful to retain the $2\ \Omega$ resistor for two reasons: first, the dependent source controlling variable appears across it, and second, we desire the current flowing through it. However, we can combine the $17\ \Omega$ and $9\ \Omega$ resistors, since they appear in series. We also see that the $3\ \Omega$ and $4\ \Omega$ resistors may be combined into a single $7\ \Omega$ resistor, which can then be used to transform the 15 V source into a $15/7$ A source as in Fig. 6(b). Finally, we note that the two $7\ \Omega$ resistors can be combined into a single $3.5\ \Omega$ resistor, which may be used to transform the $15/7$ A current source into a 7.5 V voltage source. The result is a simple loop circuit, shown in Fig.

Question 4 continued on next page...

6(c). Now, the desired current I can now be found using KVL

$$-7.5 + 3.5I - 51V_x + 28I + 9 = 0, V_x = 2I \Rightarrow I = 21.28 \text{ mA}$$