Question 1

Consider the circuit shown in Fig. 1, where $i_s(t)$ is a positive-value bounded signal and diode is ideal.

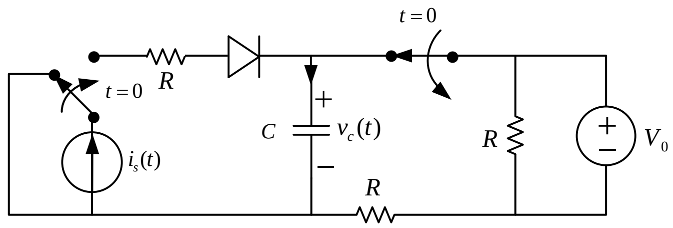

Figure 1: A circuit with a capacitor.

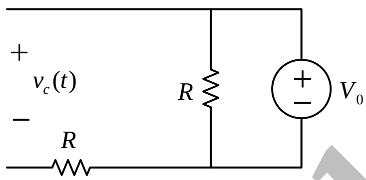

Figure 2: The circuit of Fig. 1 for $t < 0$.

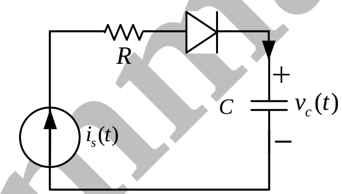

Figure 3: The circuit of Fig. 1 for $t > 0$.

(a) Find an expression for the capacitor voltage $v_c(t)$ valid for all time t.

For $t < 0$, the capacitor is fully charged by the DC voltage source and gets the voltage $v_c(t) = V_0, t < 0$, as shown in Fig. 2. Clearly, $v_c(0^-) = V_0$. The circuit for $t > 0$ is shown in Fig. 3. A discontinuity in the capacitor voltage at $t = 0$ leads to an impulse in the capacitor current and cannot be supplied by the bounded signal $i_s(t)$. So, $v_c(0^+) = v_c(0^-) = V_0$. Now, for $t > 0$,

$$v_c(t) = v_c(0+) + \frac{1}{C}\int_{0^+}^{t} i_s(\lambda)d\lambda = V_0 + \frac{1}{C}\int_{0}^{t} i_s(\lambda)d\lambda$$

Note that the diode acts like a short circuit as long as $i_s(t) > 0$.

(b) Find the steady state response of $v_c(t)$ when $i_s(t) = 1$.

We have,

$$v_c(t) = V_0 + \frac{1}{C}\int_{0^+}^{t} i_s(\lambda)d\lambda = V_0 + \frac{1}{C}\int_{0^+}^{t} d\lambda = V_0 + \frac{t}{C}$$

Clearly, $v_c(t)$ has no steady state response since $v_c(\infty) = \infty$.

(c) Find the steady state response of $v_c(t)$ when $i_s(t) = e^{-t}$.

We have,

$$v_c(t) = V_0 + \frac{1}{C}\int_{0^+}^{t} i_s(\lambda)d\lambda = V_0 + \frac{1}{C}\int_{0^+}^{t} e^{-\lambda}d\lambda = V_0 + \frac{1-e^{-t}}{C}$$

Clearly, $v_c(t)$ approaches the steady state constant value of $V_0 + \frac{1}{C}$ as time proceeds.

Question 2

Find an expression for $i_C(t), t > 1$ in Fig. 4, where

1. **The resistor is LTV with the resistance $R(t) = t$.**
2. **The inductor is LTV with the inductance $L(t) = t^2$.**
3. **The capacitor is LTI with the capacitance $C = 1$.**
4. **The voltage source is $v_s(t) = t, t > 1$.**
5. **The initial conditions are $v_C(1^+) = 0$ and $i_L(1^+) = 0$.**

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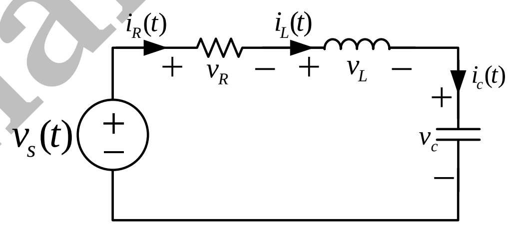

Figure 4: A second-order circuit.

Writing a KVL for the circuit loop,

$$v_R(t)+v_L(t)+v_C(t) = R(t)i_R(t)+L(t)\frac{di_L(t)}{dt}+i_L(t)\frac{dL(t)}{dt}+v_C(1^+)+\frac{1}{C}\int_{1^+}^{t} i_C(\lambda)d\lambda = v_s(t)$$

Question 2 continued on next page...

Since $i_R(t) = i_L(t) = i_C(t)$,

$$ti_C(t) + t^2\frac{di_C(t)}{dt} + 2ti_C(t) + v_C(1^+) + \int_{1^+}^{t} i_C(\lambda)d\lambda = t, t > 1$$

Taking the time derivative of the equation,

$$i_C(t) + t\frac{di_C(t)}{dt} + 2t\frac{di_C(t)}{dt} + t^2\frac{d^2i_C(t)}{dt^2} + 2i_C(t) + 2t\frac{di_C(t)}{dt} + i_C(t) = 1, t > 1$$

$$t^2\frac{d^2i_C(t)}{dt^2} + 5t\frac{di_C(t)}{dt} + 4i_C(t) = 1, t > 1$$

This is a non-homogeneous Euler differential equation with the characteristic equation

$$m^2 + (5-1)m + 4 = 0 \Rightarrow m = -2, -2$$

The solution is of the form

$$i_C(t) = K_1t^{-2} + K_2t^{-2}\ln(t) + \frac{1}{4}, t > 1$$

Initial conditions give $i_C(1^+) = i_L(1^+) = 0$ and

$$i_C(1^+) + \frac{di_C(t)}{dt}|_{t=1^+} + 2i_C(1^+) + v_C(1^+) + \int_{1^+}^{1^+} i_C(\lambda)d\lambda = 1 \Rightarrow \frac{di_C(t)}{dt}|_{t=1^+} = 1$$

So,

$$i_C(1^+) = K_1 + \frac{1}{4} = 0 \Rightarrow K_1 = -\frac{1}{4} \Rightarrow i_C(t) = -\frac{1}{4}t^{-2} + K_2t^{-2}\ln(t) + \frac{1}{4},$$

$$i'_C(t) = \frac{1}{2}t^{-3} - 2K_2t^{-3}\ln(t) + K_2t^{-2}\frac{1}{t} \Rightarrow i'_C(1^+) = K_2 + \frac{1}{2} = 1 \Rightarrow K_2 = \frac{1}{2}$$

Finally,

$$i_C(t) = -\frac{1}{4}t^{-2} + \frac{1}{2}t^{-2}\ln(t) + \frac{1}{4}, t > 1$$