Second-order Circuits

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Overview

Second-order Circuits

Circuit Types

Statement (Linear Circuit)

A linear circuit is a circuit that includes linear elements and/or independent sources.

Statement (LTI Circuit)

An LTI circuit is a circuit that includes LTI elements and/or independent sources.

Statement (Second-order Circuit)

A second-order circuit is a circuit that has two independent energy-storage elements.

Statement (Second-order LTI Circuit)

A second-order LTI circuit is an LTI circuit that has two independent energy-storage elements.

- Capacitors and inductors are energy-storage elements.

Circuit Analysis

Definition (Circuit Inputs)

Independent sources are called circuit inputs.

Definition (Circuit Initial Conditions)

The initial voltage of the capacitors and initial currents of the inductors are referred to as circuit initial conditions.

Definition (Circuit Response)

The circuit response is the voltage or current of a desired element of the circuit.

Circuit Responses

Definition (Zero-input response)

Zero-input response is defined as the response of a circuit when its inputs are identically zero.

Definition (Zero-state response)

Zero-state response is defined as the the response of a circuit when its initial conditions are zero.

Definition (Complete response)

Complete response is defined as the response of a circuit to both inputs and initial states.

Circuit Responses

Definition (Impulse Response)

The zero-state response of the LTI circuit to the impulse input is called impulse response.

Definition (Step Response)

The zero-state response of the LTI circuit to the unit step input is called step response.

Circuit Responses

Definition (Transient Response)

Transient response is the part of the circuit response that damps as time proceeds.

Definition (Steady-state response)

Steady-state response is the part of the circuit response that remains as time proceeds.

Statement (Zero-input response)

In a linear circuit, the zero-input response is a linear function of the initial conditions.

Statement (Zero-state response)

In a linear circuit, the zero-state response is a linear function of the inputs.

Statement (Complete response)

In a linear circuit, the complete response is the sum of zero-input and zero-state responses.

- Homogeneity property of a linear function:
  $i_s(t) \to y(t) \Rightarrow Ki_s(t) \to Ky(t)$
- Additivity property of a linear function:
  $i_{s1}(t) \to y_1(t), i_{s2}(t) \to y_2(t) \Rightarrow i_{s1}(t) + i_{s2}(t) \to y_1(t) + y_2(t)$

LTI Circuits

Statement (Input Shift Property)

In an LTI circuit, the zero-state response to a shifted input experiences the same shift.

- Shift property: $i_s(t), y(0^-) = 0 \to y(t), t > 0 \Rightarrow i_s(t - t_0), y(t_0^-) = 0 \to y(t - t_0), t > t_0$

Statement (Input Derivative Property)

In an LTI circuit, the zero-state response to the derivative of an input equals the derivative of the zero-state response to the input.

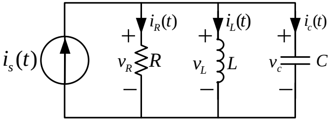
- Derivative property:
$i_s(t), y(0^-) = 0 \to y(t), t > 0 \Rightarrow \frac{di_s(t)}{dt}, y(0^-) = 0 \to \frac{dy(t)}{dt}, t > 0$

Second-order Parallel LTI RLC Circuit

Parallel RLC Circuit

Example (Second-order parallel LTI RLC circuit)

A second-order parallel LTI RLC circuit is described by a constant-coefficient second-order linear differential equation.



$$i_s(t) = i_R(t) + i_L(t) + i_C(t), \quad v_R(t) = v_L(t) = v_C(t)$$

$$i_s(t) = \frac{v_C(t)}{R} + i_L(0) + \frac{1}{L}\int_0^t v_C(\lambda)d\lambda + C\frac{dv_C(t)}{dt}$$

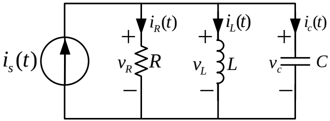
$$\Rightarrow \frac{d^2v_C(t)}{dt^2} + \frac{1}{RC}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{C}\frac{di_s(t)}{dt}$$

$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2v_C(t)}{dt^2} + 2\alpha\frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = \frac{1}{C}\frac{di_s(t)}{dt}$$

$$v_C(0^+), \quad v_C'(0^+) = \frac{i_C(0^+)}{C} = \frac{i_s(0^+) - i_L(0^+) - \frac{v_C(0^+)}{R}}{C}$$

Example (Second-order parallel LTI RLC circuit (cont.))

A second-order parallel LTI RLC circuit is described by a constant-coefficient second-order linear differential equation.



$$i_s(t) = i_R(t) + i_L(t) + i_C(t), \quad v_R(t) = v_L(t) = v_C(t)$$

$$i_s(t) = \frac{v_L(t)}{R} + i_L(t) + C\frac{dv_L(t)}{dt}, v_L(t) = L\frac{di_L(t)}{dt}$$

$$\Rightarrow \frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{1}{LC}i_s(t)$$

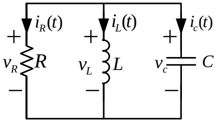
$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 i_L(t)}{dt^2} + 2\alpha\frac{di_L(t)}{dt} + \omega_0^2 i_L(t) = \frac{1}{LC}i_s(t)$$

$$i_L(0^+), \quad i_L'(0^+) = \frac{v_L(0^+)}{L} = \frac{v_C(0^+)}{L}$$

Parallel RLC Circuit

Example (Zero-input response of LTI RLC circuit)

Depending on the roots of the characteristic equation, zero-input response of an LTI RLC circuit takes various forms.



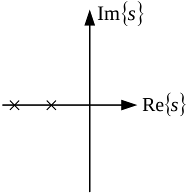
$$\frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = 0, t > 0; \quad v_C(0^+) = V_0, v_C'(0^+) = -\frac{I_0 + V_0/R}{C}$$

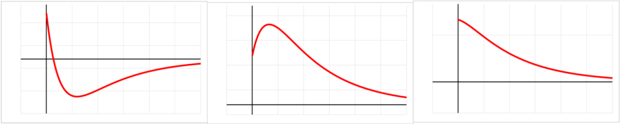
$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC}, Q = \frac{\omega_0}{2\alpha}, \quad s^2 + 2\alpha s + \omega_0^2 = 0$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Example (Zero-input response of LTI RLC circuit (cont.))

If $\alpha > 0$, $\omega_0 > 0$, and $\alpha > \omega_0 \equiv Q < 0.5$, the zero-input response of an LTI RLC circuit has overdamped form.

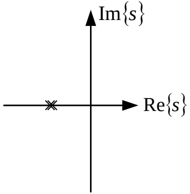




$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}, \quad s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \in \mathbb{R}$$

Example (Zero-input response of LTI RLC circuit (cont.))

If $\alpha > 0$, $\omega_0 > 0$, and $\alpha = \omega_0 \equiv Q = 0.5$, the zero-input response of an LTI RLC circuit has critically-damped form.

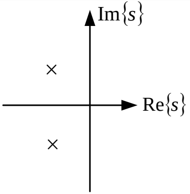


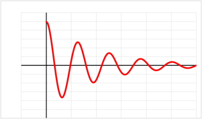
$$v_C(t) = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t}, \quad s_1, s_2 = -\alpha \in \mathbb{R}$$

Parallel RLC Circuit

Example (Zero-input response of LTI RLC circuit (cont.))

If $\alpha > 0$, $\omega_0 > 0$, and $\alpha < \omega_0 \equiv Q > 0.5$, the zero-input response of an LTI RLC circuit has underdamped form.



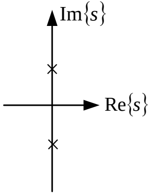


$$v_C(t) = K_1 e^{-\alpha t}\cos(\omega_d t) + K_2 e^{-\alpha t}\sin(\omega_d t), \quad s_1, s_2 = -\alpha \pm j\omega_d \in \mathbb{C}, \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$v_C(t) = K_3 e^{-\alpha t}\cos(\omega_d t + K_4), \quad s_1, s_2 = -\alpha \pm j\omega_d \in \mathbb{C}, \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Example (Zero-input response of LTI RLC circuit (cont.))

If $\alpha = 0$, $\omega_0 > 0$, or equivalently $Q = +\infty$, the zero-input response of an LTI RLC circuit has loss-less form.

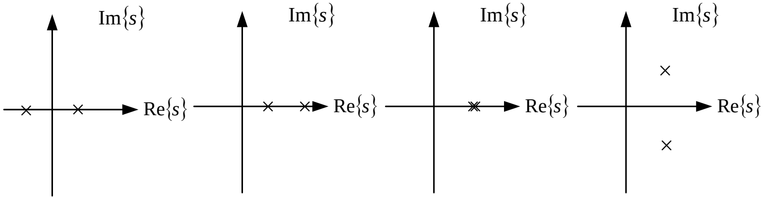


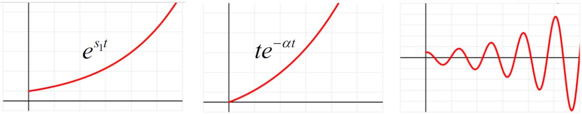
$$v_C(t) = K_1 \cos(\omega_0 t) + K_2 \sin(\omega_0 t), \quad s_1, s_2 = \pm j\omega_0 \in \mathbb{C}$$
$$v_C(t) = K_3 \cos(\omega_0 t + K_4), \quad s_1, s_2 = \pm j\omega_0 \in \mathbb{C},$$

Parallel RLC Circuit

Example (Zero-input response of LTI RLC circuit (cont.))

If $\alpha < 0$, $\omega_0 > 0$ or $\omega_0^2 < 0$, the zero-input response of an LTI RLC circuit has unstable form.



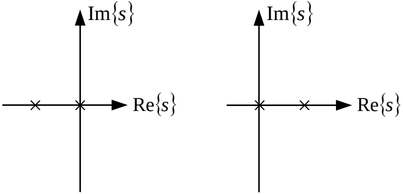


$$Ke^{s_1 t}, Kte^{-\alpha t}, Ke^{-\alpha t}\cos(\omega_d t + \theta)$$

Parallel RLC Circuit

Example (Zero-input response of LTI RLC circuit (cont.))

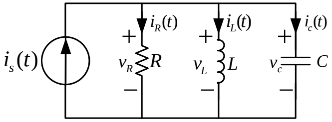
If $\omega_0 = 0$, the zero-input response of an LTI RLC circuit has a constant part and can be stable or unstable.

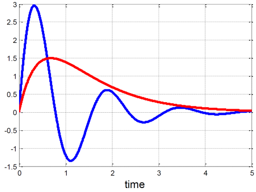


$K, Ke^{s_1 t}$

Example (Step response for parallel LTI RLC circuit)

Different step responses in a second-order parallel LTI RLC circuit may have different steady state value.





$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = \frac{1}{C}\frac{di_s(t)}{dt} = \frac{1}{C}\frac{du(t)}{dt} = \frac{1}{C}\delta(t)$$

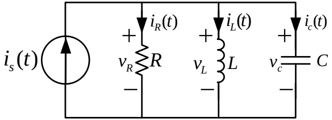
$$v_C(0^+) = v_C(0^-) = 0, \quad v_C'(0^+) = \frac{i_s(0^+) - i_L(0^+) - \frac{v_C(0^+)}{R}}{C} = \frac{1 - i_L(0^-) - \frac{v_C(0^-)}{R}}{C} = \frac{1}{C}$$

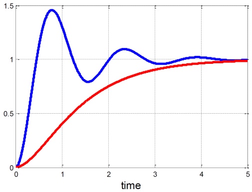
$$\frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = 0, t > 0; \quad v_C(0^+) = 0, v_C'(0^+) = \frac{1}{C}$$

$$\alpha, \omega_0 > 0, \alpha < \omega_0 \Rightarrow s_v(t) = (K_1 e^{-\alpha t}\cos(\omega_d t) + K_2 e^{-\alpha t}\sin(\omega_d t))u(t) = \frac{e^{-\alpha t}}{\omega_d C}\sin(\omega_d t)u(t)$$

Example (Step response for parallel LTI RLC circuit (cont.))

Different step responses in a second-order parallel LTI RLC circuit may have different steady state value.





$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 i_L(t)}{dt^2} + 2\alpha \frac{di_L(t)}{dt} + \omega_0^2 i_L(t) = \frac{1}{LC} i_s(t) = \frac{1}{LC} u(t)$$

$$i_L(0^+) = i_L(0^-) = 0, \quad i_L'(0^+) = \frac{v_C(0^+)}{L} = \frac{v_C(0^-)}{L} = 0$$

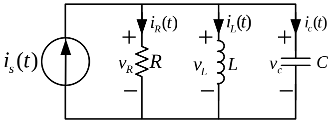
$$\frac{d^2 i_L(t)}{dt^2} + 2\alpha \frac{di_L(t)}{dt} + \omega_0^2 i_L(t) = \frac{1}{LC}, t > 0; \quad i_L(0^+) = 0, i_L'(0^+) = 0$$

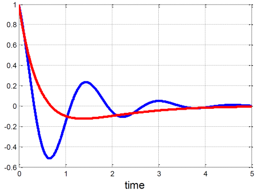
$$\alpha, \omega_0 > 0, \alpha > \omega_0 \Rightarrow s_i(t) = (K_1 e^{s_1 t} + K_2 e^{s_2 t} + 1)u(t) = (\frac{s_2}{s_1 - s_2} e^{s_1 t} - \frac{s_1}{s_1 - s_2} e^{s_2 t} + 1)u(t)$$

Parallel RLC Circuit

Example (Impulse response for parallel LTI RLC circuit)

Different impulse responses in a second-order parallel LTI RLC circuit may or may not experience discontinuity.





$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = \frac{1}{C}\frac{di_s(t)}{dt} = \frac{1}{C}\frac{d\delta(t)}{dt} = \frac{1}{C}\delta'(t)$$

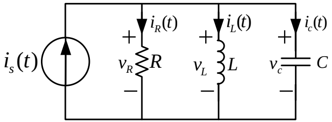
$$v_C(0^-) = i_L(0^-) = 0 \Rightarrow v_C(0^+) = \frac{1}{C}, v_C'(0^+) = \frac{-1}{RC^2}$$

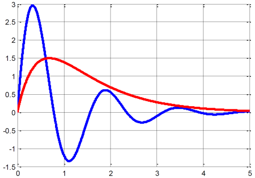
$$\frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = 0, t > 0; \quad v_C(0^+) = \frac{1}{C}, v_C'(0^+) = \frac{-1}{RC^2}$$

$$\alpha, \omega_0 > 0, \alpha < \omega_0 \Rightarrow h_v(t) = \frac{e^{-\alpha t}}{\omega_d C}[\omega_d \cos(\omega_d t) - \alpha \sin(\omega_d t)]u(t) = \frac{ds_v(t)}{dt}$$

Example (Impulse response for parallel LTI RLC circuit (cont.))

Different impulse responses in a second-order parallel LTI RLC circuit may or may not experience discontinuity.





$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 i_L(t)}{dt^2} + 2\alpha \frac{di_L(t)}{dt} + \omega_0^2 i_L(t) = \frac{1}{LC} i_s(t) = \frac{1}{LC}\delta(t)$$

$$v_C(0^-) = i_L(0^-) = 0 \Rightarrow i_L(0^+) = 0, \quad i_L'(0^+) = \frac{1}{LC}$$

$$\frac{d^2 i_L(t)}{dt^2} + 2\alpha \frac{di_L(t)}{dt} + \omega_0^2 i_L(t) = 0, t > 0; \quad i_L(0^+) = 0, i_L'(0^+) = \frac{1}{LC}$$

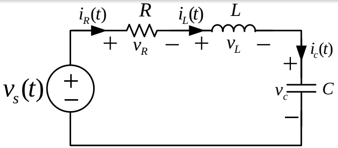
$$\alpha, \omega_0 > 0, \alpha > \omega_0 \Rightarrow h_i(t) = (\frac{s_1 s_2}{s_1 - s_2} e^{s_1 t} - \frac{s_1 s_2}{s_1 - s_2} e^{s_2 t}) u(t) = \frac{ds_i(t)}{dt}$$

Second-order Series LTI RLC Circuit

Series RLC Circuit

Example (Second-order series LTI RLC circuit)

A second-order series LTI RLC circuit is described by a constant-coefficient second-order linear differential equation.



$$v_s(t) = v_R(t) + v_L(t) + v_C(t), \quad i_R(t) = i_L(t) = i_C(t)$$

$$v_s(t) = Ri_C(t) + L\frac{di_C(t)}{dt} + v_C(0) + \frac{1}{C}\int_0^t i_C(\lambda)d\lambda$$

$$\Rightarrow \frac{d^2 i_C(t)}{dt^2} + \frac{R}{L}\frac{di_C(t)}{dt} + \frac{1}{LC}i_C(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

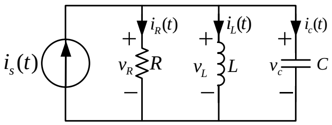
$$2\alpha = \frac{R}{L}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 i_C(t)}{dt^2} + 2\alpha\frac{di_C(t)}{dt} + \omega_0^2 i_C(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

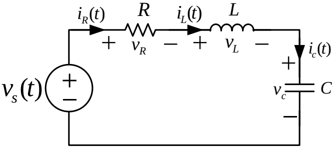
$$i_C(0^+) = i_L(0^+), \quad i_C'(0^+) = i_L'(0^+) = \frac{v_L(0^+)}{L} = \frac{v_s(0^+) - v_C(0^+) - Ri_L(0^+)}{L}$$

Duality

Example (Duality)

Duality can facilitate circuit analysis.





$$2\alpha = \frac{R}{L}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 i_C(t)}{dt^2} + 2\alpha \frac{di_C(t)}{dt} + \omega_0^2 i_C(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

$$2\alpha = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC} \Rightarrow \frac{d^2 v_C(t)}{dt^2} + 2\alpha \frac{dv_C(t)}{dt} + \omega_0^2 v_C(t) = \frac{1}{C}\frac{di_s(t)}{dt}$$

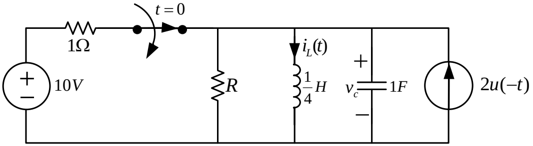
Quantity	i	v	i_s	v_s	ϕ	q	R	G	L	C
Dual	v	i	v_s	i_s	q	ϕ	G	R	C	L

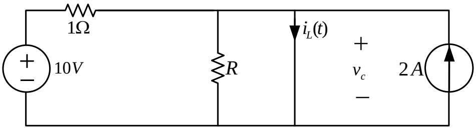
Second-order LTI Circuit

LTI RLC Circuit

Example (Second-order LTI RLC circuit with two switches)

An LTI RLC circuit may be simplified to a simple series or parallel LTI RLC circuit.



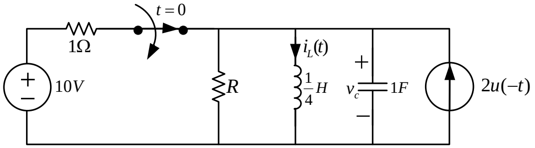


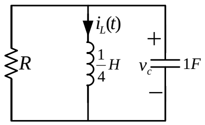
$$v_c(t) = 0, t < 0, i_L(t) = \frac{10}{1} + 2 = 12, t < 0 \Rightarrow v_c(0^-) = 0, i_L(0^-) = 12$$

LTI RLC Circuit

Example (Second-order LTI RLC circuit with two switches (cont.))

An LTI RLC circuit may be simplified to a simple series or parallel LTI RLC circuit.





$$v_c(0^+) = v_c(0^-) = 0$$

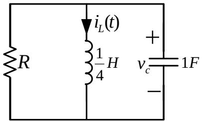
$$v_c'(0^+) = \frac{i_c(0^+)}{C} = \frac{-i_L(0^+) - \frac{v_R(0^+)}{R}}{1} = -i_L(0^+) - \frac{v_c(0^+)}{R} = -12$$

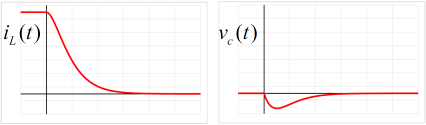
$$i_L(0^+) = i_L(0^-) = 12$$

$$i_L'(0^+) = \frac{v_L(0^+)}{L} = \frac{v_c(0^+)}{0.25} = 0$$

Example (Second-order LTI RLC circuit with two switches (cont.))

An LTI RLC circuit may be simplified to a simple series or parallel LTI RLC circuit.





$$R = \frac{1}{5} \Rightarrow 2\alpha = \frac{1}{RC} = 5, \omega_0^2 = \frac{1}{LC} = 4, Q = \frac{\omega_0}{2\alpha} = 0.4 < 0.5$$

$$\frac{d^2 i_L(t)}{dt^2} + 5\frac{di_L(t)}{dt} + 4i_L(t) = 0, t > 0, i_L(0^+) = 12, i_L'(0^+) = 0 \Rightarrow s^2 + 5s + 4 = 0$$

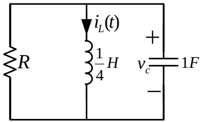
$$\Rightarrow s_1, s_2 = -4, -1 \Rightarrow i_L(t) = k_1 e^{-t} + k_2 e^{-4t} \Rightarrow i_L(t) = 16e^{-t} - 4e^{-4t}, t > 0$$

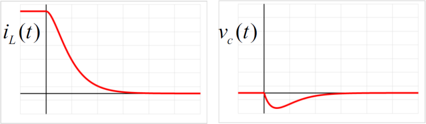
$$\frac{d^2 v_c(t)}{dt^2} + 5\frac{dv_c(t)}{dt} + 4v_c(t) = 0, t > 0, v_c(0^+) = 0, v_c'(0^+) = -12 \Rightarrow s^2 + 5s + 4 = 0$$

$$\Rightarrow s_1, s_2 = -4, -1 \Rightarrow v_c(t) = k_1 e^{-t} + k_2 e^{-4t} \Rightarrow v_c(t) = -4e^{-t} + 4e^{-4t}, t > 0$$

Example (Second-order LTI RLC circuit with two switches (cont.))

An LTI RLC circuit may be simplified to a simple series or parallel LTI RLC circuit.





$$R = \frac{1}{4} \Rightarrow 2\alpha = \frac{1}{RC} = 4, \omega_0^2 = \frac{1}{LC} = 4, Q = \frac{\omega_0}{2\alpha} = 0.5 = 0.5$$

$$\frac{d^2 i_L(t)}{dt^2} + 4\frac{di_L(t)}{dt} + 4i_L(t) = 0, t > 0, i_L(0^+) = 12, i_L'(0^+) = 0 \Rightarrow s^2 + 4s + 4 = 0$$

$$\Rightarrow s_1, s_2 = -2, -2 \Rightarrow i_L(t) = k_1 e^{-2t} + k_2 te^{-2t} \Rightarrow i_L(t) = 12e^{-2t} + 24te^{-2t}, t > 0$$

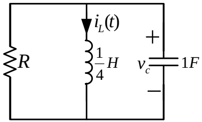
$$\frac{d^2 v_c(t)}{dt^2} + 4\frac{dv_c(t)}{dt} + 4v_c(t) = 0, t > 0, v_c(0^+) = 0, v_c'(0^+) = -12 \Rightarrow s^2 + 4s + 4 = 0$$

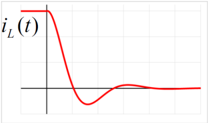
$$\Rightarrow s_1, s_2 = -2, -2 \Rightarrow v_c(t) = k_1 e^{-2t} + k_2 te^{-2t} \Rightarrow v_c(t) = 0e^{-2t} - 12te^{-2t}, t > 0$$

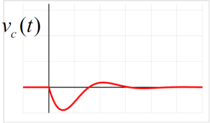
LTI RLC Circuit

Example (Second-order LTI RLC circuit with two switches (cont.))

An LTI RLC circuit may be simplified to a simple series or parallel LTI RLC circuit.







$$R = \frac{1}{2} \Rightarrow 2\alpha = \frac{1}{RC} = 2, \omega_0^2 = \frac{1}{LC} = 4, Q = \frac{\omega_0}{2\alpha} = 1 > 0.5$$

$$\frac{d^2 i_L(t)}{dt^2} + 2\frac{di_L(t)}{dt} + 4i_L(t) = 0, t > 0, i_L(0^+) = 12, i_L'(0^+) = 0 \Rightarrow s^2 + 2s + 4 = 0$$

$$\Rightarrow s_1, s_2 = -1 \pm j\sqrt{3} \Rightarrow i_L(t) = k_1 e^{-t}\cos(\sqrt{3}t) + k_2 e^{-t}\sin(\sqrt{3}t)$$

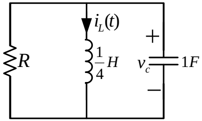
$$\Rightarrow i_L(t) = 12e^{-t}\cos(\sqrt{3}t) + 4\sqrt{3}e^{-t}\sin(\sqrt{3}t) = 8\sqrt{3}e^{-t}\cos(\sqrt{3}t - 30^\circ), t > 0$$

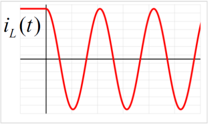
$$\frac{d^2 v_c(t)}{dt^2} + 2\frac{dv_c(t)}{dt} + 4v_c(t) = 0, t > 0, v_c(0^+) = 0, v_c'(0^+) = -12 \Rightarrow s^2 + 2s + 4 = 0 \Rightarrow$$

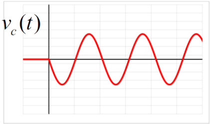
$$s_1, s_2 = -1 \pm j\sqrt{3} \Rightarrow v_c(t) = k_1 e^{-t}\cos(\sqrt{3}t) + k_2 e^{-t}\sin(\sqrt{3}t) \Rightarrow v_c(t) = -4\sqrt{3}e^{-t}\sin(\sqrt{3}t), t > 0$$

Example (Second-order LTI RLC circuit with two switches (cont.))

An LTI RLC circuit may be simplified to a simple series or parallel LTI RLC circuit.







$$R = \infty \Rightarrow 2\alpha = \frac{1}{RC} = 0, \omega_0^2 = \frac{1}{LC} = 4, Q = \frac{\omega_0}{2\alpha} = \infty$$

$$\frac{d^2 i_L(t)}{dt^2} + 4i_L(t) = 0, t > 0, i_L(0^+) = 12, i_L'(0^+) = 0 \Rightarrow s^2 + 4 = 0$$

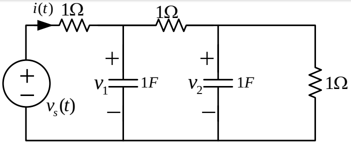
$$\Rightarrow s_1, s_2 = \pm j2 \Rightarrow i_L(t) = k_1 \cos(2t) + k_2 \sin(2t) = 12\cos(2t), t > 0$$

$$\frac{d^2 v_c(t)}{dt^2} + 4v_c(t) = 0, t > 0, v_c(0^+) = 0, v_c'(0^+) = -12 \Rightarrow s^2 + 4 = 0$$

$$\Rightarrow s_1, s_2 = \pm j2 \Rightarrow v_c(t) = k_1 \cos(2t) + k_2 \sin(2t) \Rightarrow v_c(t) = -6\sin(2t), t > 0$$

Example (Second-order LTI circuit with two capacitors)

Step and impulse responses can be found in different ways.



$$\frac{dv_1}{dt} + \frac{v_1(t) - v_s(t)}{1} + \frac{v_1(t) - v_2(t)}{1} = 0, \quad \frac{dv_2}{dt} + \frac{v_2(t) - v_1(t)}{1} + \frac{v_2(t)}{1} = 0$$

$$v_2(t) = \frac{dv_1}{dt} + 2v_1(t) - v_s(t), \quad v_1(t) = \frac{dv_2}{dt} + 2v_2(t)$$

$$\frac{dv_1}{dt} = v_2(t) - 2v_1(t) + v_s(t), \quad \frac{dv_2}{dt} = v_1(t) - 2v_2(t)$$

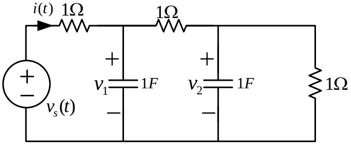
$$\frac{d^2v_1}{dt^2} + 4\frac{dv_1}{dt} + 3v_1(t) = 2v_s(t) + \frac{dv_s}{dt}, \quad \frac{d^2v_2}{dt^2} + 4\frac{dv_2}{dt} + 3v_2(t) = v_s(t)$$

$$i(t) = \frac{v_s(t) - v_1(t)}{1} \Rightarrow \frac{d^2i}{dt^2} + 4\frac{di}{dt} + 3i(t) = v_s(t) + 3\frac{dv_s}{dt} + \frac{d^2v_s}{dt^2}$$

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 3i(t) = u(t) + 3\delta(t) + \delta'(t)$$

Example (Second-order LTI circuit with two capacitors (cont.))

Step and impulse responses can be found in different ways.



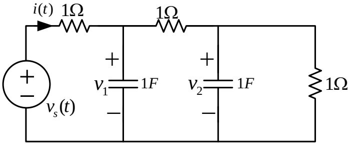
$v_1(0^-) = 0, v_2(0^-) = 0 \Rightarrow v_1(0^+) = 0, v_2(0^+) = 0, i(0^+) = v_s(0^+) - v_1(0^+) = u(0^+) = 1$

$v_1'(0^+) = v_2(0^+) - 2v_1(0^+) + v_s(0^+) = 1, v_2'(0^+) = v_1(0^+) - 2v_2(0^+) = 0$

$i'(0^+) = v_s'(0^+) - v_1'(0^+) = -1$

Example (Second-order LTI circuit with two capacitors (cont.))

Step and impulse responses can be found in different ways.



$$\frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 3i(t) = u(t) + 3\delta(t) + \delta'(t), i(0^-) = 0, i'(0^-) = 0$$

$$\frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 3i(t) = 1, i(0^+) = 1, i'(0^+) = -1$$

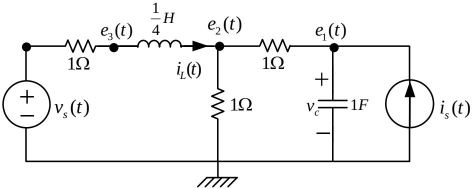
$$s^2 + 4s + 3 = 0 \Rightarrow s_1, s_2 = -1, -3$$

$$s_i(t) = i(t) = k_1 e^{-t} + k_2 e^{-3t} + \frac{1}{3} \Rightarrow s_i(t) = [\frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} + \frac{1}{3}]u(t)$$

$$h_i(t) = \frac{ds_i(t)}{dt} = [-\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}]u(t) + \delta(t)$$

Example (D operator)

D operator can be used to find the describing differential equation of a circuit.



$$Dx(t) = \frac{dx(t)}{dt}, D^{-1}x(t) = \int_0^t x(\lambda)d\lambda$$

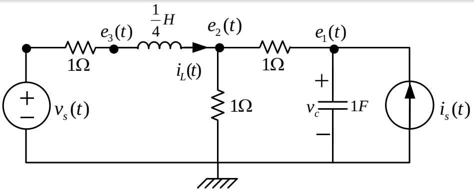
$$D(Dx(t)) = D^2x(t) = \frac{d^2x(t)}{dt^2}, D(D^{-1}x(t)) = D^{-1}(Dx(t)) = x(t), Dk = 0$$

$$\begin{cases} -i_s(t) + \frac{de_1}{dt} + \frac{e_1(t)-e_2(t)}{1} = 0 \\ \frac{e_2(t)-e_1(t)}{1} + \frac{e_2(t)}{1} - i_L(0) - 4\int_0^t (e_3(\lambda) - e_2(\lambda))d\lambda = 0 \\ i_L(0) + 4\int_0^t (e_3(\lambda) - e_2(\lambda))d\lambda + \frac{e_3(t)-v_s(t)}{1} = 0 \end{cases}$$

Second-order LTI Circuit

Example (D operator (cont.))

D operator can be used to find the describing differential equation of a circuit.



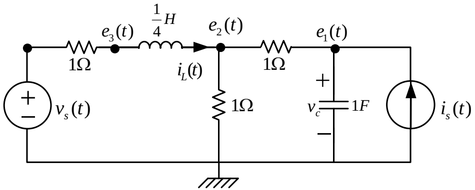
$$\begin{bmatrix} D+1 & -1 & 0 \\ -1 & 2+4D^{-1} & -4D^{-1} \\ 0 & -4D^{-1} & 1+4D^{-1} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} i_s(t) \\ i_L(0) \\ v_s(t) - i_L(0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} D+1 & -1 & 0 \\ -D & 2D+4 & -4 \\ 0 & -4 & D+4 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} i_s(t) \\ 0 \\ Dv_s(t) \end{bmatrix} \Rightarrow e_1(t) = \frac{\begin{vmatrix} i_s(t) & -1 & 0 \\ 0 & 2D+4 & -4 \\ Dv_s(t) & -4 & D+4 \end{vmatrix}}{\begin{vmatrix} D+1 & -1 & 0 \\ -D & 2D+4 & -4 \\ 0 & -4 & D+4 \end{vmatrix}}$$

Second-order LTI Circuit

Example (D operator (cont.))

D operator can be used to find the describing differential equation of a circuit.



$$\begin{vmatrix} D+1 & -1 & 0 \\ -D & 2D+4 & -4 \\ 0 & -4 & D+4 \end{vmatrix} e_1(t) = \begin{vmatrix} i_s(t) & -1 & 0 \\ 0 & 2D+4 & -4 \\ Dv_s(t) & -4 & D+4 \end{vmatrix}$$

$$\Rightarrow (2D^2 + 13D + 8)e_1(t) = 2(D+6)i_s(t) + 4v_s(t)$$

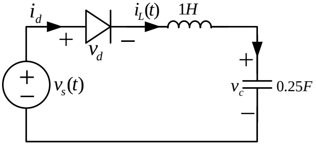
$$2\frac{d^2e_1}{dt^2} + 13\frac{de_1}{dt} + 8e_1(t) = 2\frac{di_s}{dt} + 12i_s(t) + 4v_s(t)$$

Second-order NTV Circuit

Second-order NTI circuit

Example (Second-order NTI circuit)

In the following NTI circuit with ideal diode, $v_s(t) = \delta(t)$ and $v_c(0^-) = v_c'(0^-) = 0$, the capacitor voltage remains constant for $t > 0.25\pi$.



$$\text{Diode on}: 0.25\frac{d^2v_c}{dt^2} + v_c(t) = v_s(t)$$

$$\Rightarrow \frac{d^2v_c}{dt^2} + 4v_c(t) = 4\delta(t), v_c(0^-) = 0, v_c'(0^-) = 0 \Rightarrow \frac{d^2v_c}{dt^2} + 4v_c(t) = 0, v_c(0^+) = 0, v_c'(0^+) = 4$$

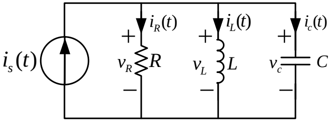
$$\Rightarrow v_c(t) = 2\sin(2t)u(t) \Rightarrow i_c(t) = 0.25\frac{dv_c(t)}{dt} = \cos(2t)u(t) \Rightarrow i_c(\frac{\pi}{4}^+) < 0$$

$$v_c(t) = \begin{cases} 2\sin(2t), & 0 < t < \frac{\pi}{4} \\ 2, & t > \frac{\pi}{4} \end{cases}, \quad \begin{cases} \text{Diode on}: i_D(t) = \cos(2t) > 0, & 0 < t < \frac{\pi}{4} \\ \text{Diode off}: v_D(t) = -2 < 0, & t > \frac{\pi}{4} \end{cases}$$

NTV Circuit

Example (State equation)

An NTV circuit can be described using state equations.



$$i_s(t) = \frac{v_c(t)}{R} + i_L(t) + c\frac{dv_c}{dt}, \quad v_c(t) = L\frac{di_L}{dt}$$

$$\begin{cases} \frac{dv_c}{dt} = \frac{i_s(t)}{C} - \frac{v_c(t)}{RC} - \frac{i_L(t)}{C} \\ \frac{di_L}{dt} = \frac{v_c(t)}{L} \end{cases} \Rightarrow \frac{d\boldsymbol{X}(t)}{dt} = \begin{bmatrix} \frac{-1}{RC} & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \boldsymbol{X}(t) + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s(t), \boldsymbol{X}(t) = \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix}$$

$$\begin{cases} \frac{d\boldsymbol{X}(t)}{dt} = \boldsymbol{A}\boldsymbol{X}(t) + \boldsymbol{B}\boldsymbol{W}(t), \boldsymbol{X}(0) = \boldsymbol{X}_0 \\ y(t) = \boldsymbol{C}^T\boldsymbol{X}(t) + \boldsymbol{D}^T\boldsymbol{W}(t) \end{cases}$$

$$\begin{cases} \frac{d\boldsymbol{X}(t)}{dt} = f(\boldsymbol{X}(t), \boldsymbol{W}(t), t) \\ y(t) = g(\boldsymbol{X}(t), \boldsymbol{W}(t), t) \end{cases}$$

The End