Sinusoidal Steady State Analysis

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Overview

Sinusoidal Steady State

LTI Circuits with Sinusoidal Input

Statement (Describing Differential Equation)

An LTI circuit with sinusoidal input is described by a non-homogeneous sinusoidally-driven constant-coefficient linear differential equation.

- Single-input LTI circuit: $\begin{cases} \sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j w^{(j)}(t), t > 0^- \\ y(0^-), y'(0^-), \cdots, y^{(n-1)}(0^-) \end{cases}$
- Sinusoidal input: $w(t) = A\cos(\omega t + \theta)u(t)$
- Single-input sinusoidal LTI circuit: $\begin{cases} \sum_{i=0}^{n} a_i y^{(i)}(t) = B\cos(\omega t + \phi), t > 0 \\ y(0^+), y'(0^+), \cdots, y^{(n-1)}(0^+) \end{cases}$
- Natural frequencies: $\sum_{i=1}^{n} a_i s_i = 0$
- Complete response: $y(t) = y_h(t) + y_p(t) = y_{tr}(t) + y_{ss}(t)$
- Homogeneous response: $y_h(t) = \sum_{i=1}^{n} k_i e^{s_i t}, t > 0$
- Particular response: $y_p(t) = \begin{cases} C\cos(\omega t + D), & s_i \neq j\omega, \forall i \\ Ct\cos(\omega t + D), & s_i = j\omega, \exists i \end{cases}$

Natural Frequency

Definition (Natural Frequencies)

The roots of the characteristic equation of the LTI circuit differential equation are called natural frequencies.

- Natural frequencies: $F(s) = \sum_{i=1}^{n} a_i s_i = 0, a_i > 0$

Statement (Natural Frequencies)

Natural frequencies appear in real or complex conjugate forms.

- Complex conjugate natural frequencies: $F(s_i) = 0 \Rightarrow F(s_i^*) = 0$

Natural Frequency

Statement (Strictly Passive LTI Circuit)

Natural frequencies of a strictly passive LTI circuit fall within the LHS of the complex plane.

Statement (Passive LTI Circuit)

Natural frequencies of a passive LTI circuit fall within the LHS of the complex plane or are simple conjugate pure imaginary values on the $j\omega$-axis.

Statement (Active LTI Circuit)

An active LTI circuit has at least one natural frequency on the RHS of the complex plane or one repeated natural frequency on the $j\omega$ axis.

Strictly Passive LTI Circuits

Statement (Sinusoidal Steady State for Strictly Passive LTI Circuits)

A strictly passive LTI circuit with sinusoidal input achieves sinusoidal steady state.

- Natural frequencies: $\Re\{s_i\} < 0, \forall i$
- Complete response: $y(t) = \sum_{i=1}^{n} k_i e^{s_i t} + C\cos(\omega t + D), t > 0$
- Sinusoidal steady state response: $y_{ss}(t) = C\cos(\omega t + D), t > 0$

Passive LTI Circuits

Statement (Steady State for Passive LTI Circuits)

A passive LTI circuit with sinusoidal input may achieve steady state.

1. Natural frequencies: $s_{1,2} = \pm j\omega_0, \omega_0 \neq \omega; \quad \Re\{s_i\} < 0, i = 3, 4, \cdots$
   - Complete response:
     $y(t) = C_0 \cos(\omega_0 t + D_0) + \sum_{i=3}^{n} k_i e^{s_i t} + C \cos(\omega t + D), t > 0$
   - Steady state response: $y_{ss}(t) = C_0 \cos(\omega_0 t + D_0) + C \cos(\omega t + D), t > 0$
2. Natural frequencies: $s_{1,2} = \pm j\omega_0, \omega_0 = \omega; \quad \Re\{s_i\} < 0, i = 3, 4, \cdots$
   - Complete response:
     $y(t) = C_0 \cos(\omega_0 t + D_0) + \sum_{i=3}^{n} k_i e^{s_i t} + Ct \cos(\omega t + D), t > 0$

Passive LTI Circuits

Statement (Active LTI Circuits)

An active LTI circuit doesn't have steady state.

- Natural frequencies: $\Re\{s_i\} > 0, \exists i$
- Complete response: $y(t) = \sum_{i=1}^{n} k_i e^{s_i t} + C\cos(\omega t + D), t > 0$

Example (Sinusoidal steady state for series RC circuit)

The series passive RC circuit achieves sinusoidal steady state.

$$\begin{cases} \frac{dv_c}{dt} + \frac{v_c(t)}{RC} = \frac{i_s(t)}{C} = \frac{A\cos(\omega t+\theta)u(t)}{C} \\ v_c(0^-) = V_0 \end{cases}$$

$$\begin{cases} \frac{dv_c}{dt} + \frac{v_c(t)}{RC} = \frac{A\cos(\omega t+\theta)}{C}, t > 0 \\ v_c(0^+) = v_c(0^-) = V_0 \end{cases}$$

$$v_c(t) = v_{ch}(t) + v_{cp}(t) = Ke^{-\frac{t}{RC}} + B\cos(\omega t + \phi), t > 0$$

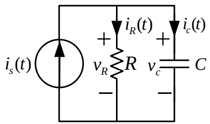
$$-B\omega\sin(\omega t + \phi) + \frac{B\cos(\omega t + \phi)}{RC} = \frac{A\cos(\omega t + \theta)}{C}$$

$$\begin{cases} B = \frac{AR}{\sqrt{1+(RC\omega)^2}} \\ \phi = \theta - \tan^{-1}(RC\omega) \end{cases}$$

$$V_0 = Ke^0 + B\cos(\phi) \Rightarrow K = V_0 - B\cos(\phi)$$

$$v_c(t) = (V_0 - B\cos(\phi))e^{-\frac{t}{RC}} + B\cos(\omega t + \phi), t > 0$$

$$v_{ss}(t) = B\cos(\omega t + \phi), t > 0$$



Phasors

Phasors

Definition (Phasors)

The phasor $X = Ae^{j\theta} = A\angle\theta$ can fully describe the sinusoidal signal $x(t) = A\cos(\omega t + \theta) = \Re\{Xe^{j\omega t}\}$ with the known frequency ω.

- Sinusoidal signal: $A\cos(\omega t + \theta) \equiv Ae^{j\theta} = A\angle\theta$
- Phasor amplitude: $|A|$
- Phasor phase: $\theta + \begin{cases} 0, & A \geq 0 \\ \pi, & A < 0 \end{cases}$
- Sinusoidal signal: $A\sin(\omega t + \theta) \equiv Ae^{j(\theta - 90^\circ)} = A\angle\theta - 90^\circ$
- Sinusoidal signal: $-A\sin(\omega t + \theta) \equiv Ae^{j(\theta + 90^\circ)} = A\angle\theta + 90^\circ$
- Differentiation: $\frac{d(Xe^{j\omega t})}{dt} = j\omega Xe^{j\omega t}$
- Real-part and differentiation: $\Re\{\frac{d(Xe^{j\omega t})}{dt}\} = \frac{d\Re\{Xe^{j\omega t}\}}{dt}$

Example (Phasor calculation)

Phasor can facilitate calculations.

$$x(t) = A\cos(\omega t + \theta), \quad X = Ae^{j\theta}$$

$$y(t) = \sum_{k=0}^{m} a_k \frac{d^k x(t)}{dt^k} = B\cos(\omega t + \phi) \Rightarrow Y = Be^{j\phi} = \sum_{k=0}^{m} a_k (j\omega)^k X = Ae^{j\theta} \sum_{k=0}^{m} a_k (j\omega)^k$$

Example (Phasor calculation)

Phasor can facilitate calculations.

$$v(t) = 2\cos(3t + \underline{/30^\circ}) - 4\sin(3t + \underline{/45^\circ}) + 3\sin(3t) - 10\cos(3t)$$

$$\Re\{Ve^{j3t}\} = \Re\{2\underline{/30^\circ}e^{j3t}\} - \Re\{4\underline{/-90^\circ + 45^\circ}e^{j3t}\} + \Re\{3\underline{/-90^\circ}e^{j3t}\} - \Re\{10\underline{/0^\circ}e^{j3t}\}$$

$$V = (2\underline{/30^\circ}) - (4\underline{/-90^\circ + 45^\circ}) + (3\underline{/-90^\circ}) - (10\underline{/0^\circ})$$

$$V = (2\cos(30^\circ) - 4\cos(-45^\circ) + 0 - 10) + j(2\sin(30^\circ) - 4\sin(-45^\circ) - 3 - 0)$$

$$V = -11.096 + j0.83 = 11.127\underline{/175.72^\circ}$$

$$v(t) = 11.127\cos(3t + 175.72^\circ)$$

Phasor Calculation

Example (Phasor calculation)

Phasor can facilitate calculations.

$$\sum_{k=0}^{n} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k w}{dt^k}, w(t) = A\cos(\omega t + \theta)$$

$$w(t) = \Re\{We^{j\omega t}\} = \Re\{Ae^{j\theta}e^{j\omega t}\}, \quad y_p(t) = \Re\{Y_p e^{j\omega t}\} = \Re\{Be^{j\phi}e^{j\omega t}\}$$

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k}\left(\Re\{Y_p e^{j\omega t}\}\right) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k}\left(\Re\{We^{j\omega t}\}\right)$$

$$\sum_{k=0}^{n} a_k \Re\left\{\frac{d^k}{dt^k}\left(Y_p e^{j\omega t}\right)\right\} = \sum_{k=0}^{m} b_k \Re\left\{\frac{d^k}{dt^k}\left(We^{j\omega t}\right)\right\}$$

$$\Re\left\{\sum_{k=0}^{n} a_k \frac{d^k}{dt^k}\left(Y_p e^{j\omega t}\right)\right\} = \Re\left\{\sum_{k=0}^{m} b_k \frac{d^k}{dt^k}\left(We^{j\omega t}\right)\right\}$$

$$\Re\left\{\left[\sum_{k=0}^{n} a_k (j\omega)^k Y_p\right] e^{j\omega t}\right)\right\} = \Re\left\{\left[\sum_{k=0}^{m} b_k (j\omega)^k W\right] e^{j\omega t}\right\}$$

$$Y_p \sum_{k=0}^{n} a_k (j\omega)^k = W \sum_{k=0}^{m} b_k (j\omega)^k \Rightarrow Y_p = \frac{\sum_{k=0}^{m} b_k (j\omega)^k}{\sum_{k=0}^{n} a_k (j\omega)^k} W$$

Example (Sinusoidal steady state analysis using phasors)

Sinusoidal steady state can be found using phasor calculation.

$v_s(t) = 2\cos(2t)u(t), i_L(0^-) = 4, v_c(0^-) = 2$

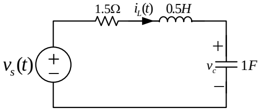
$$\frac{d^2 v_c}{dt^2} + 3\frac{dv_c}{dt} + 2v_c(t) = 4\cos(2t)u(t), v_c(0^-) = 2, i_L(0^-) = 4$$

$$\begin{cases} \frac{d^2 v_c}{dt^2} + 3\frac{dv_c}{dt} + 2v_c(t) = 4\cos(2t), t > 0 \\ v_c(0^+) = v_c(0^-) = 2, v_c'(0^+) = i_c(0^+) = i_L(0^+) = i_L(0^-) = 4 \end{cases}$$

$$v_c(t) = v_h + v_p = K_1 e^{-t} + K_2 e^{-2t} + B\cos(2t + \phi), t > 0$$

$$W = 2e^{j0}, w(t) = 2\cos(2t), \quad V_p = Be^{j\phi}, v_p(t) = B\cos(2t + \phi)$$

$$\left((j2)^2 + 3(j2) + 2\right)V_p = 2W = 4e^{j0} \Rightarrow V_p = \frac{4}{-2 + 6j} = 0.64e^{-j108.4^\circ}$$



$$\begin{cases} B = 0.64 \\ \phi = -108.4^\circ \end{cases} \Rightarrow v_p(t) = v_{css}(t) = 0.64\cos(2t - 108.4^\circ)$$

$$\begin{cases} 2 = K_1 + K_2 + 0.64\cos(-108.4^\circ) \\ 4 = -K_1 - 2K_2 - 1.28\sin(-108.4^\circ) \end{cases}$$

$$v_c(t) = 7.2e^{-t} - 5e^{-2t} + 0.64\cos(2t - 108.4^\circ), t > 0$$

Impedance and Admittance

Impedance and Admittance

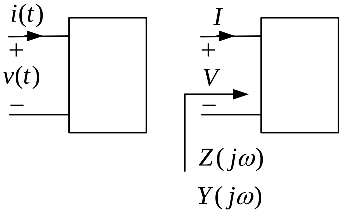


Figure: Impedance $Z(j\omega) = R(j\omega) + jX(j\omega) = \frac{V(j\omega)}{I(j\omega)}$ and admittance $Y(j\omega) = G(j\omega) + jB(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \frac{1}{Z(j\omega)}$ for a one-port in-rest network. $R(j\omega)$, $X(j\omega)$, $G(j\omega)$, and $B(j\omega)$ stand for resistance, reactance, conductance, and susceptance. The impedance and admittance are not phasors and do not have equivalent time-domain signals.

- Port voltage and current signals: $v(t) = V_m \cos(\omega t + \theta), i(t) = I_m \cos(\omega t + \phi)$
- Port voltage and current phasors: $V = V_m \underline{/\theta}, I = I_m \underline{/\phi}$
- One-port impedance and admittance: $Z(j\omega) = \frac{1}{Y(j\omega)} = \frac{V(j\omega)}{I(j\omega)}$

Resistor

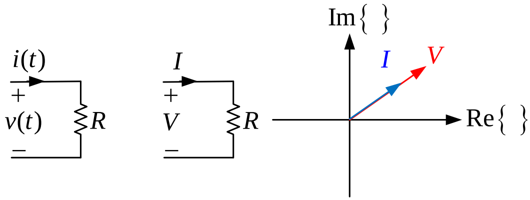


Figure: The current phasor I in an LTI resistor has no phase difference with voltage phasor V. An LTI resistor is described by the impedance $Z = R$ or admittance $Y = \frac{1}{R}$.

$$v(t) = V_m \cos(\omega t + \theta) = Ri(t) = RI_m \cos(\omega t + \phi)$$

$$V = V_m e^{j\theta} = RI_m e^{j\phi} = RI \Rightarrow \begin{cases} |V| = |RI| = |R||I| \\ \underline{/V} = \underline{/I} \end{cases}$$

Capacitor

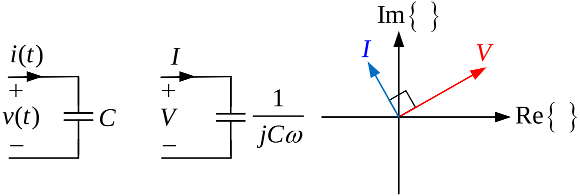


Figure: The current phasor I in an LTI capacitor leads the voltage phasor V by 90°. An LTI capacitor is described by the impedance $Z = \frac{1}{j\omega C}$ or admittance $Y = j\omega C$.

$$i(t) = C\frac{dv(t)}{dt} = C\frac{dV_m\cos(\omega t+\theta)}{dt} = -CV_m\omega\sin(\omega t+\theta) = CV_m\omega\cos(\omega t+\theta+90^\circ)$$

$$I = I_m e^{j\phi} = CV_m\omega e^{j(\theta+90^\circ)} = j\omega C V_m e^{j\theta} \Rightarrow \begin{cases} |V| = |\frac{I}{j\omega C}| = \frac{|I|}{|\omega C|} \\ \underline{/V} = \underline{/I} - 90^\circ \end{cases}$$

Inductor

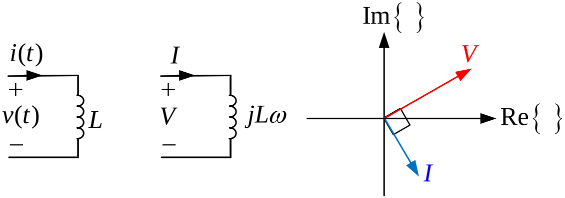


Figure: The current phasor I in an LTI inductor lags the voltage phasor V by 90°. An LTI inductor is described by the impedance $Z = j\omega L$ or admittance $Y = \frac{1}{j\omega L}$.

$$v(t) = V_m \cos(\omega t + \theta) = L\frac{di(t)}{dt} = -LI_m\omega \sin(\omega t + \phi) = LI_m\omega \cos(\omega t + \phi + 90^\circ)$$

$$V = V_m e^{j\theta} = LI_m\omega e^{j(\phi + 90^\circ)} = j\omega LI_m e^{j\theta} \Rightarrow \begin{cases} |V| = |Ij\omega L| = |I||\omega L| \\ \underline{/V} = \underline{/I} + 90^\circ \end{cases}$$

Passive One-port

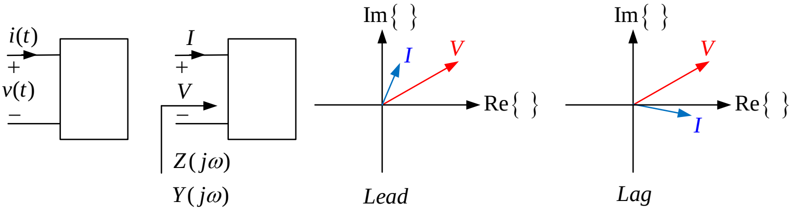


Figure: A passive one-port has the impedance $Z = R + jX = \frac{1}{Y}, R \geq 0$.

- Passive load: $Z = R + jX = |Z|e^{j\angle Z}, \quad R = \Re\{Z\} \geq 0 \equiv -90^\circ \leq \angle Z \leq 90^\circ$
- Resistive load: $Z = R + jX = |Z|e^{j\angle Z}, \quad X = 0 \equiv \angle Z = 0$
- Inductive (lagging) load: $Z = R + jX = |Z|e^{j\angle Z}, \quad X > 0 \equiv 0 < \angle Z \leq 90^\circ$
- Capacitive (leading) load: $Z = R + jX = |Z|e^{j\angle Z}, \quad X < 0 \equiv -90^\circ \leq \angle Z < 0$

Sinusoidal Steady State Analysis

KVL/KCL

- KCL in phasor-domain: $\sum_k i_k(t) = \sum_k [i_{hk}(t) + i_{pk}(t)] = 0 \Rightarrow \sum_k i_{pk}(t) = 0, t \gg 0 \Rightarrow \sum_k \Re\{I_k e^{j\omega t}\} = 0 \Rightarrow \Re\{e^{j\omega t} \sum_k I_k\} = 0 \Rightarrow \sum_k I_k = 0$
- KVL in phasor-domain: $\sum_k v_k(t) = \sum_k [v_{hk}(t) + v_{pk}(t)] = 0 \Rightarrow \sum_k v_{pk}(t) = 0, t \gg 0 \Rightarrow \sum_k \Re\{V_k e^{j\omega t}\} = 0 \Rightarrow \Re\{e^{j\omega t} \sum_k V_k\} = 0 \Rightarrow \sum_k V_k = 0$

Series and Parallel Connections

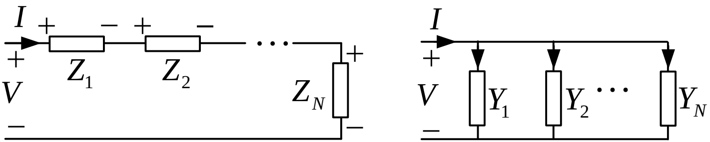


Figure: Series and parallel connection of impedances. The Delta-Wye conversion is also valid in phasor domain.

- Series connection: $V = \sum_k V_k = \sum_k Z_k I_k = I \sum_k Z_k \Rightarrow Z = \frac{V}{I} = \sum_k Z_k$
- Parallel connection: $I = \sum_k I_k = \sum_k Y_k V_k = V \sum_k Y_k \Rightarrow Y = \frac{I}{V} = \sum_k Y_k$

Source Transformation

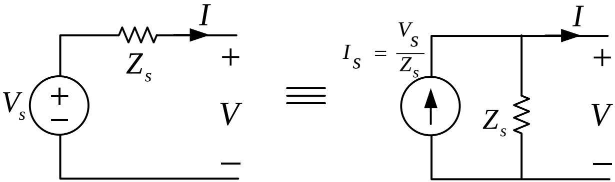


Figure: Source transformation.

$$V = V_s - Z_s I \Rightarrow I = \frac{V_s}{Z_s} - \frac{V}{Z_s}$$

Thevenin/Norton Equivalent Circuit

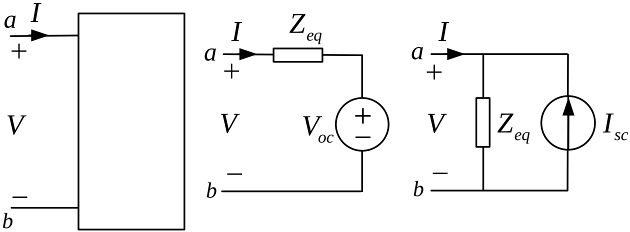


Figure: Phasor-domain Thevenin/Norton Equivalent Circuit for an LTI circuit.

$$V_{oc} = V|_{I=0}, \quad I_{sc} = -I|_{V=0}, \quad Z_{eq} = \frac{V}{I}|_{W=0}, \quad V_{oc} = Z_{eq} I_{sc}$$

Superposition

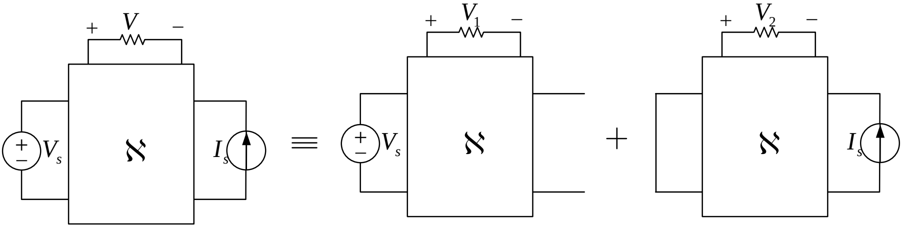
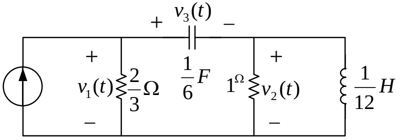


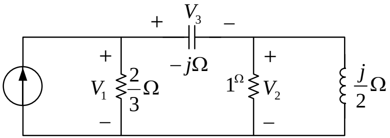
Figure: If the independent sources have the same frequency, the phasor-domain superposition for an LTI circuit yields $V = V_1 + V_2$.

Example (Phasor-domain node analysis)

Node analysis can be used in phasor domain.



$i_s(t) = 160\cos(6t + 30^\circ)$



$I_s = 160\angle 30^\circ$

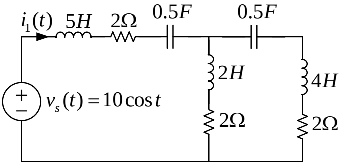
$$\begin{cases} -160\underline{/30^\circ} + \frac{V_1}{2} + \frac{V_1 - V_2}{-j} = 0 \\ \frac{V_2 - V_1}{-j} + \frac{V_2}{1} + \frac{V_2}{0.5j} = 0 \end{cases}$$

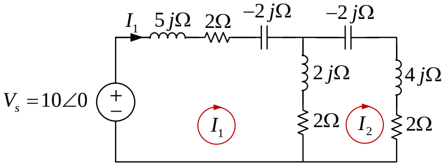
$$\Rightarrow \begin{bmatrix} 1.5 + j & j \\ j & 1 - j \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 160\underline{/30^\circ} \\ 0 \end{bmatrix}$$

$$\Rightarrow V_1 = \frac{\begin{vmatrix} 160\underline{/30^\circ} & j \\ 0 & 1 - j \end{vmatrix}}{\begin{vmatrix} 1.5 + j & j \\ j & 1 - j \end{vmatrix}} = 63.6 - 7.6j = 64\underline{/-6.9^\circ} \Rightarrow v_1(t) = 64\cos(6t - 6.9^\circ)$$

Example (Phasor-domain mesh analysis)

Mesh analysis can be used in phasor domain.





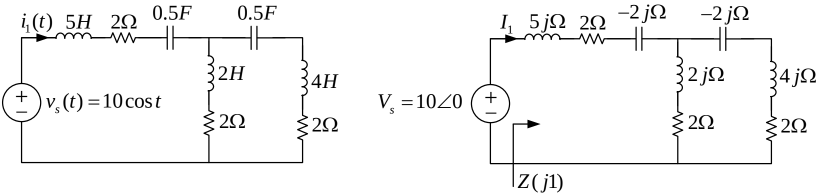
$$\begin{cases} -10 + (5j + 2 - 2j)I_1 + (2j + 2)(I_1 - I_2) = 0 \\ (2j + 2)(I_2 - I_1) + (-2j + 4j + 2)I_2 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 4 + 5j & -2 - 2j \\ -2 - 2j & 4 + 4j \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\Rightarrow I_1 = \frac{\begin{vmatrix} 10 & -2 - 2j \\ 0 & 4 + 4j \end{vmatrix}}{\begin{vmatrix} 4 + 5j & -2 - 2j \\ -2 - 2j & 4 + 4j \end{vmatrix}} = \frac{10}{3 + 4j} = 1.2 - 1.6j = 2\underline{/-53.1^\circ} \Rightarrow i_1(t) = 2\cos(t - 53.1^\circ)$$

Example (Input impedance)

Series and parallel connections can facilitate circuit analysis.

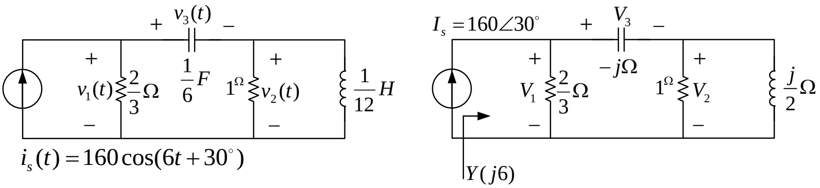


$$Z(j1) = (5j + 2 - 2j) + ((2j + 2)||(-2j + 4j + 2)) = 3 + 4j$$

$$I_1 = \frac{V_s}{Z(j1)} = \frac{10}{3 + 4j} = 1.2 - 1.6j = 2\underline{/-53.1^\circ} \Rightarrow i_1(t) = 2\cos(t - 53.1^\circ)$$

Example (Input admittance)

Series and parallel connections can facilitate circuit analysis.



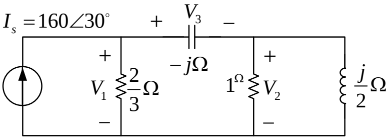
$$Y(j6) = \frac{3}{2} + \frac{1}{(-j) + (1||0.5j)} = 2 + 1.5j = 2.5\underline{/36.9^\circ}$$

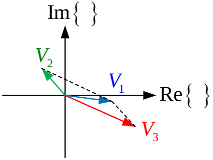
$$V_1 = \frac{I_s}{Y(j6)} = \frac{160\underline{/30^\circ}}{2.5\underline{/36.9^\circ}} = 64\underline{/-6.9^\circ} \Rightarrow v_1(t) = 64\cos(6t - 6.9^\circ)$$

SSS Analysis

Example (Voltage division)

Voltage and current division rules can facilitate circuit analysis.





$$i_s(t) = 160\cos(6t + 30^\circ) \Rightarrow I_s = 160\underline{/30^\circ}$$

$$V_1 = \frac{I_s}{\frac{3}{2} + \frac{1}{(-j)+(1||0.5j)}} = 63.6 - 7.6j = 64\underline{/-6.9^\circ} \Rightarrow v_1(t) = 64\cos(6t - 6.9^\circ)$$

$$V_2 = \frac{1||0.5j}{(-j) + (1||0.5j)} V_1 = -28 + 35.6j = 45.2\underline{/128^\circ} \Rightarrow v_2(t) = 45.2\cos(6t + 128^\circ)$$

$$V_3 = \frac{-jV_1}{(-j) + (1||0.5j)} = V_1 - V_2 = 91.6 - 43.2j = 101.2\underline{/-25.3^\circ} \Rightarrow v_3(t) = 101.2\cos(6t - 25.3^\circ)$$

Example (First-order circuit)

Sinusoidal steady state can be simply found using phasor analysis.

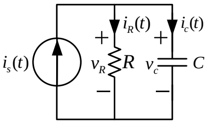
$$i_s(t) = A\cos(\omega t + \theta) \Rightarrow I_s = A\angle\theta$$

$$I_s = \frac{V_c}{R} + \frac{V_c}{\frac{1}{j\omega C}} = (\frac{1}{R} + j\omega C)V_c$$

$$\Rightarrow V_c = \frac{I_s}{\frac{1}{R} + j\omega C} = \frac{RA\angle\theta}{1 + j\omega RC}$$

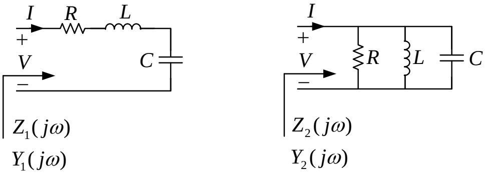
$$\Rightarrow V_c = \frac{AR}{\sqrt{1 + (RC\omega)^2}}\angle\theta - \tan^{-1}(RC\omega)$$

$$v_c(t) = \frac{AR}{\sqrt{1 + (RC\omega)^2}}\cos(\omega t + \theta - \tan^{-1}(RC\omega))$$



Example (Duality)

Duality can facilitate phasor analysis.



$$Z_1(j\omega) = R + j\omega L + \frac{1}{j\omega C} = R + j(L\omega - \frac{1}{C\omega})$$

$$Y_1(j\omega) = \frac{1}{Z_1(j\omega)} = \frac{1}{R + j(L\omega - \frac{1}{C\omega})} = \frac{R}{R^2 + (L\omega - \frac{1}{C\omega})^2} - j\frac{L\omega - \frac{1}{C\omega}}{R^2 + (L\omega - \frac{1}{C\omega})^2}$$

$$Y_2(j\omega) = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} = \frac{1}{R} + j(C\omega - \frac{1}{L\omega})$$

$$Z_2(j\omega) = \frac{1}{Y_2(j\omega)} = \frac{1}{\frac{1}{R} + j(C\omega - \frac{1}{L\omega})} = \frac{\frac{1}{R}}{\frac{1}{R^2} + (C\omega - \frac{1}{L\omega})^2} - j\frac{C\omega - \frac{1}{L\omega}}{\frac{1}{R^2} + (C\omega - \frac{1}{L\omega})^2}$$

Example (Thevenin/Norton equivalent circuits)

Thevenin/Norton equivalent circuits can be found using phasor analysis.

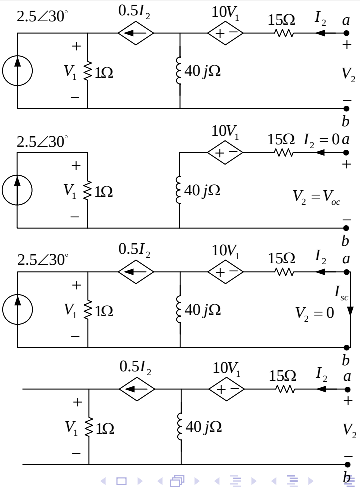
$$\begin{cases} V_1 = 2.5\underline{/30^\circ} \\ V_{oc} = -10V_1 = 25\underline{/-150^\circ} \end{cases}$$

$$\begin{cases} 2.5\underline{/30^\circ} + 0.5I_2 = \frac{V_1}{1} \\ 15I_2 - 10V_1 + 40j(I_2 - 0.5I_2) = 0 \end{cases}$$

$$I_2 = -1.12\underline{/-33.5^\circ} \Rightarrow I_{sc} = -I_2 = 1.12\underline{/-33.5^\circ}$$

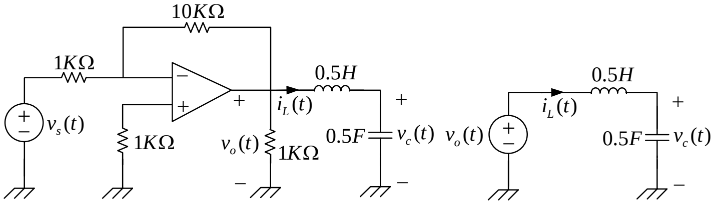
$$\begin{cases} 0.5I_2 = \frac{V_1}{1} \\ V_2 = 15I_2 - 10V_1 + 40j(I_2 - 0.5I_2) = 0 \end{cases}$$

$$V_2 = (10 + 20j)I_2 \Rightarrow Z_{eq} = \frac{V_2}{I_2} = 10 + 20j$$



Example (SSS analysis for op-amp circuits)

Phasor analysis can be used to determine sinusoidal steady state of op-amp circuits.



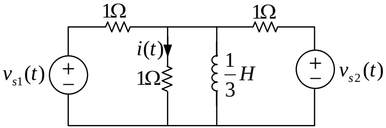
$$v_o(t) = -10v_s(t),\, v_s(t) = -0.1\sin(t) \Rightarrow v_o(t) = \sin(t) \Rightarrow V_o = -j$$

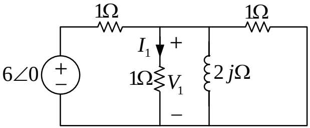
$$V_c = \frac{-2j}{0.5j - 2j} V_o = -\frac{4}{3}j \Rightarrow v_c(t) = \frac{4}{3}\cos(t - 90^\circ) = \frac{4}{3}\sin(t)$$

SSS Analysis

Example (Superposition)

Phasor-domain superposition is valid if the sinusoidal sources have the same frequency.



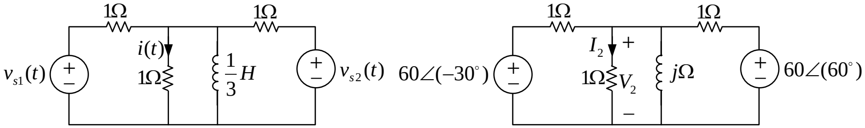


$v_{s1}(t) = 6\cos(6t) + 60\cos(3t - 30^\circ), \quad v_{s2}(t) = 60\cos(3t + 60^\circ), \quad i(t) = i_1(t) + i_2(t)$

$$I_1 = \frac{V_1}{1} = \frac{(1)||(1)||(2j)}{1 + (1)||(1)||(2j)} 6\underline{/0^\circ} = \frac{12j}{1+6j} = 1.97\underline{/9.5^\circ} \Rightarrow i_1(t) = 1.97\cos(6t + 9.5^\circ)$$

Example (Superposition (cont.))

Phasor-domain superposition is valid if the sinusoidal sources have the same frequency.



$$I_2 = \frac{V_2}{1} = I_{21} + I_{22} = \frac{V_{21}}{1} + \frac{V_{22}}{1} = \frac{(1)||(1)||(j)}{1+(1)||(1)||(j)} 60\underline{/-30^\circ} + \frac{(1)||(1)||(j)}{1+(1)||(1)||(j)} 60\underline{/60^\circ}$$

$$I_2 = \frac{j}{1+3j} 60\sqrt{2}\underline{/15^\circ} = 26.83\underline{/33.7^\circ} \Rightarrow i_2(t) = 26.83\cos(3t + 33.7^\circ)$$

$$i(t) = i_1(t) + i_2(t) = 1.97\cos(6t + 9.5^\circ) + 26.83\cos(3t + 33.7^\circ)$$

Sinusoidal Steady State Power

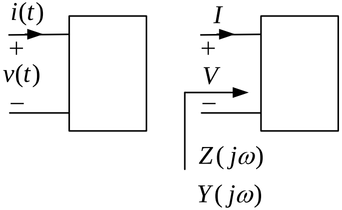


Figure: A one-port with sinusoidal voltage and currents.

- Port voltage and current signals: $v(t) = V_m \cos(\omega t + \theta), i(t) = I_m \cos(\omega t + \phi)$
- Port voltage and current phasors: $V = V_m \angle\theta, I = I_m \angle\phi$
- Absorbed instantaneous power: $p(t) = v(t)i(t) = V_m \cos(\omega t + \theta) I_m \cos(\omega t + \phi)$
- Absorbed instantaneous power: $p(t) = \frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi) + \frac{V_m I_m}{2} \cos(\theta - \phi)$

Average (Real) Power

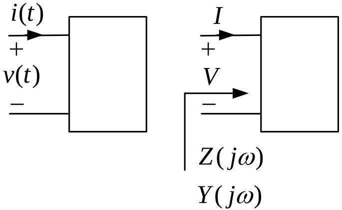


Figure: A passive one-port with sinusoidal voltage and currents.

- Port voltage and current signals: $v(t) = V_m \cos(\omega t + \theta), i(t) = I_m \cos(\omega t + \phi)$
- Port voltage and current phasors: $V = V_m \angle \theta, I = I_m \angle \phi$
- Absorbed instantaneous power: $p(t) = \frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi) + \frac{V_m I_m}{2} \cos(\theta - \phi)$
- Average (real) power: $P_{av} = \frac{1}{T} \int_0^T p(t)dt = \frac{V_m I_m}{2} \cos(\theta - \phi)$
- Average (real) power: $P_{av} = \frac{V_m I_m}{2} \cos(\angle V - \angle I) = \frac{V_m I_m}{2} \cos(\angle Z)$
- Passive load average power: $-90^\circ \leq \angle Z \leq 90^\circ \Rightarrow P_{av} \geq 0$

Resistive Load Average Power

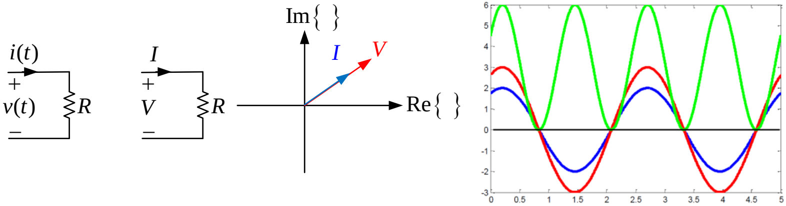
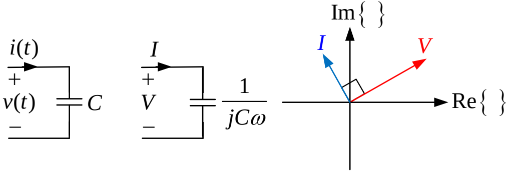


Figure: Average power for a resistive passive load.

- Average (real) power: $P_{av} = \frac{V_m I_m}{2}\cos(\angle V - \angle I) = \frac{V_m I_m}{2}\cos(\angle Z)$
- Resistive load average power: $P_{av} = \frac{V_m I_m}{2}\cos(0^\circ) = \frac{V_m I_m}{2} = \frac{1}{2} R I_m^2$

Capacitive Load Average Power



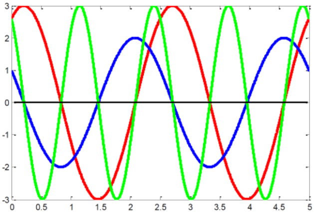


Figure: Average power for a capacitive passive load.

- Average (real) power: $P_{av} = \frac{V_m I_m}{2} \cos(\angle V - \angle I) = \frac{V_m I_m}{2} \cos(\angle Z)$
- Capacitive load average power: $P_{av} = \frac{V_m I_m}{2} \cos(-90^\circ) = 0$

Inductive Load Average Power

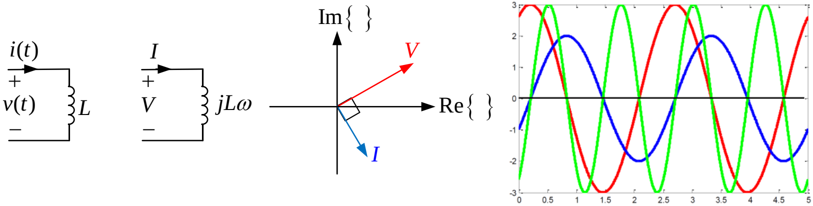


Figure: Average power for a inductive passive load.

- Average (real) power: $P_{av} = \frac{V_m I_m}{2} \cos(\angle V - \angle I) = \frac{V_m I_m}{2} \cos(\angle Z)$
- Inductive load average power: $P_{av} = \frac{V_m I_m}{2} \cos(90^\circ) = 0$

Complex Power

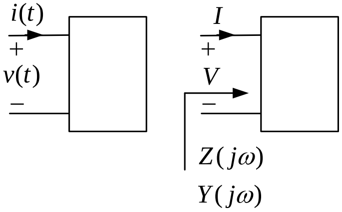


Figure: A one-port with sinusoidal voltage and currents.

- Port voltage and current signals: $v(t) = V_m \cos(\omega t + \theta), i(t) = I_m \cos(\omega t + \phi)$
- Port voltage and current phasors: $V = V_m \underline{/\theta}, I = I_m \underline{/\phi}$
- Apparent (complex) power (VA): $P = \frac{1}{2} V I^* = \frac{1}{2} Z |I|^2 = \frac{1}{2} R |I|^2 + j \frac{1}{2} X |I|^2$
- Apparent (complex) power (VA):
  $P = \frac{V_m e^{j\theta} I_m e^{-j\phi}}{2} = \frac{V_m I_m}{2} \left[\cos(\theta - \phi) + j \sin(\theta - \phi) \right]$
- Apparent (complex) power (VA): $P = P_{av} + jQ$

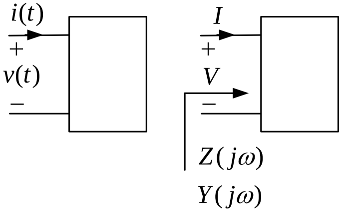


Figure: A one-port with sinusoidal voltage and currents.

- Apparent (complex) power (VA):
  $P = \frac{1}{2}VI^* = \frac{1}{2}Z|I|^2 = \frac{1}{2}R|I|^2 + j\frac{1}{2}X|I|^2 = P_{av} + jQ$
- Average (real) power (W): $P_{av} = \Re\{P\} = \frac{V_m I_m}{2}\cos(\underline{/Z}) = \frac{1}{2}R|I|^2$
- Reactive (imaginary) power (VAR): $Q = \Im\{P\} = \frac{V_m I_m}{2}\sin(\underline{/Z}) = \frac{1}{2}X|I|^2$

Reactive Power

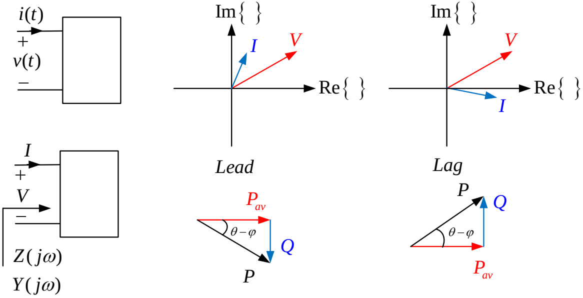


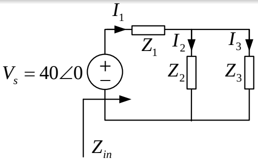
Figure: Power triangle.

- Apparent (complex) power (VA): $P = P_{av} + jQ$
- Power factor: $PF = \cos(\angle Z) = \cos(\angle V - \angle I), 0 \leq PF \leq 1$
- Resistive load: $X = 0 \equiv Q = 0 \equiv PF = 1$
- Capacitive (leading) load: $X < 0 \equiv Q < 0 \equiv 0 \leq PF < 1, lead$
- Inductive (lagging) load: $X > 0 \equiv Q > 0 \equiv 0 \leq PF < 1, lag$

SSS Powers

Example (Power Conservation)

Sum of the absorbed complex (real, reactive) powers is zero.



$Z_1 = 1.5\Omega, \quad Z_2 = j\Omega, \quad Z_3 = (1 - j2)\Omega$

$Z_{in} = 2 + 1.5j \Rightarrow I_1 = 16\angle{-36.9^\circ}, I_2 = 25.3\angle{-55.4^\circ}, I_3 = 11.3\angle{98.1^\circ}$

$$P_1 = \frac{Z_1|I_1|^2}{2} = (192 + 0j)VA \Rightarrow P_{av1} = 192W, Q_1 = 0VAR, PF_1 = 1$$

$$P_2 = \frac{Z_2|I_2|^2}{2} = (0 + 320j)VA \Rightarrow P_{av2} = 0W, Q_2 = 320VAR, PF_2 = 0, lag$$

$$P_3 = \frac{Z_3|I_3|^2}{2} = (64 - 128j)VA \Rightarrow P_{av3} = 64W, Q_3 = -28VAR, PF_3 = 0.45, lead$$

$$P_s = -\frac{V_s I_1^*}{2} = (-256 - 192j)VA \Rightarrow P_{avs} = -256W, Q_s = -192VAR$$

$$\sum_k P_k = 0, \quad \sum_k P_{avk} = 0, \quad \sum_k Q_k = 0, \quad \sum_k |P_k| \neq 0$$

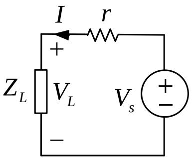
Example (Transmission line loss)

Loads with low power factor increase transmission line loss.

$$V_m = |V_L| = 300V, \quad P_{avL} = 900W, \quad PF_L = 1, \quad r = 1\Omega$$

$$P_{avL} = \frac{V_m I_m}{2}\cos(\angle Z) \Rightarrow 900 = \frac{300 I_m}{2} \times 1 \Rightarrow I_m = 6A$$

$$P_{avr} = \frac{1}{2} r I_m^2 = 18 \Rightarrow \eta = \frac{900}{900+18} = 98\%$$



$$V_m = |V_L| = 300V, \quad P_{avL} = 900W, \quad PF_L = 0.5, \quad r = 1\Omega$$

$$P_{avL} = \frac{V_m I_m}{2}\cos(\angle Z) \Rightarrow 900 = \frac{300 I_m}{2} \times 0.5 \Rightarrow I_m = 12A$$

$$P_{avr} = \frac{1}{2} r I_m^2 = 72 \Rightarrow \eta = \frac{900}{900+72} = 93\%$$

Example (Maximum power transfer)

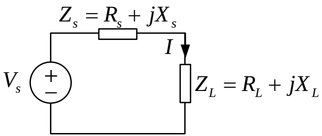
In a simple power transmission circuit with the transmission line impedance Z_s, the load $Z_L = Z_s^*$ absorbs the maximum power from the source.

$$I = \frac{V_s}{Z_s + Z_L} = \frac{V_s}{(R_s + R_L) + j(X_s + X_L)}$$

$$P_{avL} = \frac{\Re\{Z_L\}}{2}|I|^2 = \frac{1}{2}\frac{R_L}{(R_L + R_s)^2 + (X_L + X_s)^2}|V_s|^2$$

$$\begin{cases} \frac{\partial P_{avL}}{\partial R_L} = 0 \\ \frac{\partial P_{avL}}{\partial X_L} = 0 \end{cases} \Rightarrow \begin{cases} R_L = \sqrt{R_s^2 + (X_s + X_L)^2} \\ X_L = -X_s \end{cases}$$

$$\Rightarrow Z_L = Z_s^* = R_s - jX_s, \quad \max\{P_{avL}\} = \frac{|V_s|^2}{8R_L}$$



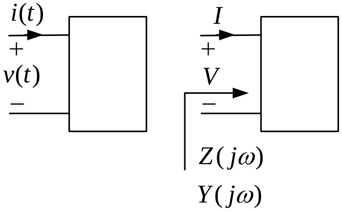


Figure: An LTI one-port with sinusoidal voltages and currents having a same frequency.

- Port voltage: $v(t) = v_1(t) + v_2(t) = V_{m1}\cos(\omega_1 t + \theta_1) + V_{m2}\cos(\omega_2 t + \theta_2), \omega_1 = \omega_2$
- Port current: $i(t) = i_1(t) + i_2(t) = I_{m1}\cos(\omega_1 t + \phi_1) + I_{m2}\cos(\omega_2 t + \phi_2), \omega_1 = \omega_2$
- Overall port voltage: $v(t) = V_m \cos(\omega_1 t + \theta)$
- Overall port current: $i(t) = I_m \cos(\omega_1 t + \phi)$
- Instantaneous power: $p(t) = \frac{V_m I_m}{2}\cos(2\omega t + \theta + \phi) + \frac{V_m I_m}{2}\cos(\theta - \phi)$
- Average power: $P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{V_m I_m}{2}\cos(\theta - \phi)$

Additive Property of Average Power

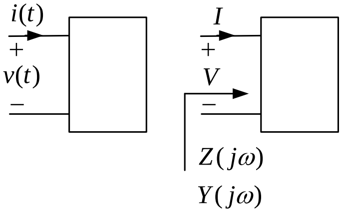
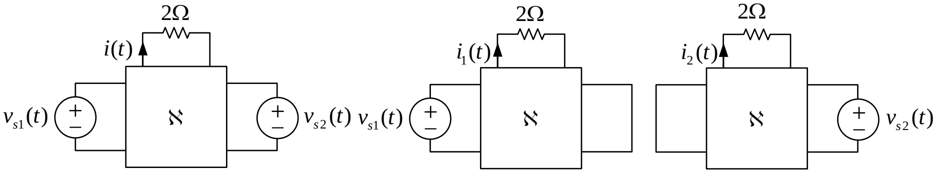


Figure: An LTI one-port with sinusoidal voltage and currents having different frequencies.

- Port voltage: $v(t) = v_1(t) + v_2(t) = V_{m1}\cos(\omega_1 t + \theta_1) + V_{m2}\cos(\omega_2 t + \theta_2), \omega_1 \neq \omega_2$
- Port current: $i(t) = i_1(t) + i_2(t) = I_{m1}\cos(\omega_1 t + \phi_1) + I_{m2}\cos(\omega_2 t + \phi_2), \omega_1 \neq \omega_2$
- Instantaneous power:
  $p(t) = \frac{V_{m1}I_{m1}}{2}\cos(2\omega_1 t + \theta_1 + \phi_1) + \frac{V_{m1}I_{m1}}{2}\cos(\theta_1 - \phi_1) + \frac{V_{m2}I_{m2}}{2}\cos(2\omega_2 t + \theta_2 + \phi_2) + \frac{V_{m2}I_{m2}}{2}\cos(\theta_2 - \phi_2) + \frac{V_{m1}I_{m2}}{2}\cos((\omega_1 + \omega_2)t + \theta_1 + \phi_2) + \frac{V_{m1}I_{m2}}{2}\cos((\omega_1 - \omega_2)t + \theta_1 - \phi_2) + \frac{V_{m2}I_{m1}}{2}\cos((\omega_2 + \omega_1)t + \theta_2 + \phi_1) + \frac{V_{m2}I_{m1}}{2}\cos((\omega_2 - \omega_1)t + \theta_2 - \phi_1)$
- Average power: $P_{av} = \frac{1}{T}\int_0^T p(t)dt, \quad T = LCM(T_1, T_2)$
- Average power: $P_{av} = \frac{V_{m1}I_{m1}}{2}\cos(\theta_1 - \phi_1) + \frac{V_{m2}I_{m2}}{2}\cos(\theta_2 - \phi_2) = P_{av1} + P_{av2}$

Example (Additive property of average power)

The average power is identical for the same and different frequency conditions in the circuit below provided that $\phi_1 - \phi_2 = \pm 90^\circ$, where $I_1 = |I_1|\underline{/\phi_1}$ and $I_2 = |I_2|\underline{/\phi_2}$.



$$\omega_1 \neq \omega_2 \Rightarrow P_{av} = P_{av1} + P_{av2} = \frac{1}{2}R\left[|I_1|^2 + |I_2|^2\right]$$

$$\omega_1 = \omega_2 \Rightarrow P_{av} = \frac{1}{2}R|I|^2 = \frac{1}{2}R|I_1 + I_2|^2 = \frac{1}{2}R(I_1 + I_2)(I_1 + I_2)^* = \frac{1}{2}R\left[|I_1|^2 + |I_2|^2 + 2\Re\{I_1 I_2^*\}\right]$$

$$\omega_1 = \omega_2 \Rightarrow P_{av} = \frac{1}{2}R\left[|I_1|^2 + |I_2|^2 + 2I_1 I_2 \cos(\phi_1 - \phi_2)\right] \Rightarrow \phi_1 - \phi_2 = \pm 90^\circ$$

Effective Values

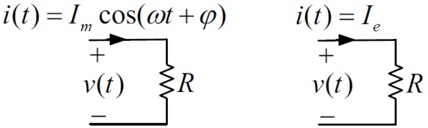


Figure: Effective value is equivalent to root mean square (RMS) value.

- Average power: $P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{R}{T}\int_0^T i^2(t)dt = \frac{1}{2}RI_m^2$
- Effective power: $P = RI_e^2$
- Effective current: $P = P_{av} \Rightarrow I_e = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} = \frac{I_m}{\sqrt{2}}$
- Average power: $P_{av} = \frac{1}{2}RI_m^2 = RI_e^2$

Effective Values

Example (Effective value of the sum of two sinusoidal voltages)

The effective value of the sum of two sinusoidal voltages depends on whether the voltages have equal or non-equal frequencies.

$$v(t) = V_{m1}\cos(\omega_1 t + \theta_1) + V_{m2}\cos(\omega_2 t + \theta_2), \quad T = LCM(T_1, T_2)$$

$$V_{av} = V_{dc} = \frac{1}{T}\int_0^T v(t)dt = 0$$

$$V_{rms} = V_e = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} = \begin{cases} \frac{\sqrt{V_{m1}^2+V_{m2}^2}}{\sqrt{2}}, & \omega_1 \neq \omega_2 \\ \frac{V_m}{\sqrt{2}} = \frac{\sqrt{V_{m1}^2+V_{m2}^2+2V_{m1}V_{m2}\cos(\theta_1-\theta_2)}}{\sqrt{2}}, & \omega_1 = \omega_2 \end{cases}$$

Network Function

Network Function

Definition (Network Function)

In a single input LTI circuit with sinusoidal steady state, the network function $H(j\omega) = \frac{Y(j\omega)}{W(j\omega)}$ is defined as the ratio of the phasor $Y(j\omega)$ of a desired response to the phasor $W(j\omega)$ of the sinusoidal input with the frequency ω.

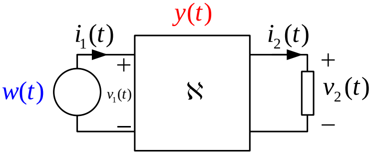


Figure: Definition of network function for a single input LTI circuit.

Types of Network Functions

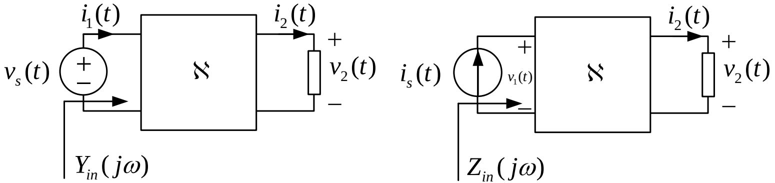
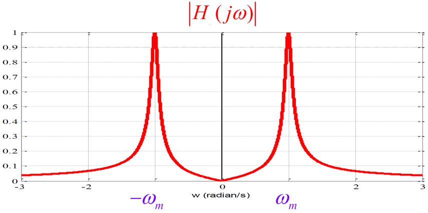


Figure: Different types of network functions.

- Driving point impedance: $H(j\omega) = \frac{V_1}{I_s}$
- Driving point admittance: $H(j\omega) = \frac{I_1}{V_s}$
- Transfer impedance: $H(j\omega) = \frac{V_2}{I_s}$
- Transfer admittance: $H(j\omega) = \frac{I_2}{V_s}$
- Voltage gain: $H(j\omega) = \frac{V_2}{V_s}$
- Current gain: $H(j\omega) = \frac{I_2}{I_s}$

Properties of Network Functions



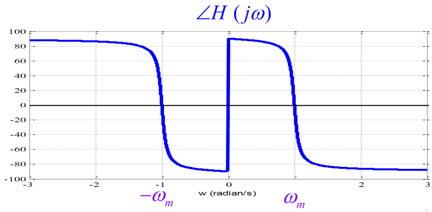


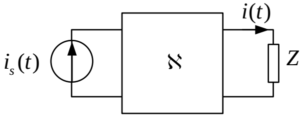
Figure: Due to Hermitian symmetry, the amplitude and phase of a network functions is usually plotted for $\omega \geq 0$. The amplitude is usually plotted in logarithmic scale.

- Network function: $H(j\omega) = |H(j\omega)|\underline{/H(j\omega)} = \Re\{H(j\omega)\} + j\Im\{H(j\omega)\}$
- Hermitain symmetry: $|H(-j\omega)| = |H(j\omega)|, \quad \underline{/H(-j\omega)} = -\underline{/H(j\omega)}$
- Hermitain symmetry: $\Re\{H(-j\omega)\} = \Re\{H(j\omega)\}, \quad \Im\{H(-j\omega)\} = -\Im\{H(j\omega)\}$
- Output response: $Y(j\omega) = H(j\omega)W(j\omega) \Rightarrow \begin{cases} |Y(j\omega)| = |H(j\omega)||W(j\omega)| \\ \underline{/Y(j\omega)} = \underline{/H(j\omega)} + \underline{/W(j\omega)} \end{cases}$
- Output response:

  $w(t) = \sum_k A_k \cos(\omega_k t + \theta_k) \Rightarrow y(t) = \sum_k A_k |H(j\omega_k)| \cos(\omega_k t + \theta_k + \underline{/H(j\omega_k)})$

Example (Network Function)

Network function can be used to calculate the circuit response to different sinusoidal inputs.



$$i_s(t) = 3\cos(t) + \cos(2t + 20^\circ), \quad H(j\omega) = \frac{I}{I_s} = \frac{1 - j\omega}{1 + j\omega}, \quad Z = R = 2\Omega$$

$$i_{s1}(t) = 3\cos(t) \equiv I_{s1} = 3\angle 0^\circ \Rightarrow I_1 = H(j1)I_{s1} = \frac{1 - j1}{1 + j1} 3\angle 0^\circ = 3\angle -90^\circ$$

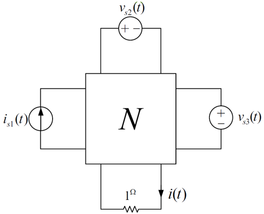
$$i_{s2}(t) = \cos(2t + 20^\circ) \equiv I_{s2} = 1\angle 20^\circ \Rightarrow I_2 = H(j2)I_{s2} = \frac{1 - j2}{1 + j2} 1\angle 20^\circ = 1\angle -107^\circ$$

$$i(t) = i_1(t) + i_2(t) = 3\sin(t) + \cos(2t - 107^\circ)$$

$$\omega_1 \neq \omega_2 \Rightarrow P_{av} = P_{av1} + P_{av2} = \frac{1}{2}R|I_1|^2 + \frac{1}{2}R|I_2|^2 = 3^2 + 1^2 = 10W$$

Example (Network Function)

In a multi-input LTI circuit, the impact of each input on a desired response is described by a network function.



$$H_1(j\omega) = \frac{I}{I_{s1}}\Big|_{\substack{V_{s2}=0\\ V_{s3}=0}} = \frac{2+j\omega}{1+j\omega}, H_2(j\omega) = \frac{I}{V_{s2}}\Big|_{\substack{I_{s1}=0\\ V_{s3}=0}} = \frac{3-2j\omega}{1+j\omega}, H_3(j\omega) = \frac{I}{V_{s3}}\Big|_{\substack{I_{s1}=0\\ V_{s2}=0}} = \frac{4+j\omega}{1+j\omega}$$

$$i_{s1}(t) = \cos(t), \quad v_{s2}(t) = 2\cos(t), \quad v_{s3}(t) = 3\cos(t)$$

$$I_1 = H_1(j1)I_{s1} = \frac{2+j1}{1+j1}1\angle 0^\circ, \quad I_2 = H_2(j1)V_{s2} = \frac{3-j2}{1+j1}2\angle 0^\circ, \quad I_3 = H_3(j1)V_{s3} = \frac{4+j1}{1+j1}3\angle 0^\circ$$

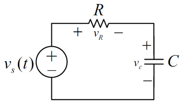
$$I = I_1 + I_2 + I_3 = 10 - 10j = 10\sqrt{2}\angle -45^\circ \Rightarrow i(t) = 10\sqrt{2}\cos(t-45^\circ) = 10\cos(t) + 10\sin(t)$$

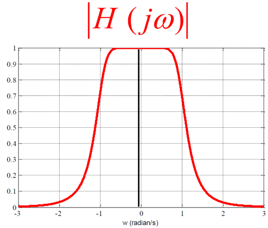
$$P_{av} = \frac{1}{2}R|I|^2 = 100$$

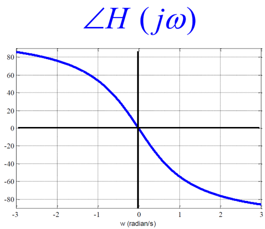
Network Function

Example (Network Function)

Phasor analysis can be used to determine network functions.







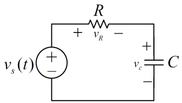
$$H(j\omega) = \frac{V_c}{V_s} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + jRC\omega}$$

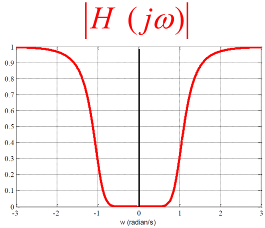
$$|H(j\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}, \quad \underline{/H(j\omega)} = -\tan^{-1}(RC\omega)$$

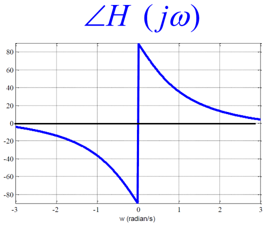
Network Function

Example (Network Function)

Phasor analysis can be used to determine network functions.







$$H(j\omega) = \frac{V_R}{V_s} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{jRC\omega}{1 + jRC\omega}$$

$$|H(j\omega)| = \frac{RC|\omega|}{\sqrt{1 + (RC\omega)^2}}, \quad \angle H(j\omega) = \pm 90^\circ - \tan^{-1}(RC\omega)$$

Filters

Filters

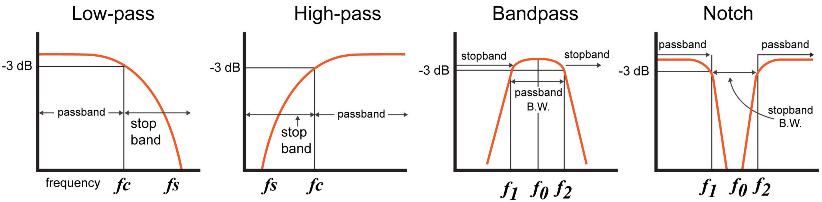


Figure: Frequency response of different filters. Note that the horizontal axis may be frequency $f = \frac{\omega}{2\pi}$ or angular frequency ω.

- Filter types
  - Lowpass filter (LPF)
  - Highpass filter (HPF)
  - Bandpass filter (BPF)
  - Bandstop filter (Notch)
- 3-dB cut off frequency: $|H(j\omega_c)| = \frac{\max\{|H(j\omega)|\}}{\sqrt{2}}$

Parallel RLC Circuit

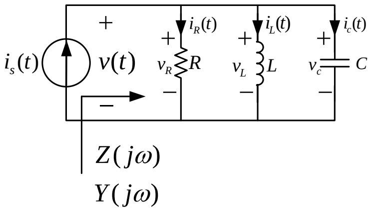
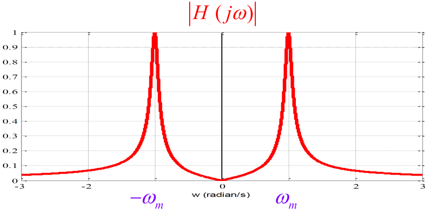


Figure: A Parallel RLC circuit with natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, damping factor $\alpha = \frac{1}{2RC}$, and quality factor $Q = \frac{\omega_0}{2\alpha} = \frac{R\sqrt{C}}{\sqrt{L}}$ can act like a BPF.

- Network function: $H(j\omega) = \frac{V}{I_s} = Z(j\omega) = \frac{1}{\frac{1}{R}+j\left(c\omega - \frac{1}{L\omega}\right)} = \frac{R}{1+jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$
- Frequency response:
  $|H(j\omega)| = \frac{R}{\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}, \quad \angle H(j\omega) = -\tan^{-1}\left(Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})\right)$
- Amplitude response: $20\log_{10}(|H(j\omega)|) = 20\log_{10}\left(\frac{R}{\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}\right)$
- Phase response: $\angle H(j\omega) = -\tan^{-1}\left(Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})\right)$



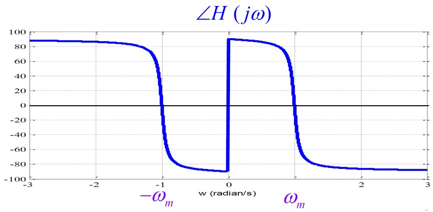


Figure: Typical frequency response of a parallel RLC circuit.

- 3-dB cut-off frequencies: $|H(j\omega)| = \frac{|H|_{max}}{\sqrt{2}} \Rightarrow \frac{R}{\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}} = \frac{R}{\sqrt{2}}$
- 3-dB cut-off frequencies: $\omega_{1,2} = \omega_0\left(\frac{\pm 1}{2Q} + \sqrt{1+\frac{1}{4Q^2}}\right)$
- 3-dB bandwidth: $\Delta\omega = |\omega_2 - \omega_1| = \frac{\omega_0}{Q} = 2\alpha$
- Quality factor: $Q = \frac{\omega_0}{2\alpha} = \frac{\omega_0}{\Delta\omega}$
- Central frequency: $\omega_0 = \sqrt{\omega_1\omega_2} = \frac{1}{\sqrt{LC}}$

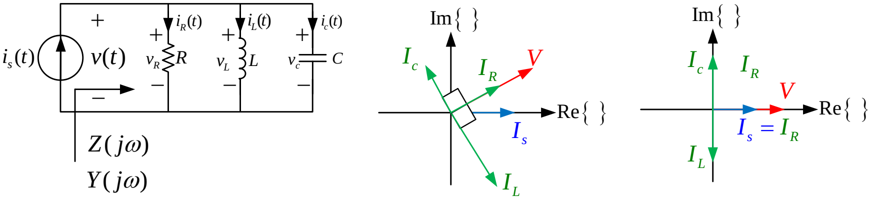


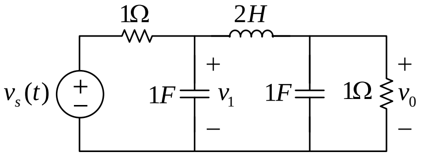
Figure: Resonance in a parallel RLC circuit.

- Network function: $H(j\omega) = \frac{1}{\frac{1}{R}+j\left(c\omega-\frac{1}{L\omega}\right)} = \frac{R}{1+jQ\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)}$
- Resonance frequency: $c\omega - \frac{1}{L\omega} = 0 \Rightarrow \omega_r = \frac{1}{\sqrt{LC}}$

Filtering Response

Example (Third-order lowpass filter)

Filtering response can be investigated using phasor analysis.



$$\frac{V_1 - V_s}{1} + \frac{V_1}{1/j\omega} + \frac{V_1 - V_o}{j2\omega} = 0, \quad \frac{V_o}{1} + \frac{V_o}{1/j\omega} + \frac{V_o - V_1}{j2\omega} = 0$$

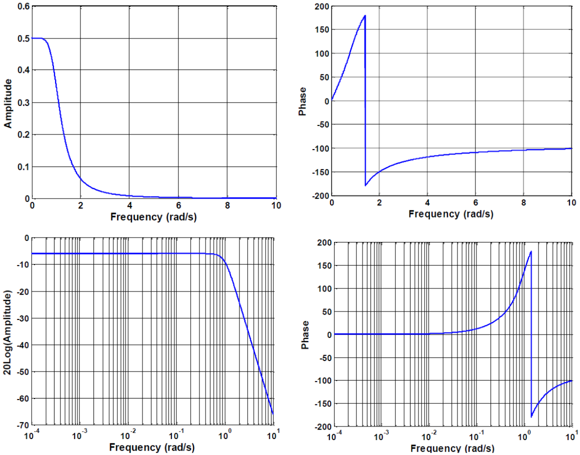
$$H(j\omega) = \frac{V_o}{V_s} = \frac{1}{2(1-2\omega^2) + j2\omega(\omega^2 - 2)}$$

$$|H(j\omega)| = \frac{1}{2\sqrt{1+\omega^6}}, \quad \angle H(j\omega) = -\tan^{-1}\left(\frac{\omega(\omega^2-2)}{1-2\omega^2}\right)$$

$$|H(j\omega)| = \frac{1}{\sqrt{2}}\frac{1}{2} \Rightarrow \omega_c = 1$$

Example (Third-order lowpass filter (cont.))

Filtering response can be investigated using phasor analysis.



Filtering Response

Example (Third-order lowpass filter (cont.))

Filtering response can be investigated using phasor analysis.

$$v_s(t) = 2 + 4\cos(t) - 6\sin(\frac{\sqrt{2}}{2}t + 20^\circ) + 3\cos(\sqrt{2}t - 40^\circ) + 8\cos(10t + 30^\circ)$$

$$v_o(t) = 2|H(j0)|\cos(0t + \underline{/H(j0)}) + 4|H(j1)|\cos(t + \underline{/H(j1)})$$
$$- 6|H(j\frac{\sqrt{2}}{2})|\sin(\frac{\sqrt{2}}{2}t + 20^\circ + \underline{/H(j\tfrac{\sqrt{2}}{2})}) + 3|H(j\sqrt{2})|\cos(\sqrt{2}t - 40^\circ + \underline{/H(j\sqrt{2})})$$
$$+ 8|H(j10)|\cos(10t + 30^\circ + \underline{/H(j10)})$$

$$v_o(t) = 2\frac{1}{2}\cos(0t + 0^\circ) + 4\frac{1}{2\sqrt{2}}\cos(t + \frac{3\pi}{4}) - 6\frac{\sqrt{2}}{3}\sin(\frac{\sqrt{2}}{2}t + 20^\circ + \frac{\pi}{2})$$
$$+ 3\frac{1}{6}\cos(\sqrt{2}t - 40^\circ + \pi) + 8\frac{1}{2000}\cos(10t + 30^\circ - 91.2^\circ)$$

The End