MATHEMATICAL QUESTIONS

Question 1

For the lumped circuit in Fig. 1.

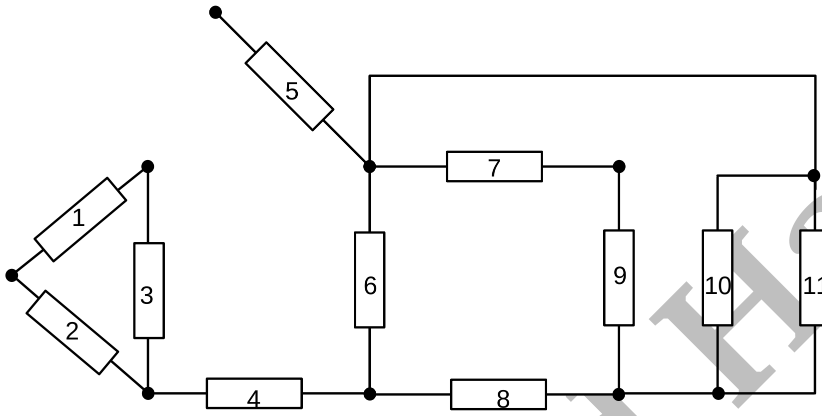
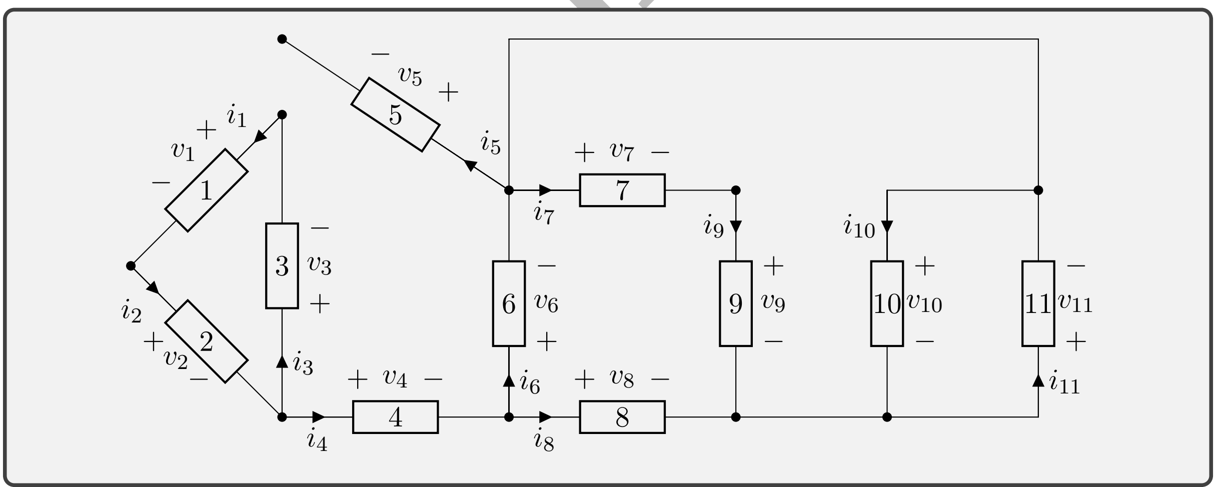


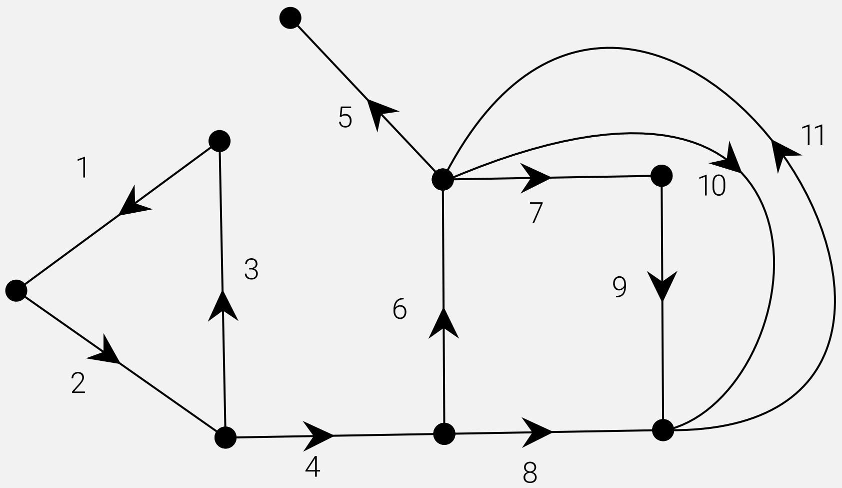
Figure 1: A sample lumped circuit.

(a) Choose a suitable voltage polarity and current direction for each element such that the passive sign convention is held.



(b) Draw the equivalent circuit graph.

Question 1 continued on next page. . .



(c) Determine the number of nodes, branches, and meshes.

There are $n = 8$ nodes, $b = 11$ branches, and $l = 4$ meshes.

(d) Write node KCL equations.

$$
\begin{gathered}
i_1 - i_3 = 0 \\
-i_1 + i_2 = 0 \\
-i_2 + i_3 + i_4 = 0 \\
-i_4 + i_6 + i_8 = 0 \\
-i_8 - i_9 - i_{10} + i_{11} = 0 \\
-i_{11} + i_{10} + i_7 + i_5 - i_6 = 0 \\
-i_7 + i_9 = 0 \\
-i_5 = 0
\end{gathered}
$$

(e) Write mesh KVL equations.

$$
\begin{gathered}
-v_8 + v_6 + v_7 + v_9 = 0 \\
-v_9 - v_7 + v_{10} = 0 \\
-v_{10} - v_{11} = 0 \\
-v_2 - v_1 - v_3 = 0
\end{gathered}
$$

(f) Introduce a set of linearly independent KCLs that has the largest possible cardinality.

The set with the largest cardinality includes $n-1=7$ equations. To determine the set, we can consider the equations in part (d) and omit one of the equations. So,

$$
\begin{gathered}
i_1 - i_3 = 0 \\
-i_1 + i_2 = 0 \\
-i_2 + i_3 + i_4 = 0 \\
-i_4 + i_6 + i_8 = 0 \\
-i_8 - i_9 - i_{10} + i_{11} = 0 \\
-i_{11} + i_{10} + i_7 + i_5 - i_6 = 0 \\
-i_7 + i_9 = 0
\end{gathered}
$$

we can see that if we calculate the summation of the first 7 equations in part (d), it will be equal to the 8th equation, which means it is dependent to the equations listed above.

(g) Introduce a set of linearly independent KVLs that has the largest possible cardinality.

The set with the largest cardinality includes $l=4$ equations. To determine the set, we can consider the mesh equations in part (e). So,

$$
\begin{gathered}
-v_8 + v_6 + v_7 + v_9 = 0 \\
-v_9 - v_7 + v_{10} = 0 \\
-v_{10} - v_{11} = 0 \\
-v_2 - v_1 - v_3 = 0
\end{gathered}
$$

(h) Introduce a set of linearly independent currents that has the largest possible cardinality.

The set with the largest cardinality include $l=4$ currents. i_1, i_9, i_{10}, and i_{11} can be considered as the 4 independent currents.

$$
\begin{gathered}
i_1 = i_1 \\
i_2 = i_1 \\
i_3 = i_1 \\
i_4 = -i_3 + i_2 = -i_1 + i_1 - 1 = 0 \\
i_7 = i_9 \\
i_8 = -i_9 - i_{10} + i_{11} \\
i_6 = i_4 - i_8 = 0 + i_9 + i_{10} - i_{11} \\
i_5 = i_6 + i_{11} - i_{10} - i_7 = i_9 + i_{10} - i_{11} + i_{11} - i_{10} - i_9 = 0 \\
i_9 = i_9 \\
i_{10} = i_{10} \\
i_{11} = i_{11}
\end{gathered}
$$

Question 1 continued on next page. . .

(i) Introduce a set of linearly independent voltages that has the largest possible cardinality.

The set with the largest cardinality includes $n - 1 = 7$ voltages. $v_2, v_3, v_4, v_5, v_6, v_7$, and v_8 can be considered as the 7 independent voltages.

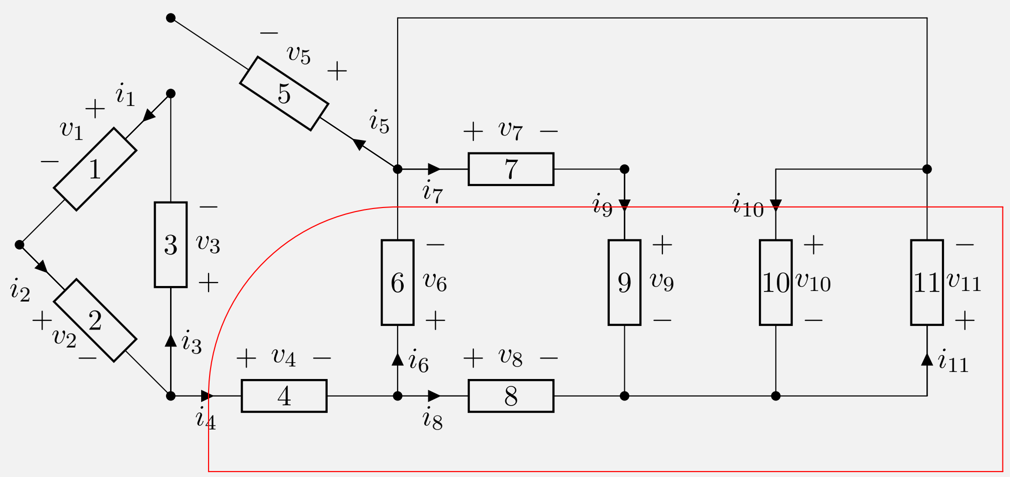
$$
\begin{aligned}
v_1 &= -v_3 - v_2 \\
v_2 &= v_2 \\
v_3 &= v_3 \\
v_4 &= v_4 \\
v_5 &= v_5 \\
v_6 &= v_6 \\
v_7 &= v_7 \\
v_8 &= v_8 \\
v_9 &= v_8 - v_6 - v_7 \\
v_{10} &= v_8 - v_6 \\
v_{11} &= -v_{10} = -v_8 + v_6
\end{aligned}
$$

(j) Write a KCL for the left node of branch 5.

$-i_5 = 0$

(k) Write a KCL for a Gaussian surface crossing branch 4.

The Gaussian surface can considered as



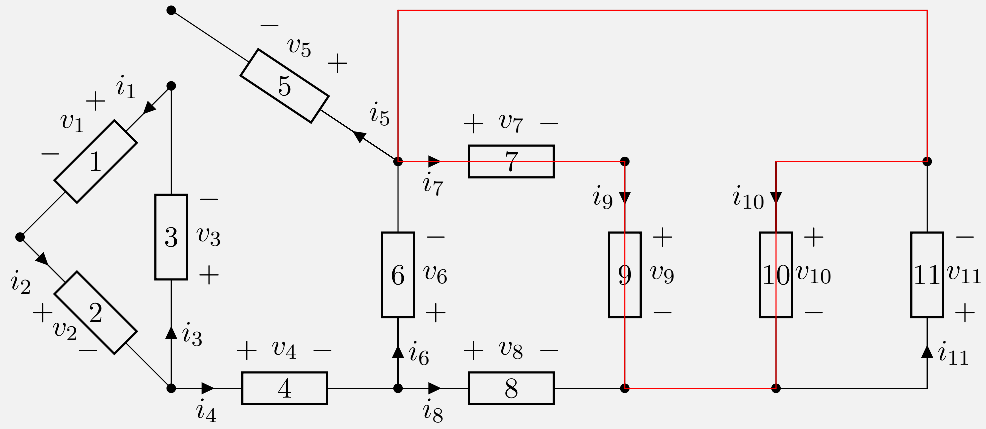
The corresponding KCL equation is

$$-i_4 + i_6 - i_9 - i_{10} + i_{11} = 0$$

(l) Write a KVL for a closed chain passing over branches 9 *and* 10.

The closed chain can considered as

Question 1 continued on next page...



The corresponding KVL equation is

$$-v_9 - v_7 + v_{10} = 0$$

(m) Write a KVL for a closed chain passing over branch 5.

No KVL can be written because there is no closed path including branch 5.

(n) Verify that Tellegen's theorem is held for this circuit.

We should verify that $\sum_{k=1}^{11} v_k i_k = 0$. Using the equations in part (h),
$v_1 i_1 + v_2 i_2 + v_3 i_3 + v_4 i_4 + v_5 i_5 + v_6 i_6 + v_7 i_7 + v_8 i_8 + v_9 i_9 + v_{10} i_{10} + v_{11} i_{11} =$
$v_1 i_1 + v_2 i_1 + v_3 i_1 + v_4 * 0 + v_5 * 0 + v_6(i_9 + i_{10} - i_{11}) + v_7 i_9 + v_8(-i_9 - i_{10} + i_{11}) + v_9 i_9 +$
$v_{10} i_{10} + v_{11} i_{11} =$
$i_1(v_1 + v_2 + v_3) + i_9(v_6 + v_7 - v_8 + v_9) + i_{10}(v_6 - v_8 + v_{10}) + i_{11}(-v_6 + v_8 + v_{11}) = 0$
where, all the parentheses include voltages which form a closed loop, so all of them equal zero based on KVL.

Question 2

An FM receiver is connected to its antenna by a piece of cable 2 m long. Considering that the receiver is tuned to 100 MHz, can you say that the instantaneous currents at the input of the receiver and at the antenna terminals are equal? If not, for what approximate cable lengths would they be equal?

First wee should find the wavelength:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8 \ m/s}{100 \times 10^6 \ Hz} = 3 \ m$$

The lumped condition is not held for the circuit dimension $l = 2$ m $\lesssim 3$ m. So we have circuit dimension=$2m$ and circuit wavelength= $3m$ and lumped condition is not held. Thus instantaneous currents at the input of the receiver and at the end of the antenna terminals are not equal at each time instant.

Question 2 continued on next page...

In order to have lumped conditions we should have $l \ll \lambda$. So, $x = \frac{1}{10}\lambda = 0.3m$ could be a good choice for the cable length.

Question 3

Consider the lumped circuit of Fig. 2.

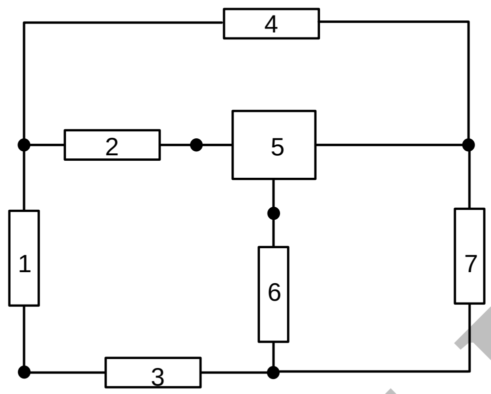
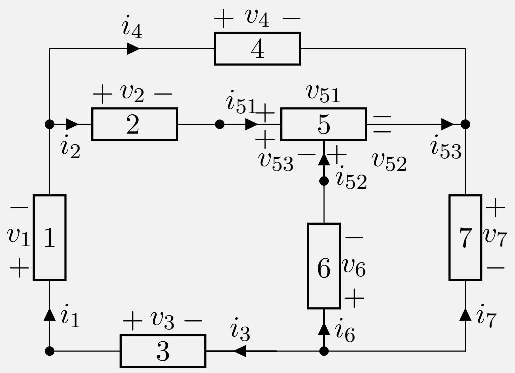


Figure 2: A lumped circuit with a three-terminal element.

(a) Label voltages and currents of different elements such that at least for two elements the passive sign convention is violated.



(b) Write node KCL equations.

$$
\begin{gathered}
i_4 + i_2 - i_1 = 0 \\
-i_4 - i_{53} - i_7 = 0 \\
i_7 + i_6 + i_3 = 0 \\
-i_3 + i_1 = 0 \\
-i_2 + i_{51} = 0 \\
-i_6 + i_{52} = 0
\end{gathered}
$$

Question 3 continued on next page…

(c) Write mesh KVL equations.

$$v_1 + v_2 + v_{53} - v_6 - v_3 = 0$$
$$v_6 + v_{52} + v_7 = 0$$
$$v_4 - v_{51} - v_2 = 0$$
$$-v_{53} + v_{51} - v_{52} = 0$$

(d) Verify the integrity of Tellegen's theorem for the circuit.

$$\sum_{k=1}^{7} v_k i_k = v_1 i_1 + v_2 i_2 + v_3 i_3 + v_4 i_4 + v_5 i_5 + v_6 i_6 + v_7 i_7$$
$$= v_1 i_1 + v_2 i_2 + v_3 * (-i_3) + v_4 i_4 + v_{51} i_{51} + v_{52} i_{52} + v_6 i_6 + v_7 * (-i_7)$$
$$= v_1(i_3) + v_2 i_2 - v_3 i_3 + v_4(i_3 - i_2) + v_{51}(i_2) + v_{52}(-i_3 - i_7) + v_6(-i_3 - i_7) - v_7 i_7$$
$$= i_2(v_2 - v_4 + v_{51}) + i_3(v_1 - v_3 + v_4 - v_{52} - v_6) + i_7(-v_{52} - v_6 - v_7) = 0$$

, which coincides with the result expected from Tellegen's theorem. Note that in the last line, all the parentheses include voltages which form a closed loop, so all of them equal zero based on KVL.

Question 4

Consider the lumped circuit of Fig.3.

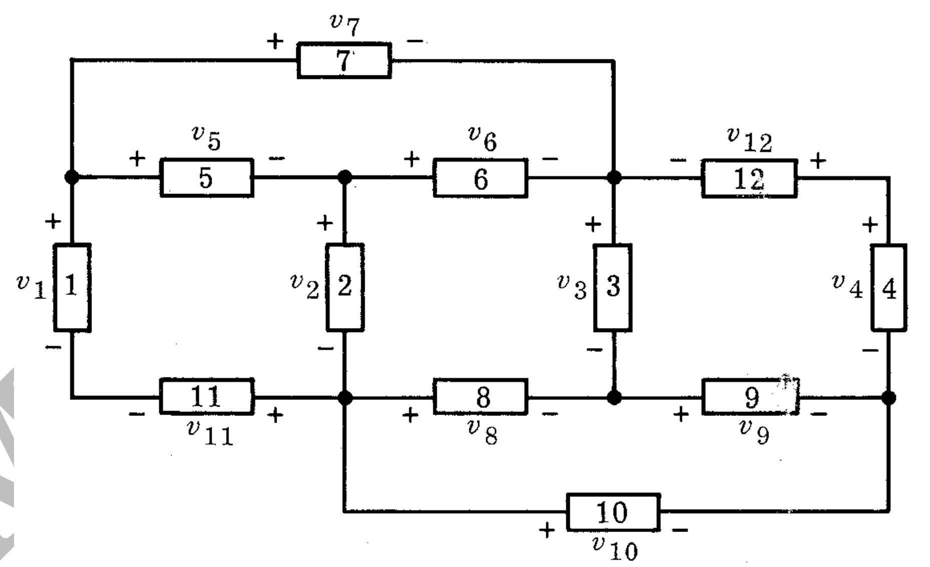


Figure 3: A planar lumped circuit with 12 elements.

(a) Assume that the voltages $v_1 = 10$, $v_2 = 5$, $v_4 = -3$, $v_6 = 2$, $v_7 = -3$, and $v_{12} = 8$ V are given. Determine as many branch voltages as possible.

By a KVL for branches 5,6,7

$$v_5 + v_6 = v_7$$

so,

$$v_5 = (-3) - 2 = -5$$

Writing KVL for branches 1,2,5,11

$$v_{11} + v_2 + v_5 = v_1$$

so,

$$v_{11} = 10 - 5 - (-5) = 10$$

Finally, A KVL for branches 2,6,12,4,10

$$v_2 + v_{10} + v_{12} = v_4 + v_6$$

so,

$$v_{10} = (-3) + 2 - 5 - 8 = -14$$

(b) Assume that the passive sign convention is held and let $i_1 = 2$, $i_7 = -5$, $i_4 = 5$, $i_{10} = -3$, and $i_3 = 1$ A. Is it possible to determine the remaining branch currents? Determine as many as you can.

Writing KCL for branches 1,5,7,

$$i_1 + i_5 + i_7 = 0$$

so,

$$i_5 = -(-5) - 2 = 3$$

Writing KCL for branches 4,12,

$$i_4 + i_{12} = 0$$

so,

$$i_{12} = -5$$

Writing KCL for branches 3,6,7,12,

$$i_3 = i_6 + i_7 + i_{12}$$

so,

$$i_6 = 1 - (-5) - (-5) = 11$$

Witing KCL for branches 2,5,6,

$$i_2 + i_6 = i_5$$

so,

$$i_2 = 3 - 11 = -8$$

Writing KCL for branches 4,9,10,

$$i_4 + i_9 + i_{10} = 0$$

Question 4 continued on next page...

so,

$$i_9 = -5 - (-3) = -2$$

Writing KCL for branches 3,8,9,

$$i_3 + i_8 = i_9$$

so,

$$i_8 = -2 - 1 = -3$$

Writing KCL for branches 2,8,10,11,

$$i_8 + i_{10} + i_{11} = i_2$$

so,

$$i_{11} = -8 - (-3) - (-3) = -2$$

(c) *Prove that* $i_1 + i_2 + i_3 + i_4 = 0$ *and* $i_7 + i_6 + i_8 + i_{10} = 0$.

Writing KCL for branches 1,5,7,

$$i_1 = -i_5 - i_7$$

Writing KCL for branches 2,5,6,

$$i_2 = i_5 - i_6$$

Writing KCL for branches 3,6,12,

$$i_3 = i_6 + i_7 + i_{12}$$

Writing KCL for branches 4,12,

$$i_4 = -i_{12}$$

summing the equations above,

$$i_1 + i_2 + i_3 + i_4 = 0$$

Writing KCL for branches 3,6,7,12

$$i_6 + i_7 = i_3 - i_{12}$$

Writing KCL for branches 3,8,9

$$i_8 = i_9 - i_3$$

Writing KCL for branches 4,9,10

$$i_{10} = -i_4 - i_9$$

, which yield,

$$i_6 + i_7 + i_8 + i_{10} = -i_4 - i_{12}$$

Now, Writing KCL for branches 4,12,

$$i_4 = -i_{12}$$

So,

$$i_6 + i_7 + i_8 + i_{10} = -i_4 - (-i_4) = 0$$

(d) Assuming the given voltages and currents in the previous parts, determine as many absorbed powers as possible.

We know currents of all branches and voltages of 1,2,4,5,6,7,10,11,and 12, so we can determine powers of 1,2,4,5,6,7,10,11, and 12 as

$$p_1 = v_1 i_1 = 20$$

$$p_2 = v_2 i_2 = -40$$

$$p_4 = v_4 i_4 = -15$$

$$p_5 = v_5 i_5 = -15$$

$$p_6 = v_6 i_6 = 22$$

$$p_7 = v_7 i_7 = 15$$

$$p_{10} = v_{10} i_{10} = 42$$

$$p_{11} = v_{11} i_{11} = -20$$

$$p_{12} = v_{12} i_{12} = -40$$

SOFTWARE QUESTIONS

Question 5

A directed graph can be represented by its adjacency matrix. In fact, for the graph $G(\mathbf{N} = \{1, 2, \cdots, n\}, \mathbf{E})$ with n nodes, the adjacency matrix is $A_{n\times n} = [a_{ij}]$, where a_{ij} is 1 if $(i, j) \in \mathbf{E}$, and 0 otherwise. Write a MATLAB/Python function that takes the adjacency matrix of a connected planar directed graph, draws the corresponding graph, and determines the number of nodes, branches, and meshes of the graph. For simplicity, assume that there is no parallel edges with the same source and destination nodes.

Question 5 continued on next page...

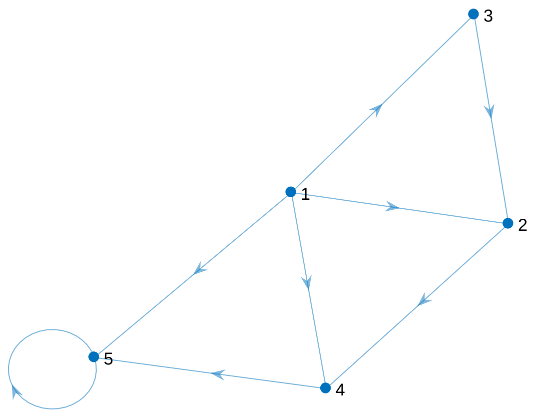


Figure 4: A directed graph with self-loop.

Here is a simple MATLAB function that plots a directed graph.

```
function showGraph(ingraph)

% convert the input adjacency matrix to matlab adjacency matrix
sg = [];
dg = [];
wg = [];
for i=1:size(ingraph,1)
    for j=1:size(ingraph,2)
        if (ingraph(i,j)~=0)
            sg = [sg i];
            dg = [dg j];
            wg = [wg 1];
        end
    end
end
G = digraph(sg,dg,wg);

% plot graph
p = plot(G);
set(gca,'XTick',[],'YTick',[])

end
```

You may use the following mfile to call the developed function and see its results.

```
clear all
clc

% sample circuit graph
adjmat = [  0 1 1 1 1;
            0 0 0 1 0;
            0 1 0 0 0;
            0 0 0 0 1;
            0 0 0 0 1];

% show the tree
showGraph(adjmat)
```

Sample output of the codes are shown in Fig. 4.

BONUS QUESTIONS

Question 6

Prove that in a connected planar graph, the number of meshes is given by $l = b - n + 1$, where b and n are the number of branches and nodes of the graph, respectively.

Consider the graph in Fig. 5.a, where $l = 1$. Here it is obvious that $l = b - n + 1$ is true. Next consider a graph which has l meshes, and assume that $l = b - n + 1$ is true. We want to show that $l = b - n + 1$ is still true if the graph is changed to have $l + 1$ meshes. We can increase the number of meshes by 1 through adding a branch between two existing nodes or by adding m branches in series which are connected to the existing graph through $m - 1$ new nodes, as shown in Fig. 5.b. For the new graph with $l + 1$ meshes, $l' = l + 1 = (b + m) - (n + m - 1) + 1 = b - n + 1 + 1$ is still satisfied, because $m - 1$ nodes and m branches have been added, resulting in one additional mesh. Therefore by induction, $l = b - n + 1$ is true in general.

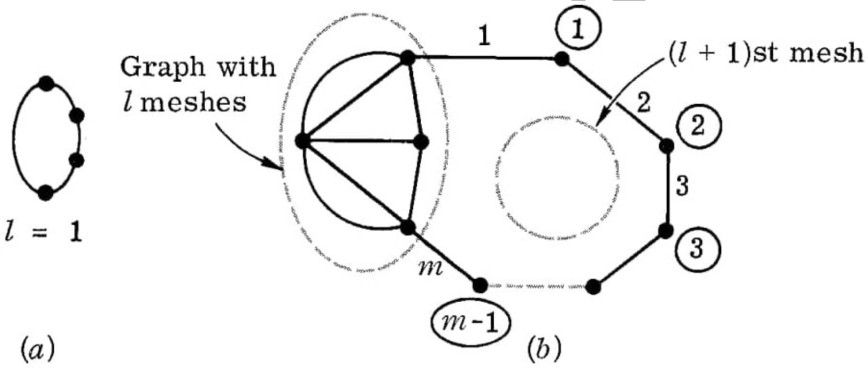


Figure 5: Indication of the proof of $l = b - n + 1$.

Question 7

Return your answers by filling the LaTeX template of the assignment.