MATHEMATICAL QUESTIONS

Question 1

Find the equivalent resistance of the ladder network in Fig. 1.

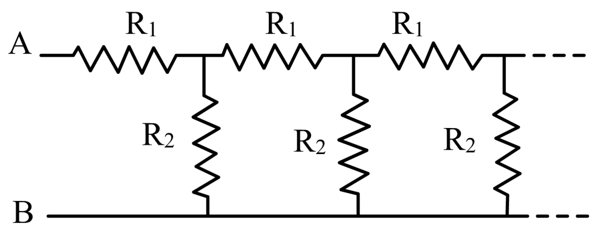


Figure 1: Ladder resistor network.

Let R be the equivalent resistor seen from the terminal point A and B. Since the ladder has infinite length, the same equivalent resistor R is seen from each vertical resistor R_2. Therefore, $R = R_1 + R_2||R = R_1 + R_2R/(R_2 + R)$. This is a second-order equation, whose acceptable solution is

$$R = \frac{R_1 + \sqrt{R_1^2 + 4R_1R_2}}{2}$$

.

Question 2

How are Δ and T resistor networks in Fig. 2 equivalent?

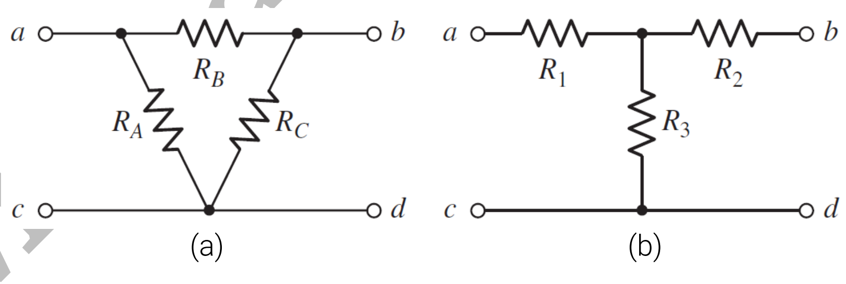
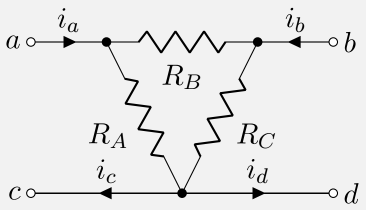
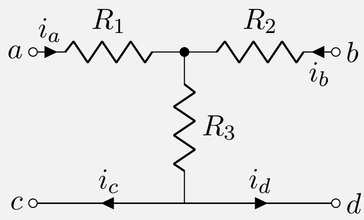


Figure 2: Two well-known equivalent resistor circuits. (a) Δ network. (b) T network.

Question 2 continued on next page...

The ports voltages and currents are labeled as





First solution: The port equations for the Δ network are

$$\begin{cases} i_1 = \frac{v_1}{R_A} + \frac{v_1 - v_2}{R_B} = v_1(\frac{1}{R_A} + \frac{1}{R_B}) - v_2\frac{1}{R_B} \\ i_2 = \frac{v_2}{R_C} + \frac{v_2 - v_1}{R_B} = -v_1\frac{1}{R_B} + v_2(\frac{1}{R_C} + \frac{1}{R_B}) \end{cases} \quad (1)$$

The port equations for the T network are

$$\begin{cases} v_1 = R_1 i_1 + R_3(i_1 + i_2) = (R_1 + R_3)i_1 + R_3 i_2 \\ v_2 = R_2 i_2 + R_3(i_1 + i_2) = R_3 i_1 + (R_2 + R_3)i_2 \end{cases}$$

or equivalently,

$$\begin{cases} i_1 = \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}v_1 - \frac{R_3}{R_1R_2+R_1R_3+R_2R_3}v_2 \\ i_2 = -\frac{R_3}{R_1R_2+R_1R_3+R_2R_3}v_1 + \frac{R_1+R_3}{R_1R_2+R_1R_3+R_2R_3}v_2 \end{cases} \quad (2)$$

Since the port equations should be identical, (1) and (2) yield

$$\begin{cases} \frac{1}{R_A} + \frac{1}{R_B} = \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3} \\ \frac{1}{R_B} = \frac{R_3}{R_1R_2+R_1R_3+R_2R_3} \\ \frac{1}{R_C} + \frac{1}{R_B} = \frac{R_1+R_3}{R_1R_2+R_1R_3+R_2R_3} \end{cases}$$

which gives

$$\begin{cases} R_A = \frac{R_1R_2+R_2R_3+R_3R_1}{R_2} \\ R_B = \frac{R_1R_2+R_2R_3+R_3R_1}{R_3} \\ R_C = \frac{R_1R_2+R_2R_3+R_3R_1}{R_1} \end{cases} \quad (3)$$

According to (3), $R_AR_2 = R_BR_3 = R_CR_1$. So,

$$R_A = \frac{R_1R_CR_1/R_A + (R_CR_1/R_A)(R_CR_1/R_B) + R_CR_1R_1/R_B}{R_CR_1/R_A} = R_1\frac{R_B + R_C + R_A}{R_B}$$

and

$$R_1 = \frac{R_AR_B}{R_A + R_B + R_C}$$

Similarly, R_2 and R_3 are found. Finally,

$$\begin{cases} R_1 = \frac{R_AR_B}{R_A+R_B+R_C} \\ R_2 = \frac{R_BR_C}{R_A+R_B+R_C} \\ R_3 = \frac{R_AR_C}{R_A+R_B+R_C} \end{cases} \quad (4)$$

Question 2 continued on next page. . .

second solution: The networks should behave the same for any values of i_a, i_b, i_c, i_d and v_a, v_b, v_c, v_d. Especially, when $i_b = 0$, then $v_c - v_a = i_a R_1 + i_a R_3 = i_a R_A || (R_B + R_C)$, which results in $R_1 + R_3 = R_A || (R_B + R_C)$. Similarly, if $i_a = 0$, $R_2 + R_3 = R_C || (R_B + R_A)$, and if $i_c + i_d = 0$, $R_1 + R_2 = R_B || (R_C + R_A)$. These equations lead to

$$R_1 = \frac{R_A R_B}{R_A + R_B + R_C}$$

,

$$R_2 = \frac{R_B R_C}{R_A + R_B + R_C}$$

, and

$$R_3 = \frac{R_A R_C}{R_A + R_B + R_C}$$

. Further,

$$R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$$

,

$$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

, and

$$R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

.

Question 3

Consider the characteristic curves of the two NTI inductors shown in Fig. 3.

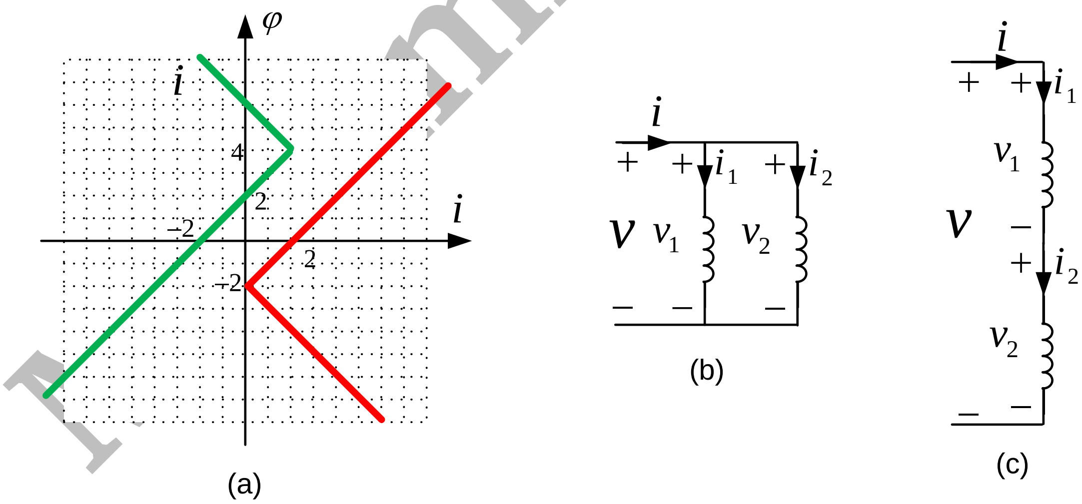
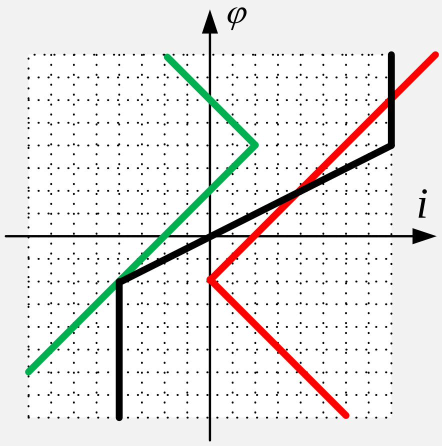


Figure 3: (a) Characteristic curves of two NTI inductors. (b) parallel connection. (c) series connection.

Question 3 continued on next page. . .

(a) Plot the characteristic curve of the parallel connection of the two inductors. Assume that the initial conditions are zero.

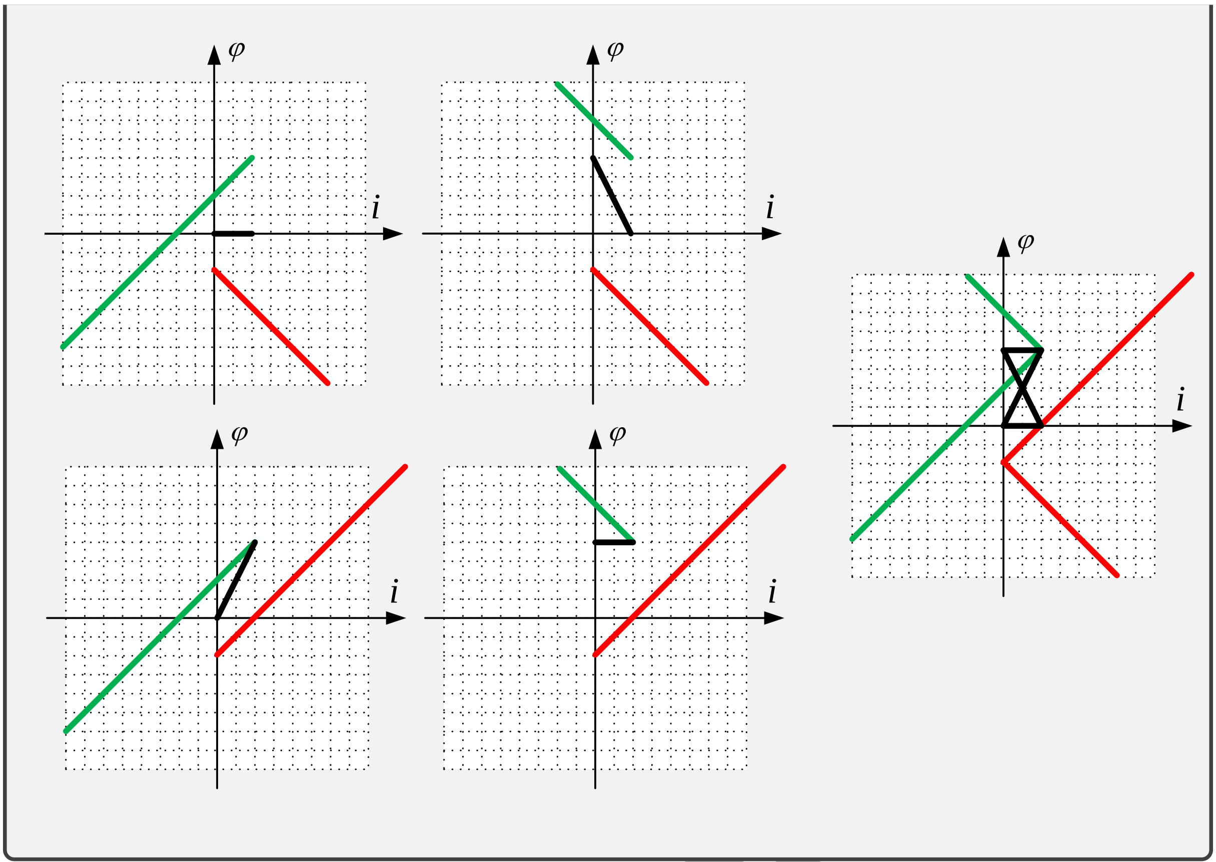
The two NTI inductors are flux-controlled. So, to obtained the characteristic curve of their parallel connection, we should sum currents at the same flux value.



(b) Plot the characteristic curve of the series connection of the two inductors. Assume that the initial conditions are zero.

At first note that the current flowing into the series connection should be confined within $[0, 2]$. Any current outside this interval is not acceptable for one of the inductors. Each characteristic curve has two arms over $[0, 2]$. So, there are four cases for the summation of the fluxes at the same current as shown in the following curve.

Question 3 continued on next page. . .



Question 4

Consider the diode circuit shown in Fig.4.

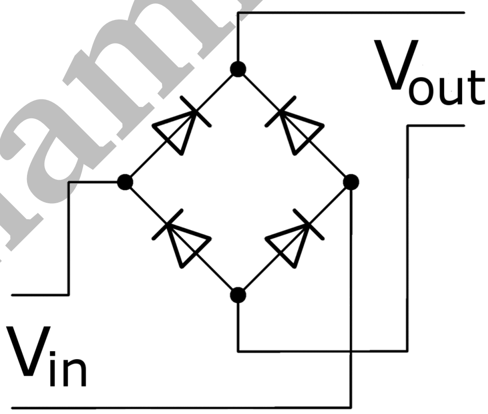


Figure 4: Full diode bridge rectifier.

(a) *Plot the input-output characteristic curve, i.e., V_{out} versus V_{in}.*

If V_{in} is positive, diodes D_1 and D_4 conduct while D_2 and D_3 are reverse biased, so

$$V_{out} = V_{in}$$

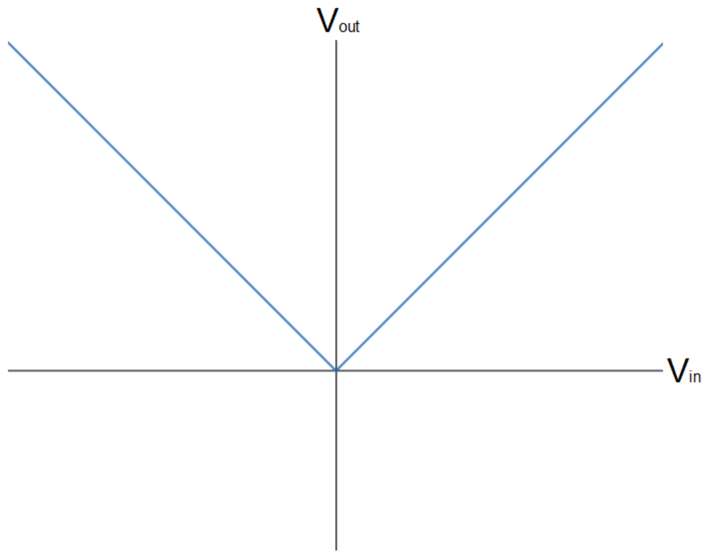
If V_{in} is negative, diodes D_2 and D_3 conduct while D_1 and D_4 are reverse biased, so

$$V_{out} = -V_{in}$$

Hence,

$$V_{out} = |V_{in}|$$

The characteristic curve is shown below.



(b) *Assume that $V_{in} = A\cos(\omega t + \theta)$. Plot V_{out} versus time t.*

According to part (a), if $V_{in} = A\cos(\omega t + \theta)$ is positive, then

$$V_{out} = A\cos(\omega t + \theta)$$

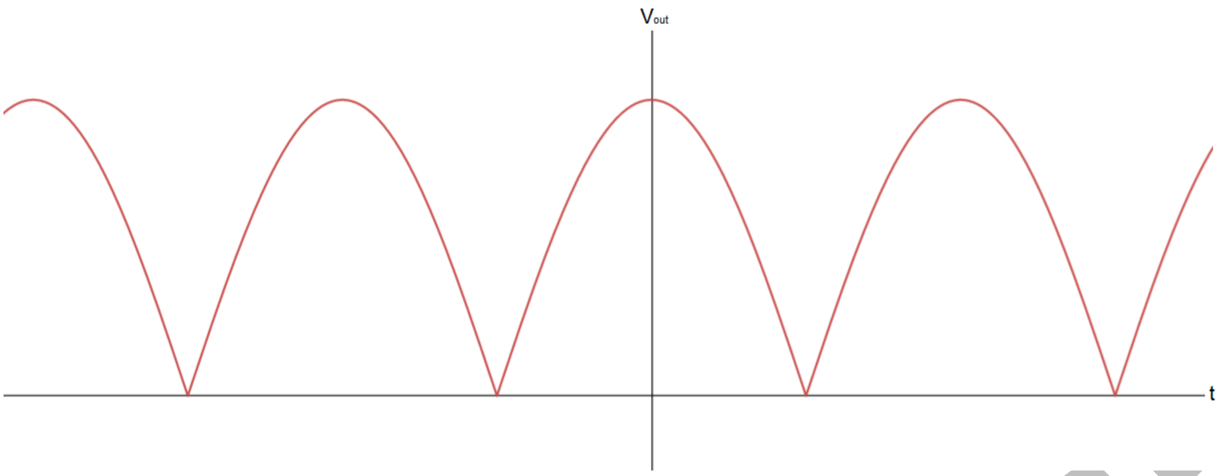
and if $V_{in} = A\cos(\omega t + \theta)$ is negative, then

$$V_{out} = -V_{in} = -A\cos(\omega t + \theta)$$

Hence,

$$V_{out} = |A\cos(\omega t + \theta)|$$

A sample plot of V_{out} is shown below.



(c) How can this circuit be used as a rectifier?

If a resistor load is connected to V_{out}, the circuit keeps the load current flowing solely in one direction. So, it acts like a rectifier.

Question 5

Two LTI capacitors and two LTI inductors are respectively going to be connected in parallel and series at $t = 0$, as shown in Fig. 5.

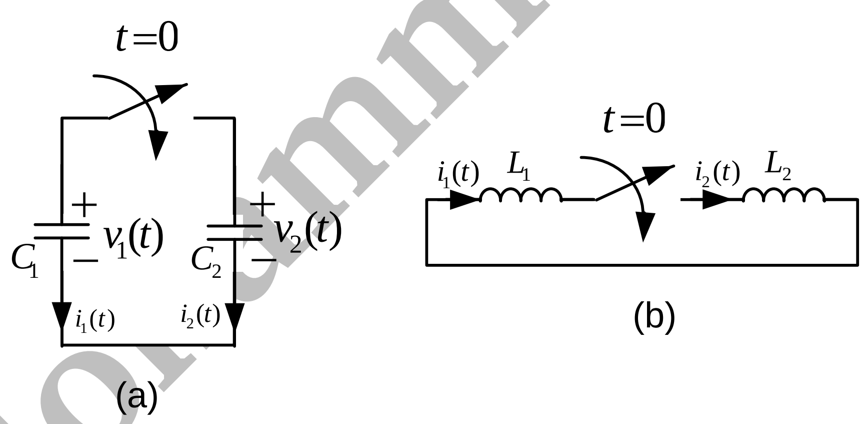


Figure 5: (a) Parallel connection of two LTI capacitors. (b) series connection of two LTI inductors.

(a) Calculate $\epsilon(0^-)$ and $\epsilon(0^+)$, the absolute electrical energy before and after connection, for the capacitors and show that $\epsilon(0^-) \leq \epsilon(0^+)$. Is the energy conserved? Provide enough explanation.

First write a kvl for 0^+ as

$$V_1(0^+) = V_2(0^+) = V(0^+)$$

Because of the conservation of charge

$$Q(0^-) = Q(0^+)$$

So, we have

$$C_1V_1(0^-) + C_2V_2(0^-) = (C_1 + C_2)V(0^+)$$

$$V(0^+) = \frac{C_1V_1(0^-) + C_2V_2(0^-)}{C_1 + C_2}$$

Thus,

$$\epsilon(0^-) = \frac{1}{2}C_1V_1^2(0^-) + \frac{1}{2}C_2V_2^2(0^-)$$

$$\epsilon(0^+) = \frac{1}{2}(C_1 + C_2)V^2(0^+) = \frac{1}{2}\frac{(C_1V_1(0^-) + C_2V_2(0^-))^2}{C_1 + C_2}$$

Finally we have,

$$\epsilon(0^+) \leq \epsilon(0^-)$$

$$\Leftrightarrow \frac{1}{2}\frac{(C_1V_1(0^-) + C_2V_2(0^-))^2}{C_1 + C_2} \leq \frac{1}{2}C_1V_1^2(0^-) + \frac{1}{2}C_2V_2^2(0^-)$$

$$\Leftrightarrow C_1^2V_1^2(0^-)+C_2^2V_2^2(0^-)2+2C_1C_2V_1(0^-)V_1(0^+) \leq C_1(C_1+C_2)V_1^2(0^-)+C_2(C_1+C_2)V_2^2(0^-)$$

$$\Leftrightarrow 2V_1(0^-)V_1(0^+) \leq V_2^2(0^-) + V_1^2(0^-)$$

$$\Leftrightarrow 0 \leq (V_2(0^-) - V_1(0^-))^2$$

The missing energy actually goes into the kinetic energy of conducting charges getting transferred from C_1 to C_2. In reality, the missing energy is dissipated in non-zero resistance of the wires.

(b) Calculate $\epsilon(0^-)$ and $\epsilon(0^+)$, the absolute magnetic energy before and after connection, for the inductors and show that $\epsilon(0^-) \leq \epsilon(0^+)$. Is the energy conserved? Provide enough explanation.

first write kcl for 0^+

$$i_1(0^+) = i_2(0^+) = i(0^+)$$

because of the conservation of flux

$$\phi(0^-) = \phi(0^+)$$

so we have,

$$L_1i_1(0^-) + L_2i_2(0^-) = (L_1 + L_2)i(0^+)$$

$$i(0^+) = \frac{L_1i_1(0^-) + L_2i_2(0^-)}{L_1 + L_2}$$

so,

$$\epsilon(0^-) = \frac{1}{2}L_1i_1^2(0^-) + \frac{1}{2}L_2i_2^2(0^-)$$

$$\epsilon(0^+) = \frac{1}{2}(L_1 + L_2)i^2(0^+) = \frac{1}{2}\frac{(L_1i_1(0^-) + L_2i_2(0^-))^2}{L_1 + L_2}$$

Finally, in a similar way as the previous part, one cane verify that

$$\epsilon(0^+) \leq \epsilon(0^-)$$

The missing energy actually goes into the kinetic energy of conducting fluxes getting transferred from L_1 to L_2. In reality, the missing energy is dissipated in non-zero resistance of the wires.

Question 6

Design a resistive one-port with linear resistors, ideal diodes, and independent sources that has the iv characteristic shown in Fig. 6.

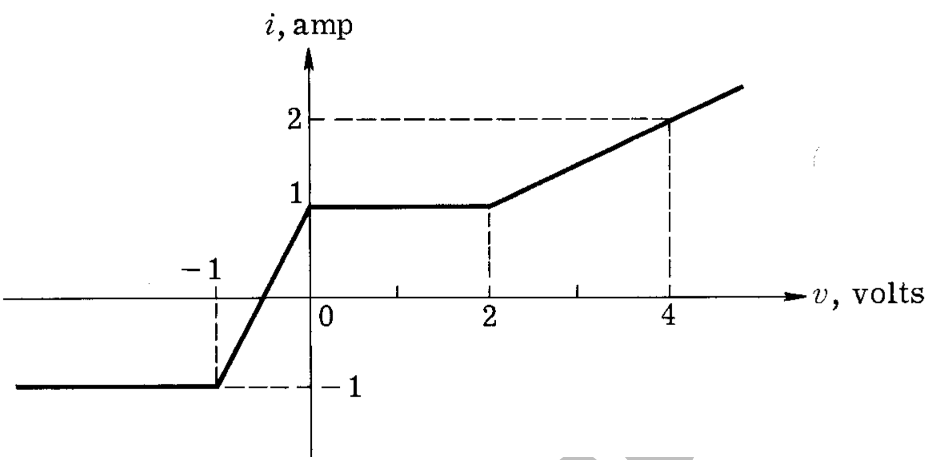


Figure 6: A desired iv characteristic curve.

First, look at the characteristic curve carefully and determine the property of each part.

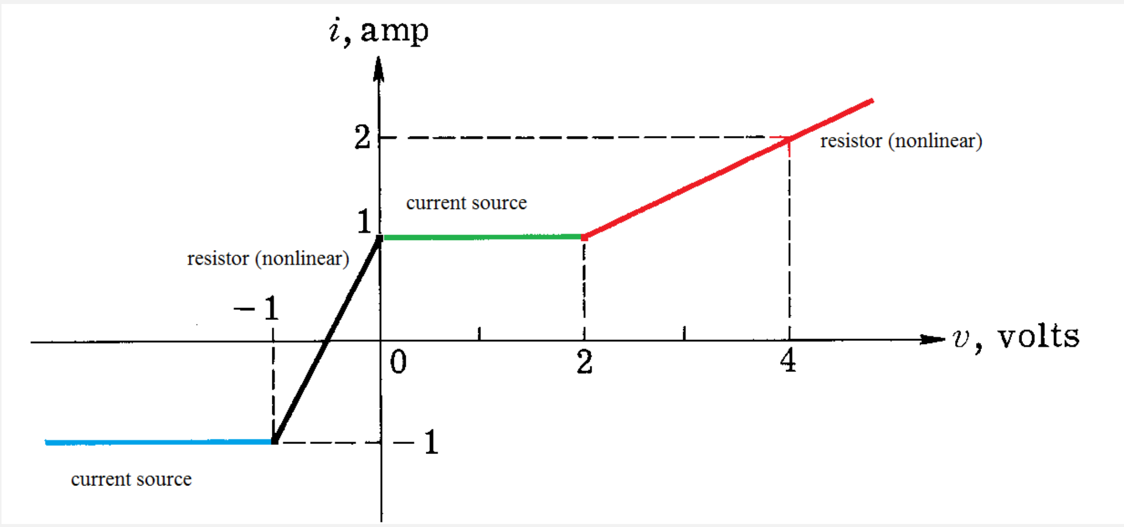


Figure 7: The property of diffrent parts of the desired characteristic curves.

Question 6 continued on next page. . .

Now, we need to synthesize some sub-circuits with the iv characteristic similar to the different parts of the desired characteristic curve. Here, are some suggestions.

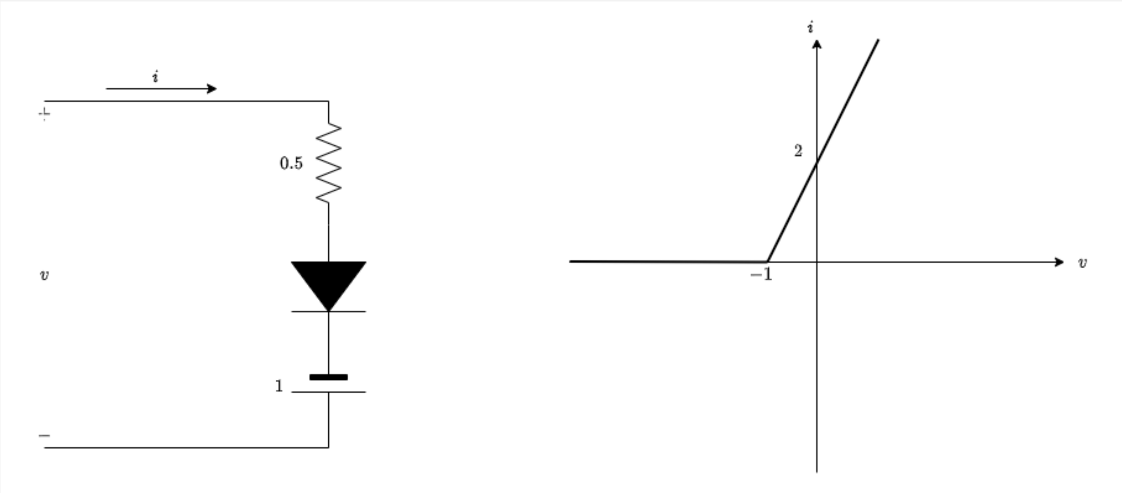


Figure 8:   A series combination of a diode, a resistor, and a voltage source and its corresponding characteristic curve.

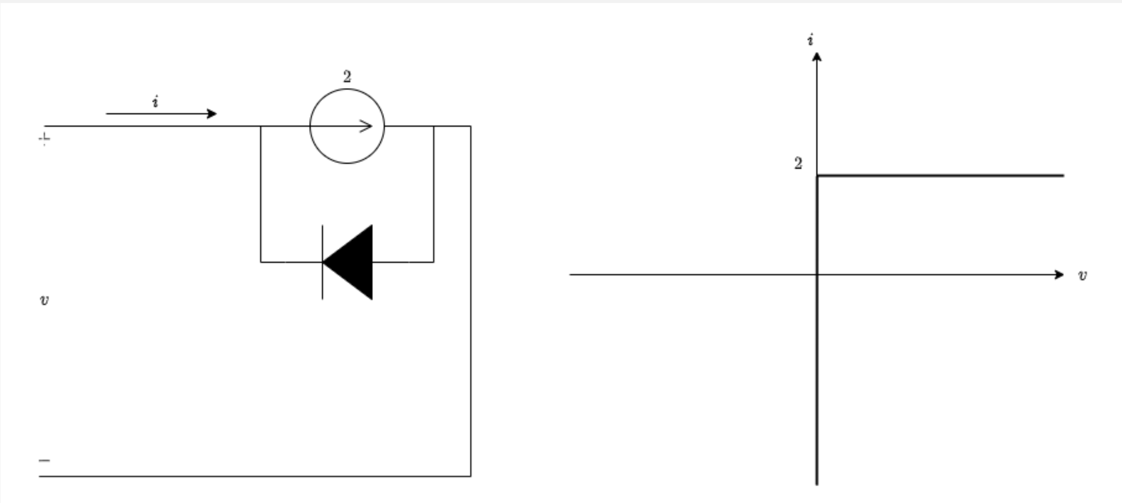


Figure 9:   A parallel combination of a diode and a current source and its corresponding characteristic curve.

Question 6 continued on next page...

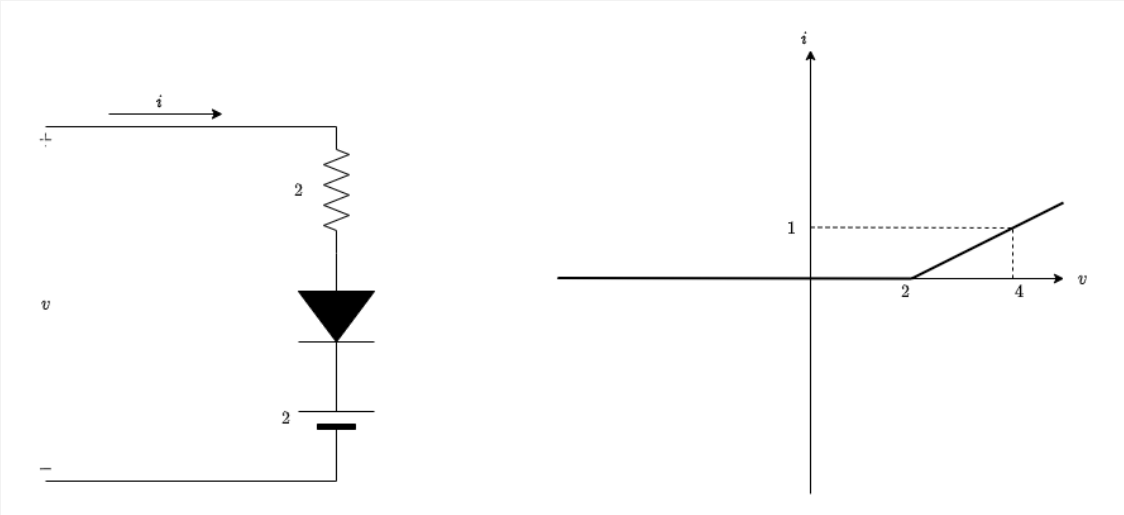


Figure 10: A series combination of a diode, a resistor, and a voltage source and its corresponding characteristic curve.

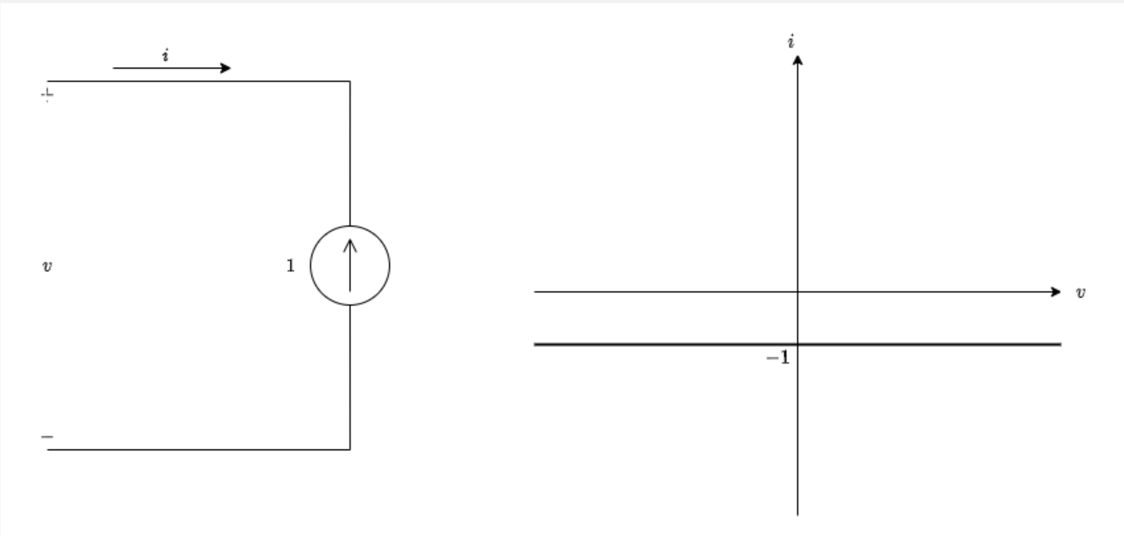


Figure 11: A current source and its corresponding characteristic curve.

Now, we try to get the desired characteristic curve using series and parallel combination of the proposed sub-circuits. Series combination of 8 and 9 results in the characteristic curve

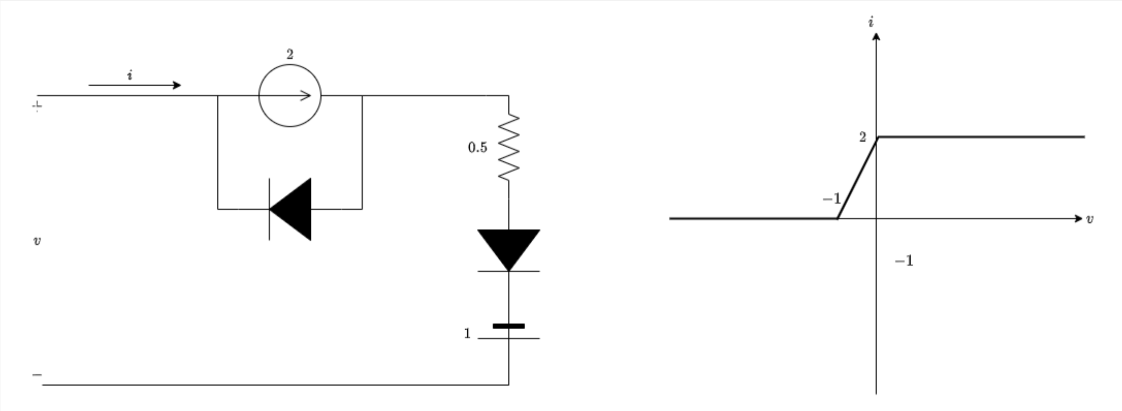


Figure 12: Series combination of 8 and 9 and its corresponding characteristic curve.

while parallel combination of 10 and 11 has the characteristic curve

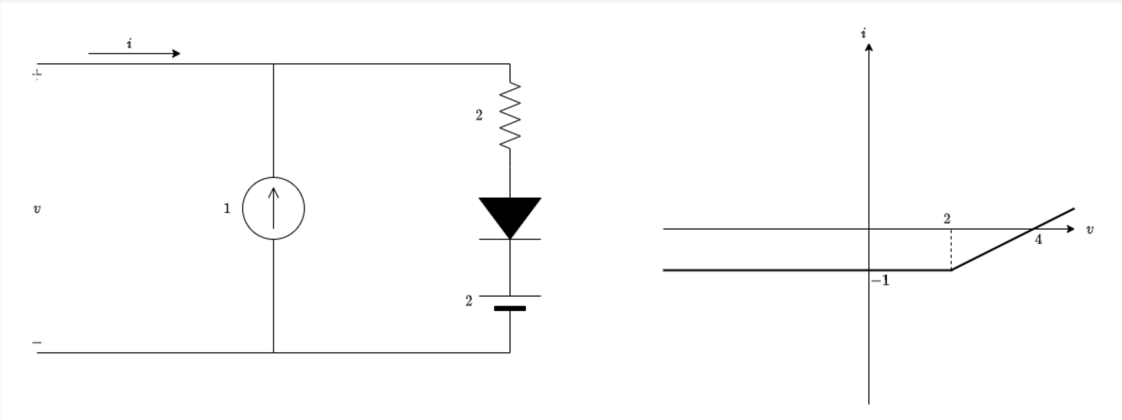


Figure 13: Parallel combination of 10 and 11 and its corresponding characteristic curve.

Finally, parallel combination of 13 and 12 provides the desired characteristic curve.

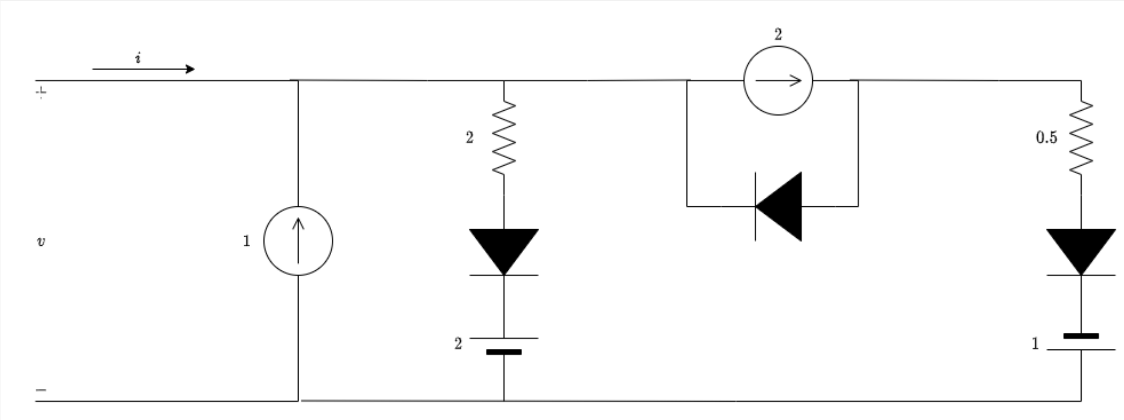


Figure 14: Synthesized circuit.

SOFTWARE QUESTIONS

Question 7

Assume that the characteristic curve of a voltage-controlled NTI capacitor is described by $q = f(v)$. Write a MATLAB/Python function that receives the characteristic curve along with an arbitrary input voltage waveform and generates the corresponding output current waveform. Plot the current waveform for a few sample NTI capacitors.

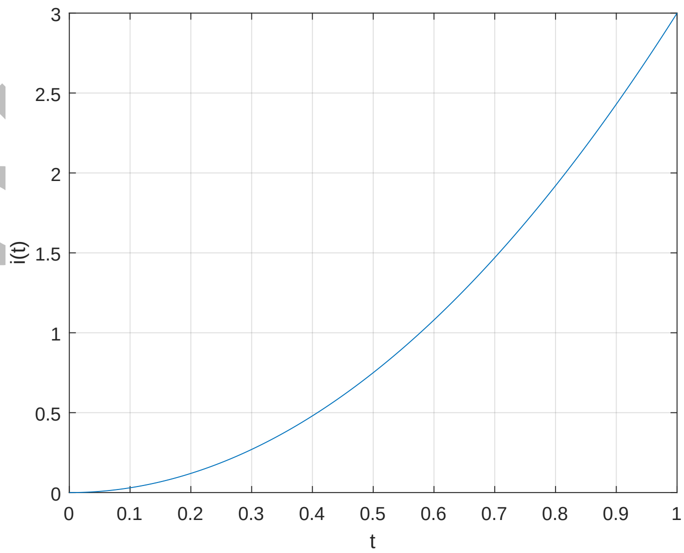


Figure 15: Sample current curve for $q(v) = v^3$ and $v(t) = t$.

Question 7 continued on next page. . .

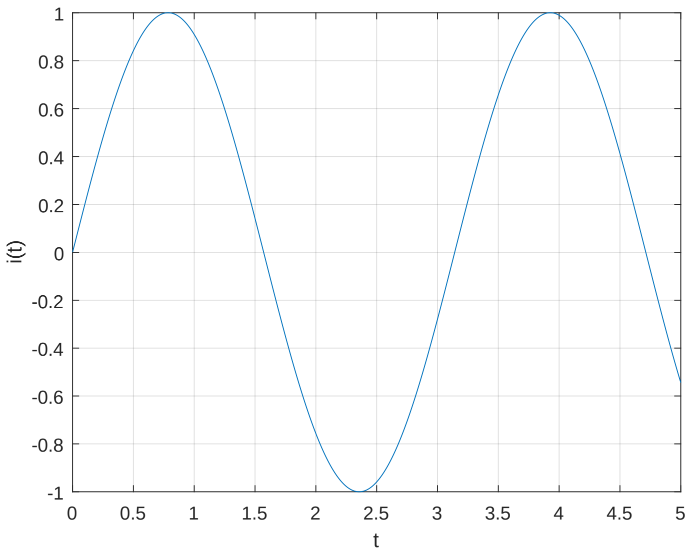


Figure 16: Sample current curve for $q(v) = v^2$ and $v(t) = \sin(t)$.

Here is a sample MATLAB implementation of the desired function.

```
function [i] = i_curve(f, v)

% take the derivative of f with respect to v
syms x;
fpv = matlabFunction(diff(f(x)));

% take the derivative of v with respect to t
syms t;
vpt = matlabFunction(diff(v(t)));

% create the current function
if (nargin(fpv)==0 && nargin(vpt)==0)
    i = @(t) fpv().*vpt();
elseif (nargin(fpv)~=0 && nargin(vpt)==0)
    i = @(t) fpv(v(t)).*vpt();
elseif (nargin(fpv)==0 && nargin(vpt)~=0)
    i = @(t) fpv().*vpt(t);
else
    i = @(t) fpv(v(t)).*vpt(t);
end

end
```

You may use the following mfile to call the developed function and see its results.

```
% clear all
% clc

% sample current curve
f = @(v) v^3;
v = @(t) t;
i = i_curve(f, v);

figure
tval=0:0.001:1;
plot(tval, i(tval))
xlabel('t')
ylabel('i(t)')
box on
grid on
```

Question 7 continued on next page...

```

% sample current curve
f = @(v) v^2;
v = @(t) sin(t);
i = i_curve(f, v);

figure
tval=0:0.001:5;
plot(tval, i(tval))
xlabel('t')
ylabel('i(t)')
box on
grid on
```

Two sample current curves are shown in Figs. 15 and 16.

BONUS QUESTIONS

Question 8

Consider the infinite grid of resistors shown in Fig. 17. All the resistors have a same resistance of R.

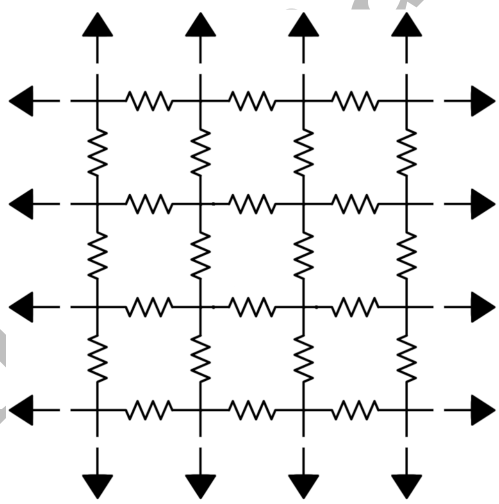


Figure 17: Infinite grid of resistors.

Question 8 continued on next page...

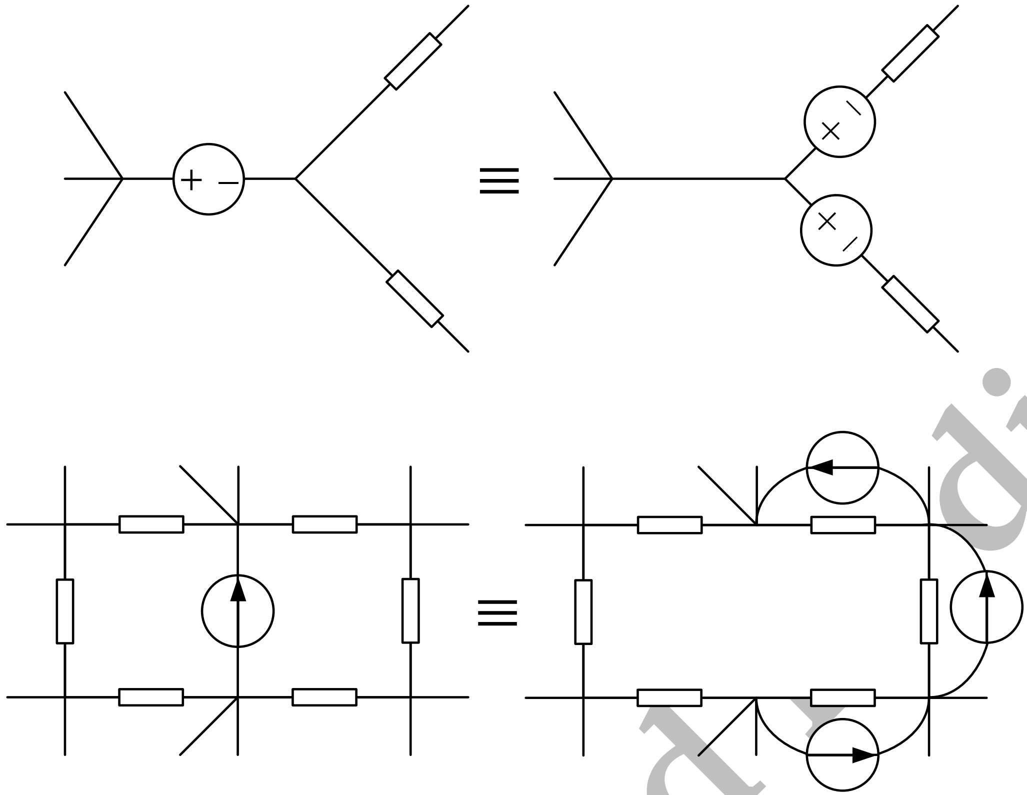


Figure 18: Source transformation.

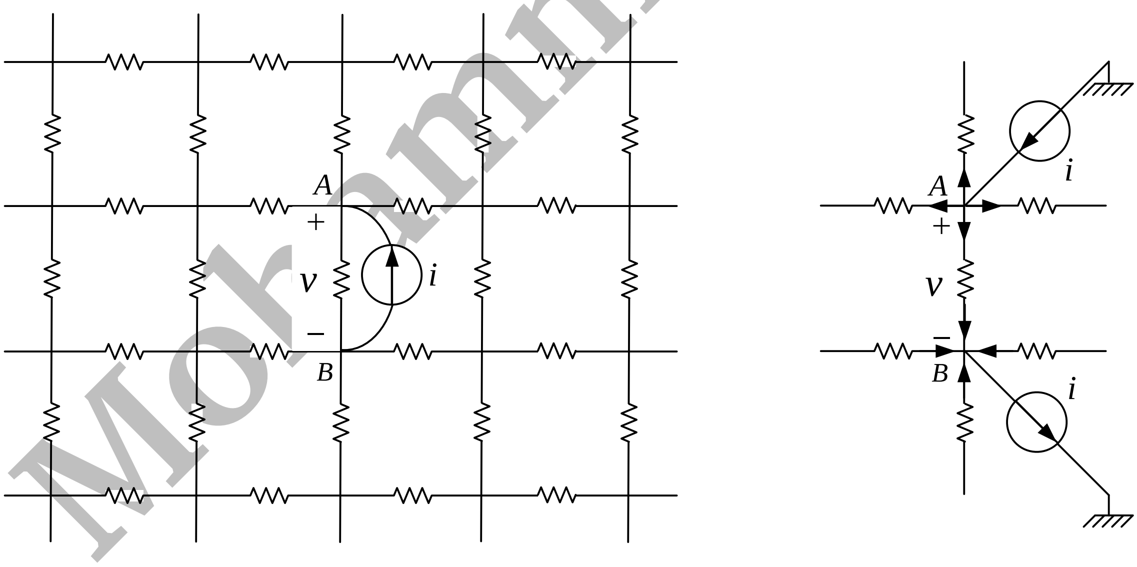


Figure 19: Applying a current source to nodes A nad B to fing the desired resistance value.

Question 8 continued on next page. . .

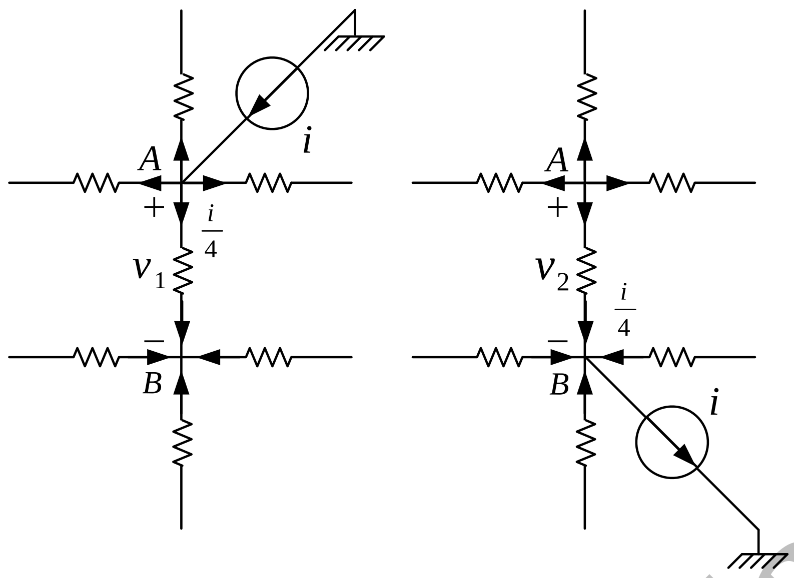


Figure 20: The resistance seen from modes A nad B.

(a) Find the resistance between any two horizontal adjacent nodes $R_{0,1}$.

At first, pay your attention to the generalized version of the source transformation shown in Fig. 18. Since these transformations do not alter any KVL and KCL, they are valid. Now, consider Fig. 19, where a current source connects to nodes A and B. Applying the source transformation, we can break the current source into two currents sources. Note that one side of each current source connects to a location at the end of infinite network. Using superposition, we can consider the effect of each current source separately, as shown in Fig. 20. Due to symmetry, we have

$$v_1 = R\frac{i}{4}, v_2 = R\frac{i}{4} \Rightarrow v = v_1 + v_2 = R\frac{i}{2}$$

So,

$$R_{in} = R_{0,1} = \frac{v}{i} = \frac{R}{2}$$

(b) Find the resistance between any two vertical adjacent nodes $R_{1,0}$.

This part is exactly the same as the previous part due to symmetry of the grid. So, $R_{1,0} = \frac{R}{2}$.

(c) Find the resistance $R_{m,n}$ *between two arbitrary nodes, which are* m *horizontal and* n *vertical steps away from each other. Note that this part may need complex mathematical calculations.*

This part needs complex math. Visit this page for a detailed description of the solution.

Question 9 continued on next page...

Question 9

Return your answers by filling the LaTeX template of the assignment.

EXTRA QUESTIONS

Question 10

Feel free to solve the following questions from the book *"Basic Circuit Theory"* by C. Desoer and E. Kuh.

1. **Chapter 3, question 3.**
2. **Chapter 3, question 6.**
3. **Chapter 3, question 8.**
4. **Chapter 3, question 11.**
5. **Chapter 3, question 12.**
6. **Chapter 3, question 15.**
7. **Chapter 3, question 19.**
8. **Chapter 3, question 20.**