MATHEMATICAL QUESTIONS

Question 1

Find an expression for $v_c(t)$ in Fig. 1 valid for all time t.

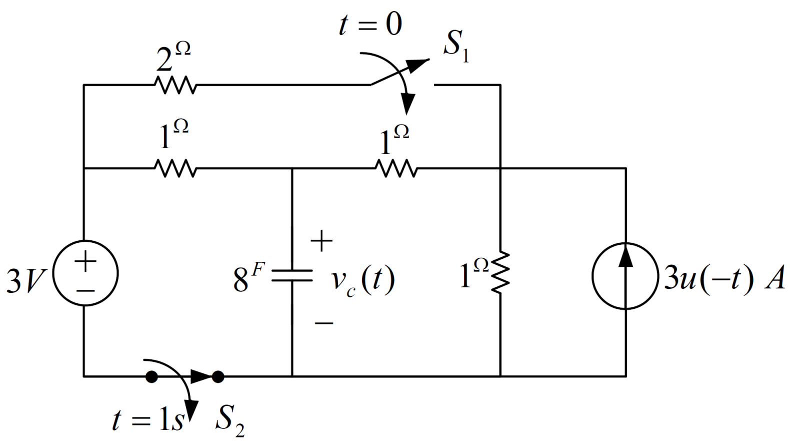
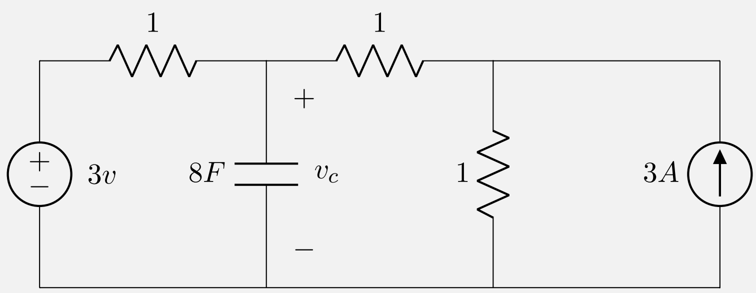


Figure 1: An RC circuit with two switches.

The circuit should be analyzed in different time intervals.

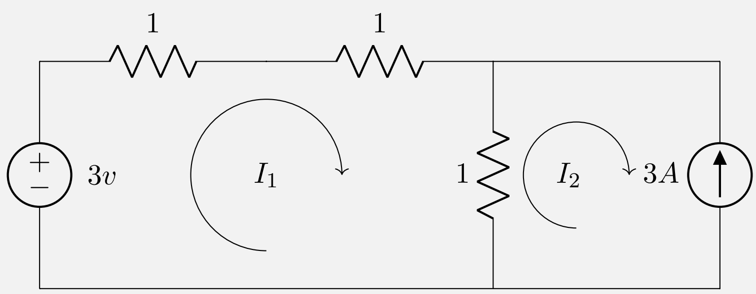
For $t < 0$:

The circuit is shown below.



The DC-driven circuit gets its steady state situation, where capacitor is open, as shown below.

Question 1 continued on next page. . .



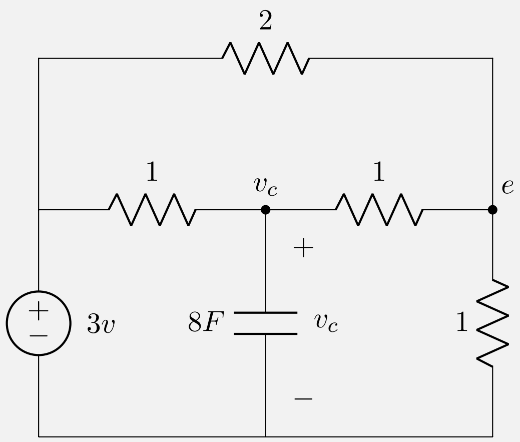
By writing mesh equations, v_c equals

$$\begin{cases} -3 + 2I_1 + (I_1 - I_2) = 0 \\ -I_2 = 3 \end{cases} \Rightarrow \qquad I_1 = 0$$

So v_c is equal to 3 V for $t < 0$ and therefore, $v_c(0^-) = 3$.

For $0 < t < 1$**:**

The circuit is shown below.



Here, the node equations at nodes v_c and e are as follows.

$$\begin{cases} (v_c - 3) + (v_c - e) + 8\frac{d(v_c - 0)}{dt} = 0 \\ \frac{e-3}{2} + (e - v_c) + (e - 0) = 0 \end{cases} \Rightarrow \qquad e = \frac{2v_c + 3}{5}$$

$$\Rightarrow (v_c - 3) + (v_c - e) + 8\frac{d(v_c - 0)}{dt} = (v_c - 3) + (v_c - \frac{2v_c + 3}{5}) + 8\frac{d(v_c - 0)}{dt} = 0$$

So, we obtain the differential equation

$$20\frac{dv_c}{dt} + 4v_c = 9, \quad v_c(0^-) = v_c(0^+) = 3$$

with the homogeneous solution

$$20s + 4 = 0 \Rightarrow s = -0.2 \Rightarrow v_{c_h} = Ae^{-0.2t}$$

, the particular solution

$$v_{c_p} = \frac{9}{4} = 2.25$$

Question 1 continued on next page...

, and the complete solution

$$v_c = v_{c_h} + v_{c_p} = Ae^{-0.2t} + 2.25$$

To find A, we substitute the initial condition in the complete response.

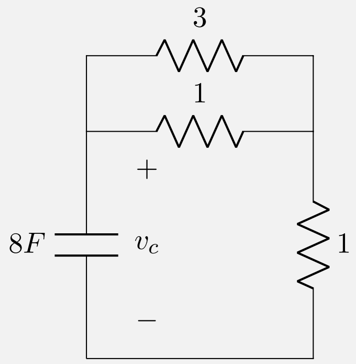
$$v_c(0^+) = 3 = A + 2.25 \Rightarrow A = 0.75$$

So

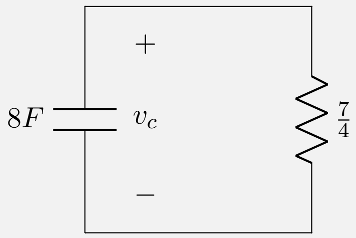
$$v_c = 0.75e^{-0.2t} + 2.25$$

. **For $t > 1$:**

The circuit is shown below:



, which is equivalent to the following circuit



The complete solution to this circuit is as

$$v_c(t) = (v_c(1^+) - v_c(\infty))e^{-\frac{t-1}{\tau}} + v_c(\infty)$$

Due to continuity of the capacitor voltage, $v_c(1) = 0.75e^{(-0.2\times 1)} + 2.25 = 2.864v$. The capacitor will be discharged in $t = \infty$, so $v_c(\infty) = 0$. Further, $\tau = RC = \frac{7}{4} \times 8 = 14$. Finally,

$$v_c = 2.864e^{-\frac{t-1}{14}}$$

.

Overall response:

$$v_c(t) = \begin{cases} 3 & t < 0 \\ 0.75e^{-0.2t} + 2.25 & 0 < t < 1 \\ 2.864e^{-\frac{t-1}{14}} & t > 1 \end{cases}$$

Question 2 continued on next page...

Question 2

Find an expression for $v(t)$ in Fig. 2 valid for all time t.

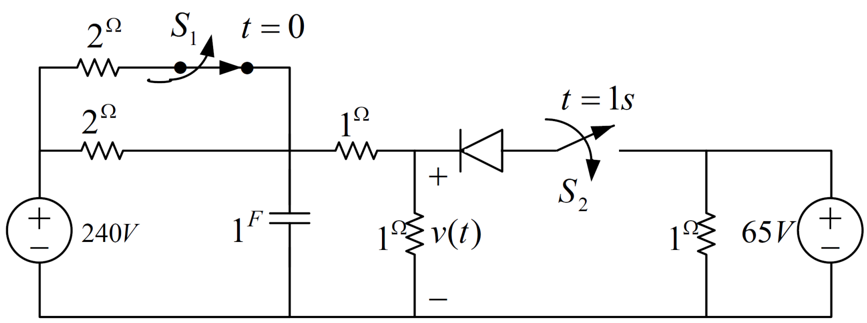
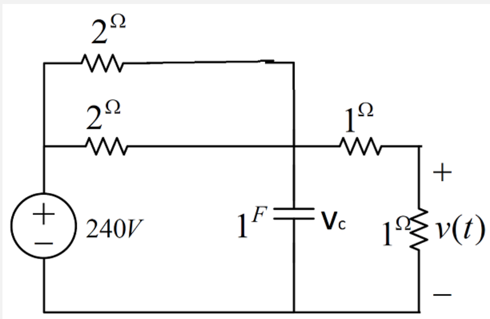


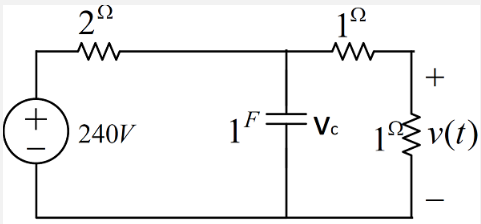
Figure 2: A nonlinear RC circuit with diode.

For $t < 0$:



$$V_c(0^-) = \frac{1+1}{(1+1)+(2||2)} \times 240 = \frac{2}{3} \times 240 = 160 \Rightarrow v(0^-) = \frac{V_c(0^-)}{2} = 80$$

For $0 < t < 1$:

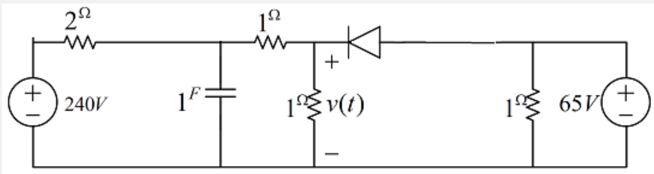


Question 2 continued on next page...

$$\begin{cases} V_c(0^+) = V_c(0^-) = 160 \\ V_c(\infty) = \dfrac{2}{2+2} \times 240 = 120 \\ R_{eq} = 2||(1+1) = 1 \Rightarrow \tau = R_{eq}C = 1 \end{cases}$$

$$V_c(t) = 120 + (160 - 120)e^{\frac{-t}{1}} = 120 + 40e^{-t} \Rightarrow v(t) = \frac{V_c(t)}{2} = 60 + 20e^{-t}$$

For $t > 1$:



Note that the diode opens when $v(t) = 60 + 20e^{-t} = 65$, which yields$t = \ln(4)$. Therefore, if $1 < t < \ln(4)$, the diode is off and clearly, $v(t) = 60 + 20e^{-t}$. Further, if $t > \ln(4)$, the diode is on and clearly $v(t) = 65$.

Overall response:

$$v(t) = \begin{cases} 80 & t < 0 \\ 60 + 20e^{-t} & 0 < t < 1 \\ 60 + 20e^{-t} & 1 < t < \ln(4) \\ 65 & t > \ln(4) \end{cases} = \begin{cases} 80 & t < 0 \\ 60 + 20e^{-t} & 0 < t < \ln(4) \\ 65 & t > \ln(4) \end{cases}$$

Question 3

Determine both i_1 and i_L in the circuit shown in Fig. 3 for $t > 0$.

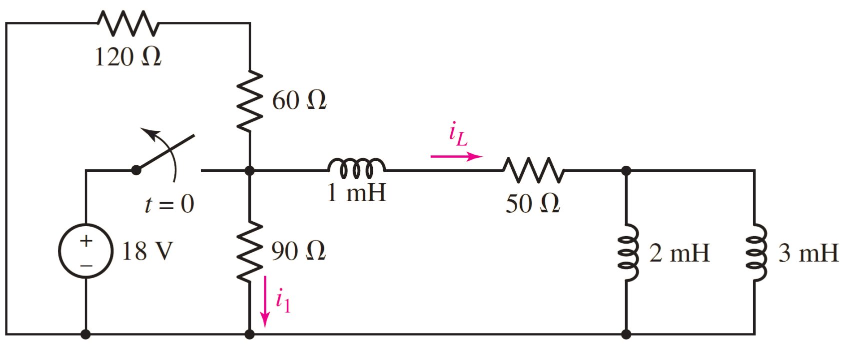


Figure 3: A circuit with multiple resistors and inductors.

After t=0, when the voltage source is disconnected, we easily calculate the equivalent inductance

$$L_{eq} = \frac{2 \times 3}{2+3} + 1 = 2.2 \text{ mH}$$

the equivalent resistance, in series with the equivalent inductance,

$$R_{eq} = \frac{90(60+120)}{90+180} + 50 = 110\,\Omega$$

and the time constant

$$\tau = \frac{L_{eq}}{R_{eq}} = \frac{2.2 \times 10^{-3}}{110} = 20\,\mu\text{s}$$

The form of the natural response is $Ke^{-50000t}$, where K is an unknown constant. Considering the circuit just prior to the switch opening at $t = 0^-$, $i_L = 18/50$ A. Since $i_L(0^+) = i_L(0^-)$, we know that $i_L = 18/50$ A, or 360 mA at $t = 0^+$ and so

$$i_L = \begin{cases} 360 \text{ mA} & t < 0 \\ 360e^{-50{,}000t} \text{ mA} & t \geq 0 \end{cases}$$

There is no restriction on i_1 changing instantaneously at $t = 0$, so its value at $t = 0^-$ (200 mA) is not relevant to finding i_1 for $t > 0$. Instead, we must find $i_1(0^+)$ through our knowledge of $i_L(0^+)$. Using current division,

$$i_1(0^+) = -i_L(0^+)\frac{120+60}{120+60+90} = -240 \text{ mA}$$

Hence,

$$i_1 = \begin{cases} 200 \text{ mA} & t < 0 \\ -240e^{-50{,}000t} \text{ mA} & t \geq 0 \end{cases}$$

Question 4

Consider the circuit shown in Fig. 4.

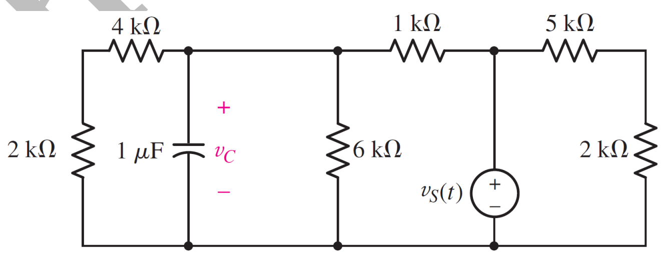


Figure 4: A first-order LIT RC circuit.

(a) *Find the step response of $v_c(t)$.*

First, we calculate the Thevenin voltage and resistor as

$$R_T = (2+4)||1||6 = 0.75 \text{ k}\Omega$$

$$V_T = \frac{(2+4)||6}{(2+4)||6+1} v_S(t) = \frac{3}{4} v_S(t)$$

Now, we have a simple series RC circuit with the governing differential equation

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau} v_c(t) = \frac{3}{4}\frac{1}{\tau}, \tau = R_T C = 0.00075, \quad t > 0$$

and the initial condition

$$v_c(0-) = 0 \Rightarrow v_c(0+) = 0$$

So, for the step response with $v_S(t) = u(t)$,

$$s(t) = \begin{cases} 0 & t < 0 \\ \frac{3}{4}(1 - e^{-\frac{4000}{3}t}) & t \geq 0 \end{cases} = \frac{3}{4}(1 - e^{-\frac{4000}{3}t}) u(t)$$

(b) *Find the impulse response of $v_c(t)$.*

$$h(t) = \frac{ds(t)}{dt} = 1000 e^{-\frac{4000}{3}t} u(t) + 0\delta(t) = 1000 e^{-\frac{4000}{3}t} u(t)$$

(c) *Find the zero-state response of $v_c(t)$ if $v_s(t) = 5u(t) + 2\delta'(t)$.*

$$h'(t) = \frac{dh(t)}{dt} = 1000\delta(t) - \frac{4000000}{3} e^{-\frac{4000}{3}t} u(t)$$

$$v_c(t) = 5s(t) + 2h'(t) = \frac{15}{4}(1 - e^{-\frac{4000}{3}t}) u(t) + 2000\delta(t) - \frac{8000000}{3} e^{-\frac{4000}{3}t} u(t)$$

(d) *Find the zero-state response of $v_c(t)$ if $v_s(t) = 2e^{-2t}u(t)$.*

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau} v_c(t) = \frac{3v_S(t)}{4\tau} = \frac{3e^{-2t}}{2\tau}, t > 0; v_c(0^+) = v_c(0^-) = 0$$

$$v_c(t) = Ke^{-\frac{t}{\tau}} + Ae^{-2t}$$

The particular solution Ae^{-2t} satisfies the differential equation. So,

$$-2Ae^{-2t} + \frac{1}{\tau} Ae^{-2t} = \frac{3e^{-2t}}{2\tau} \Rightarrow A = \frac{3}{2 - 4\tau}$$

Further, the initial condition yields

$$v_c(0^+) = K + A = 0 \Rightarrow K = -A$$

Question 4 continued on next page...

Finally,

$$v_c(t) = A(e^{-2t} - e^{-\frac{t}{\tau}}) = \frac{3}{2-4\tau}(e^{-2t} - e^{-\frac{t}{\tau}}), t > 0$$

and

$$v_c(t) = \begin{cases} 0, & t < 0 \\ \frac{3}{2-4\tau}(e^{-2t} - e^{-\frac{t}{\tau}}), & t > 0 \end{cases} = \frac{3}{2-4\tau}(e^{-2t} - e^{-\frac{t}{\tau}})u(t) = 1.502(e^{-2t} - e^{-1333t})u(t)$$

(e) *Find the zero-state response of $v_c(t)$ if $v_s(t) = 2\cos(t)u(t)$.*

$$\frac{dv_c(t)}{dt} + \frac{1}{\tau}v_c(t) = \frac{3v_S(t)}{4\tau} = \frac{3\cos(t)}{2\tau}, t > 0; v_c(0^+) = v_c(0^-) = 0$$

$$v_c(t) = Ke^{-\frac{t}{\tau}} + A\cos(t) + B\sin(t)$$

The particular solution $A\cos(t) + B\sin(t)$ satisfies the differential equation. So,

$$-A\sin(t) + B\cos(t) + \frac{A\cos(t) + B\sin(t)}{\tau} = \frac{3\cos(t)}{2\tau}$$

$$\Rightarrow \begin{cases} -A + \frac{B}{\tau} = 0 \\ B + \frac{A}{\tau} = \frac{3}{2\tau} \end{cases} \Rightarrow A = \frac{3}{2+2\tau^2}, B = \frac{3\tau}{2+2\tau^2} = \tau A$$

Further, the initial condition yields

$$v_c(0^+) = K + A = 0 \Rightarrow K = -A$$

Finally,

$$v_c(t) = -Ae^{-\frac{t}{\tau}} + A\cos(t) + \tau A\sin(t) = \frac{3}{2+2\tau^2}(\cos(t) + \tau\sin(t) - e^{-\frac{t}{\tau}}), t > 0$$

and

$$v_c(t) = \begin{cases} 0, & t < 0 \\ \frac{3}{2+2\tau^2}(\cos(t) + \tau\sin(t) - e^{-\frac{t}{\tau}}), & t > 0 \end{cases} = \frac{3}{2+2\tau^2}(\cos(t) + \tau\sin(t) - e^{-\frac{t}{\tau}})u(t)$$

$$v_c(t) = 1.4999(\cos(t) + 0.00075\sin(t) - e^{-1333t})u(t)$$

SOFTWARE QUESTIONS

Question 5

Consider the circuit shown in Fig. 5, where the diodes are 1N4148 and $v_s(t)$ is a periodic square signal with low level 0, high level 15 V, duty cycle 0.5, and period T. Simulate the circuit in PSpice and plot the capacitor voltage versus time when $\tau_1 = R_1C \ll T$ and $\tau_2 = R_2C \ll T$ and when $\tau_1 = R_1C \gg T$ and $\tau_2 = R_2C \gg T$. Discuss and compare the results.

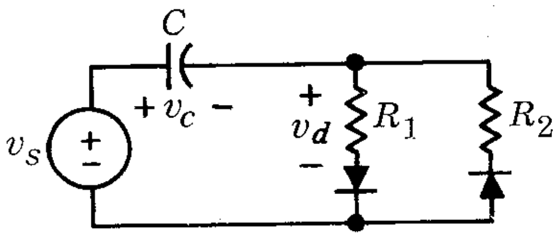


Figure 5: A first-order NTI RC circuit.

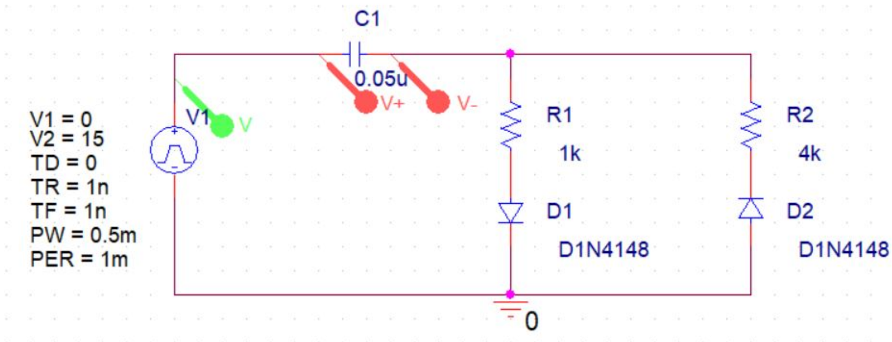


Figure 6: Schematic of the circuit shown in Fig. 5.

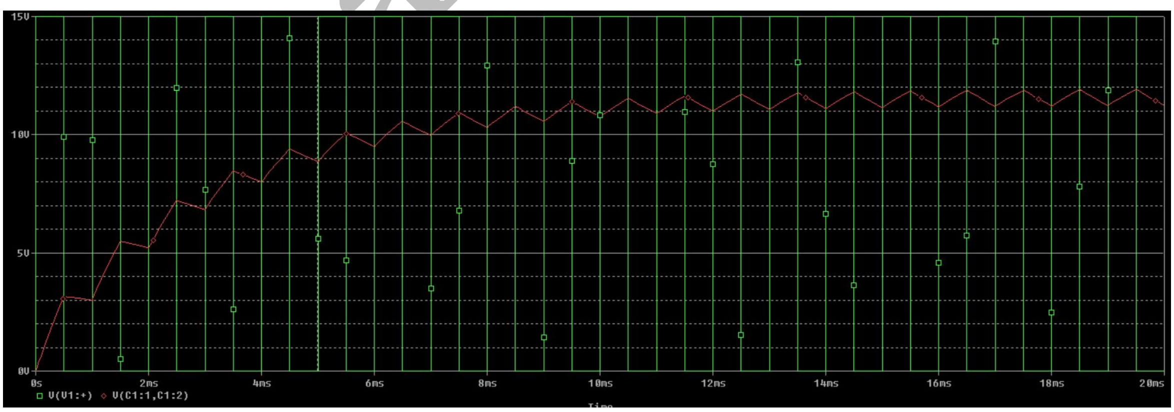


Figure 7: Input and capacitor voltages versus time for $T = 1$ ms, $C = 2$ μF, $R_1 = 1$ kΩ, and $R_2 = 4$ kΩ.

Question 5 continued on next page...

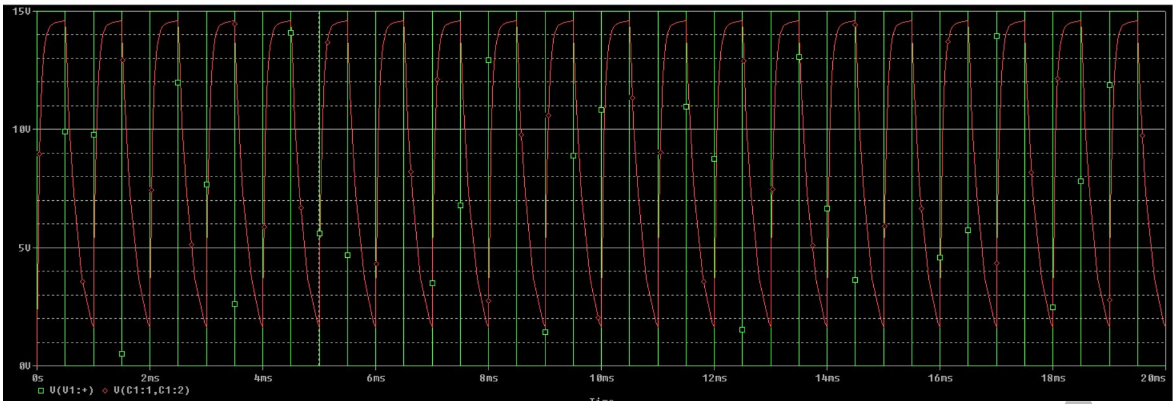


Figure 8: Input and capacitor voltages versus time for $T = 1$ ms, $C = 0.05$ μF, $R_1 = 1$ kΩ, and $R_2 = 4$ kΩ.

> The schematic of the circuit is shown in Fig. 5 while Figs. 7 and 8 shows capacitor voltage versus time for $\tau_1 = R_1C \gg T$ and $\tau_2 = R_2C \gg T$ and for $\tau_1 = R_1C \ll T$ and $\tau_2 = R_2C \ll T$, respectively. In both figures, the capacitor charges when the input voltage is high and discharges when the input voltage is zero. The time constants of the charging and discharging periods are $\tau_1 = R_1C$ and $\tau_2 = R_2C$, respectively. When the time constants are large as shown in Fig. 7, the capacitor cannot charge or discharge to the input high and low voltages, respectively. However, as shown in Fig. 8, when the time constants are small, the capacitor voltage can approach the extreme values of 0 and 15.

BONUS QUESTIONS

Question 6

Find an expression for the NTI resistor voltage in Fig. 9 valid for $t > 0$. Assume that $v_c(0^+) = V_0 = 3$.

Question 6 continued on next page...

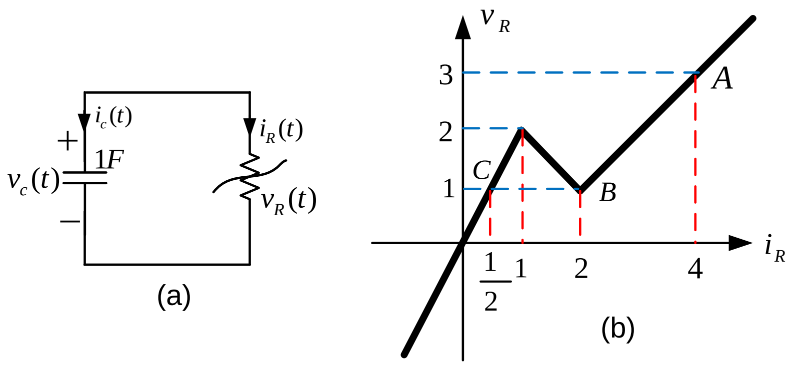


Figure 9: (a) A nonlinear RC circuit with an NTI resistor (b) Characteristic curve of the NTI resistor.

There is a piecewise linear relation between v_R and i_R as

$$v_R = \begin{cases} 2i_R & i_R \leq 1 \\ -i_R + 3 & 1 \leq i_R \leq 3 \\ i_R - 1 & 2 \leq i_R \end{cases}$$

Writing the KVL, KCL, and characteristic equations of the circuit,

$$\begin{cases} \text{KCL: } i_R = -i_C \\ \text{KVL: } v_R = v_C \\ \text{capacitor current: } i_C = C\frac{dv_C}{dt} \end{cases} \Rightarrow \boxed{\frac{dv_R}{dt} + i_R = 0}$$

We have $v_R(0^+) = v_C(0^+) = 3$ so at $t = 0^+$, we are on the third piece (point A). Hence:

$$i_R = -\frac{d}{dt}(i_R - 1) \Rightarrow i_R = i_0 e^{-t} \Rightarrow v_R = i_0 e^{-t} - 1$$
$$\Rightarrow v_R(0^+) = i_0 - 1 = 3 \Rightarrow i_0 = 4 \Rightarrow v_R = 4e^{-t} - 1$$

This solution is valid until $i_R \geq 2$ which means $t \leq t_1 = \ln 2$. At $t = t_1$, we reach point B and now we have to determine the next piece. If we go on the second piece (the one with negative slope) then

$$\frac{dv_R}{dt} > 0,\ i_R > 0$$

which violates the differential equation $\frac{dv_R}{dt} + i_R = 0$. Note that on the second piece, the voltage is ascending and when the piece it traversed from B to C as time proceeds, $\frac{dv_R}{dt} > 0$. So, we should go on the first piece on point C. At this point,

$$i_R = -\frac{d}{dt}(2i_R), v_R(t_1^+) = 1 \Rightarrow t \geq t_1:\ v_R(t) = e^{-0.5(t-t_1)}$$

Hence, for $t > 0$,

$$\boxed{v_R(t) = \begin{cases} 4e^{-t} - 1 & 0 < t \leq \ln 2 \\ e^{-0.5(t-\ln 2)} & \ln 2 \leq t \end{cases} = \begin{cases} 4e^{-t} - 1 & 0 < t \leq \ln 2 \\ \sqrt{2}e^{-0.5t} & \ln 2 \leq t \end{cases}}$$

Question 7

Return your answers by filling the LaTeX template of the assignment.

EXTRA QUESTIONS

Question 8

Feel free to solve the following questions from the book *"Engineering Circuit Analysis"* by W. Hayt, J. Kemmerly, and S. Durbin.

1. **Chapter 8, question 29.**
2. **Chapter 8, question 31.**
3. **Chapter 8, question 37.**
4. **Chapter 8, question 38.**
5. **Chapter 8, question 39.**
6. **Chapter 8, question 48.**
7. **Chapter 8, question 50.**
8. **Chapter 8, question 51.**
9. **Chapter 8, question 52.**
10. **Chapter 8, question 58.**
11. **Chapter 8, question 60.**
12. **Chapter 8, question 61.**
13. **Chapter 8, question 64.**