MATHEMATICAL QUESTIONS

Question 1

Find the step and impulse responses of the current $i(t)$ in Fig. 1.

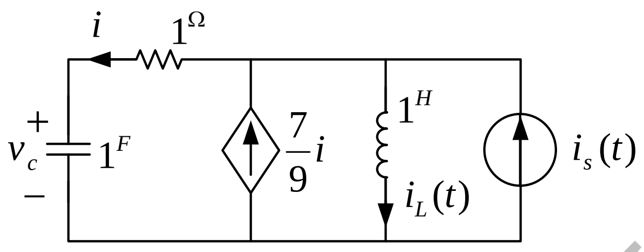

Figure 1: An LTI RLC circuit with dependent source.

Differential equation:

$$i = 1.\frac{dv_c}{dt}, \quad v_L = 1.\frac{di_L}{dt}$$

$$\begin{cases} \text{KCL: } -i - i_L + i_s + \frac{7}{9}i = 0 \Rightarrow i_L + \frac{2}{9}i = i_s \Rightarrow v_L + \frac{2}{9}\frac{di}{dt} = \frac{di_s}{dt} \\ \text{KVL: } v_L - v_c - i = 0 \end{cases}$$

$$\frac{2}{9}\frac{d^2v_c}{dt^2} + \frac{dv_c}{dt} + v_c = \frac{di_s}{dt}$$

Step Response:

$$i_s(t) = u(t) \Rightarrow \frac{d^2v_c}{dt^2} + \frac{9}{2}\frac{dv_c}{dt} + \frac{9}{2}v_c = \frac{9}{2}\frac{du(t)}{dt} = \frac{9}{2}\delta(t), t > 0^-$$

$$\frac{d^2v_c}{dt^2} + \frac{9}{2}\frac{dv_c}{dt} + \frac{9}{2}v_c = 0, t > 0^+$$

$$s^2 + \frac{9}{2}s + \frac{9}{2} = 0 \Rightarrow s_{1,2} = \frac{-3}{2}, -3 \Rightarrow v_c = K_1e^{-3t} + K_2e^{-\frac{3}{2}t}$$

$$v_c(0^-) = v_c(0^+) = 0, \quad i_L(0^-) = i_L(0^+) = 0, \quad v_c'(0^+) = \frac{9}{2}i_s(0^+) - \frac{9}{2}i_L(0^+) = \frac{9}{2}$$

$$\Rightarrow K_1 = -3, K_2 = 3 \Rightarrow v_c(t) = 3(e^{-\frac{3}{2}t} - e^{-3t})u(t)$$

$$\Rightarrow s(t) = i(t) = \frac{dv_c(t)}{dt} = 9(e^{-3t} - \frac{1}{2}e^{-\frac{3}{2}t})u(t)$$

Impulse Response:

$$\Rightarrow h(t) = \frac{ds(t)}{dt} = 27(\frac{1}{4}e^{-\frac{3}{2}t} - e^{-3t})u(t) + \frac{9}{2}\delta(t)$$

Question 2 continued on next page. . .

Question 2

Find an expression for the zero-input response of $v_1(t)$ in Fig. 2 valid for $t > 0$ if $v_1(0^+) = V_1$ and $v_2(0^+) = V_2$.

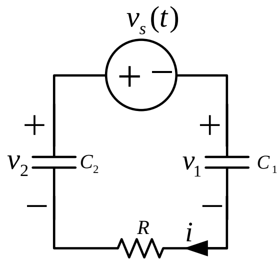

Figure 2: A circuit with two capacitors.

KVL yields

$$v_1(t) + Ri(t) - v_2(t) + v_s(t) = 0$$

while the KCL gives

$$C_1\frac{dv_1}{dt} = i, \quad C_2\frac{dv_2}{dt} = -i$$

The capacitor voltages should be continuous at $t = 0$. In fact, if the capacitor voltages experience a discontinuity at $t = 0$, the resistor current and voltage experience an impulse, which can not be supplied by the circuit. So,

$$v_1(0^-) = v_1(0^+) = V_1,\ v_2(0^-) = v_2(0^+) = V_2.$$

If

$$i = C_1\frac{dv_1}{dt} = -C_2\frac{dv_2}{dt}$$

is integrated as $\int_{0^+}^{t}$,

$$C_1(v_1(t) - v_1(0^+)) = -C_2(v_2(t) - v_2(0^+)) \Rightarrow v_2(t) = \frac{C_1V_1 + C_2V_2}{C_2} - \frac{C_1}{C_2}v_1(t)$$

From the KVL,

$$\begin{aligned}
&v_1 + RC_1\frac{dv_1}{dt} + \frac{C_1}{C_2}v_1 - \frac{C_1V_1 + C_2V_2}{C_2} + v_s = 0\\
\Rightarrow &\frac{dv_1}{dt} + (\frac{1}{RC_2} + \frac{1}{RC_1})v_1 = \frac{C_1V_1 + C_2V_2}{RC_1C_2} - \frac{v_s}{RC_1}\\
\Rightarrow &\frac{dv_1}{dt} + \frac{1}{RC_{eq}}v_1 = \frac{C_1V_1 + C_2V_2}{RC_1C_2} - \frac{v_s}{RC_1}, \quad C_{eq} = \frac{C_1C_2}{C_1 + C_2}
\end{aligned}$$

Question 2 continued on next page. . .

In the zero-input situation,

$$\frac{dv_1}{dt} + \frac{1}{RC_{eq}} v_1 = \frac{C_1V_1 + C_2V_2}{RC_1C_2}, \quad C_{eq} = \frac{C_1C_2}{C_1 + C_2}$$

Solving the differential equation yields

$$v_1(t) = A + Be^{-\frac{t}{RC_{eq}}}$$

The particular solution A satisfies the differential equation. So,

$$A = \frac{\frac{C_1V_1+C_2V_2}{RC_1C_2}}{\frac{1}{RC_{eq}}} = \frac{C_1V_1 + C_2V_2}{RC_1C_2} R \frac{C_1C_2}{C_1 + C_2} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$

Further, using the initial condition,

$$V_1 = A + B \Rightarrow B = V_1 - A$$

Finally,

$$v_1(t) = A + (V_1 - A)e^{-\frac{t}{RC_{eq}}} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2} - \frac{C_2(V_2 - V_1)}{C_1 + C_2} e^{-\frac{t}{RC_{eq}}}, t > 0$$

Question 3

Calculate $i_L(t), t > 0$ in Fig. 3, where

$$R(t) = \begin{cases} 2, & t < 0 \\ 1, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$$

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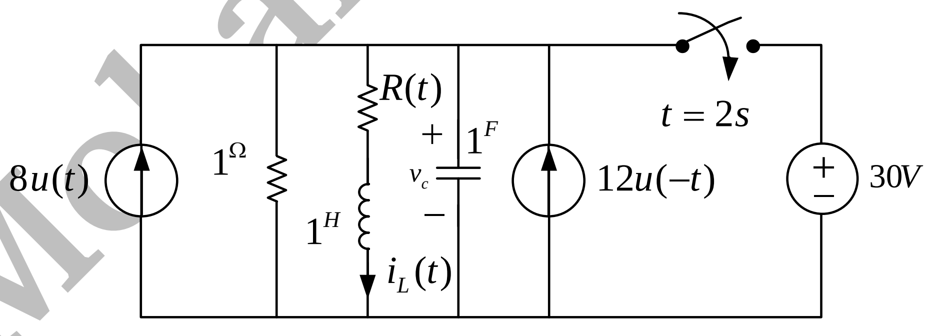
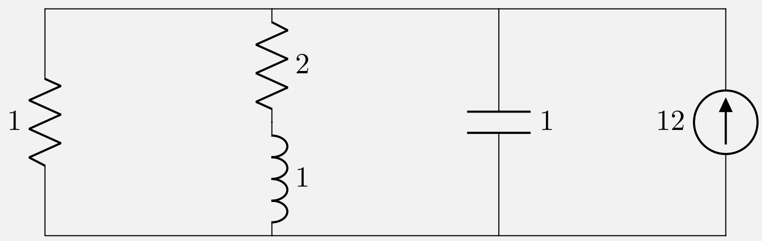
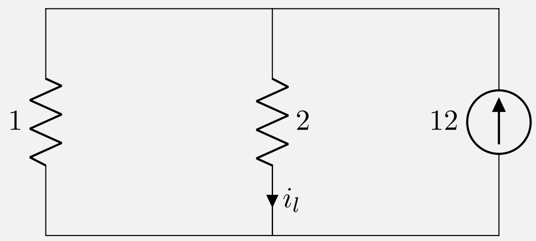


Figure 3: A second-order time-variant circuit.

The circuit should be analyzed in different time intervals:
For $t < 0$:
The circuit has been shown below,



The capacitor is open circuit and inductor is short circuit in $t = 0^-$, so the circuit is as shown below

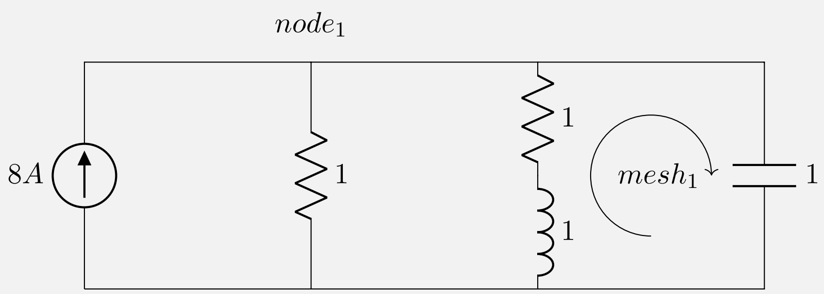


$$i_L(t) = 12 \times \frac{1}{3} = 4 \quad v_c(t) = R_2 i_l = 2 \times 4 = 8$$

Clearly,

$$i_L(0^-) = 12 \times \frac{1}{3} = 4 \quad v_c(0^-) = R_2 i_l = 2 \times 4 = 8$$

For $0 < t < 2$:
The circuit has been shown below



By writing KCL in $node_1$

$$-8 + v_c + i_L + \frac{dv_c}{dt} = 0$$

By writing KVL in $mesh_1$,

$$v_c = i_L + \frac{di_L}{dt}$$

Combining the written KCL and KVL equations and noting the continuity of $i_L(t)$,

$$\frac{d^2 i_L}{dt^2} + 2\frac{di_L}{dt} + 2i_L = 8$$

The continuity if the inductor current results in

$$i_L(0^-) = i_L(0^+) = 4$$

Question 3 continued on next page...

By writing KVl in $mesh_1$,

$$v_c(0^+) = i_L(0^+) + \frac{di_L(0^+)}{dt}, i_L(0^+) = 4, v_c(0^-) = v_c(0^+) = 8 \Rightarrow \frac{di_L(0^+)}{dt} = 4$$

Overall, the differential equation of the circuit is

$$\frac{d^2 i_L}{dt^2} + 2\frac{di_L}{dt} + 2i_L = 8 \qquad i_L(0^+) = 4 \qquad \frac{di_L(0^+)}{dt} = 4$$

with the homogeneous solution

$$s^2 + 2s + 2 = 0 \Rightarrow s = -1 + i, -1 - i \Rightarrow i_{L_h} = Ae^{-t}\sin(t) + Be^{-t}\cos(t)$$

, the particular solution

$$i_{L_p} = K$$

, and the complete solution

$$i_L = Ae^{-t}\sin(t) + Be^{-t}\cos(t) + K$$

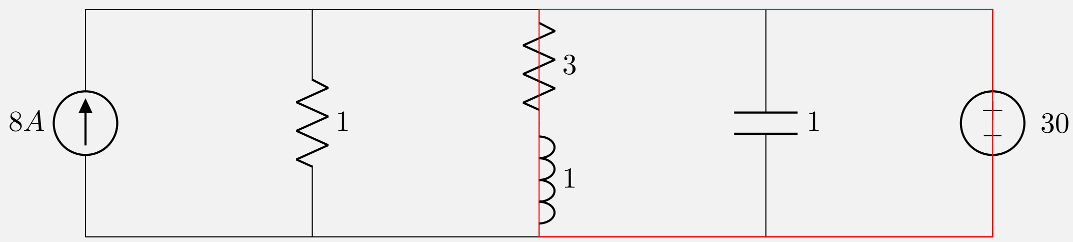
To find A, B, and K, we use initial conditions and substitute the complete solution in differential equation.

$$\begin{cases} 2K = 8 \\ i_L(0^+) = 4 = B + 4 \\ \frac{di_L(0^+)}{dt} = 4 = A - B \end{cases} \Rightarrow K = 4, A = 4, B = 0$$

Finally,

$$i_L(t) = 4e^{-t}\sin(t) + 4$$

For $t > 2$:
The circuit has been shown below



The describing differential equation is obtained by writhing KVL in red loop as

$$\frac{di_L}{dt} + 3i_L = 30$$

with the initial condition

$$i_L(2^-) = 4e^{(-2)} + 4 = 4.5 = i_L(2^+)$$

, the homogeneous solution

$$s + 3 = 0 \Rightarrow s = -3 \Rightarrow\Rightarrow i_{L_h} = Ae^{-3(t-2)}$$

Question 3 continued on next page. . .

, the particular solution

$$i_{L_p} = B$$

, and the complete solution

$$i_L = Ae^{-3(t-2)} + B$$

To find A and B, we use initial conditions and substitute the complete solution in differential equation.

$$\begin{cases} 3B = 30 \\ i_L(2^+) = 4.5 = A + B \end{cases} \Rightarrow A = -5.5, B = 10$$

So,

$$i_L(t) = -5.5e^{-3(t-2)} + 10$$

Overall response:

$$i_L(t) = \begin{cases} 4 & t < 0 \\ 4e^{-t}\sin(t) + 4 & 0 \le t < 2 \\ -5.5e^{-3(t-2)} + 10 & t \ge 2 \end{cases}$$

Question 4

Let $v_c(0^-) = 2$ V and $i_L(0^-) = 1$ A and find $v_c(t), t > 0$ in Fig. 4.

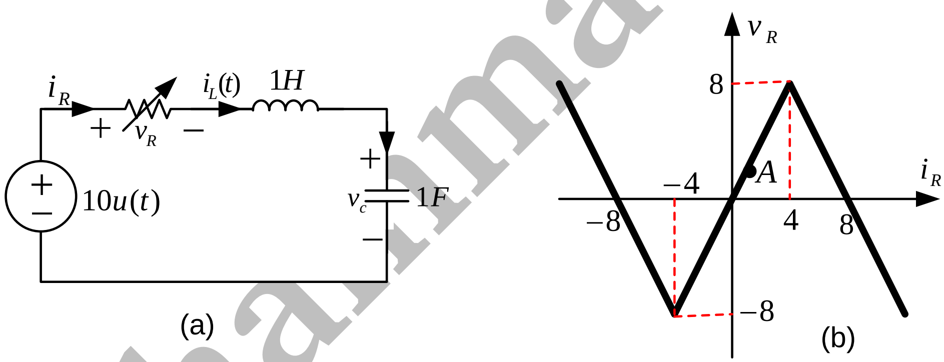


Figure 4: A nonlinear second-order circuit.

The nonlinear resistor is described by

$$v_R = \begin{cases} -2i_R + 16 & 4 \le i_R \\ 2i_R & -4 \le i_R \le 4 \\ -2i_R - 16 & i_R \le -4 \end{cases} \quad (1)$$

$$\begin{cases} i_L(0^-) = 1 \\ i_R = i_L = i_C \end{cases} \Rightarrow i_R(0^-) = 1 \Rightarrow v_R(0^-) = 2$$

Question 4 continued on next page...

KVL for $t = 0^-$ gives

$$v_R(0^-) + v_L(0^-) + v_C(0^-) = 0 \Rightarrow 2 + \frac{di_L}{dt} + 2 = 0 \Rightarrow i'_L(0^-) = -4$$

$i_L(t)$ and $v_C(t)$ can't jump at $t = 0$. So, we have, $v_C(0^-) = v_C(0^+) = 2$ and $i_L(0^-) = i_L(0^+) = 1$. Hence,

$$v_R(0^+) + v_L(0^+) + v_C(0^+) = 10u(0^+) \Rightarrow 2 + \frac{di_L}{dt} + 2 = 10 \Rightarrow i'_L(0^+) = 6$$

For $-4 < i_L(t) < 4$ and $t > 0$, we have

$$v_R(t) + v_L(t) + v_C(t) = 100$$

$$\frac{d(2i_L + \frac{di_L}{dt} + v_C(t))}{dt} = 0$$

$$\frac{d^2 i_L}{dt} + 2\frac{di_L}{dt} + i = 0$$

$$\Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_1 = -1, r_2 = -1$$

$$\Rightarrow i_L(t) = A_1 te^{-t} + A_0 e^{-t}$$

According to initial conditions $i_L(0^+) = 1$ and $i'_L(0^+) = 6$, we get $A_0 = 1$ and $A_1 = 7$. Hence,

$$i_L(t) = e^{-t} + 7te^{-t}, t > 0$$

Now, we find the minimum and the maximum of $i_L(t)$ to check whether $-4 < i_L(t) = i_R(t) < 4$ is true or not.

Minimum of $i_L(t)$:

$$\frac{di_L(t)}{dt} = (6 - 7t)e^{-t}, t > 0$$

Clearly, for $t > \frac{6}{7}$, $\frac{di_L(t)}{dt} < 0$. Therefore, $\frac{6}{7} < t$, the inductor current $i_L(t)$ is descending while for $0^+ < t < \frac{6}{7}$, the inductor current is ascending. So, the minimum is either at $t = \infty$ or $t = 0^+$ and equals $\min\{i_L(0^+), i(\infty)\} = 0$. Hence, $i_L(t) > -4, t > 0$.

Maximum of $i_L(t)$:

$$\frac{di_L(t)}{dt} = (6 - 7t)e^{-t} = 0 \Rightarrow t = \frac{6}{7}$$

So, the maximum inductor current is $i_L(\frac{6}{7}) \approx 2.97 < 4$. Hence, $i_L(t) < 4, t > 0$.

Now, for $t > 0$, we get

$$\begin{aligned} v_C(t) &= v_C(0^+) + \frac{1}{C}\int_0^t i_C(t)dt \\ &= 2 + \int_0^t i_C(t)dt \\ &= 2 + \int_0^t (e^{-t} + 7te^{-t})dt \\ &= -7te^{-t} - 8e^{-t} + 10 \end{aligned}$$

SOFTWARE QUESTIONS

Question 5

Consider the parallel RLC circuit shown in Fig. 5. Use PSpice transient simulation to plot the step responses of $v_c(t)$, $i_L(t)$, $i_R(t)$, and $i_c(t)$ for $L = 1$, $C = 1$, and $R = 0.25, 0.5, 1, 1000$.

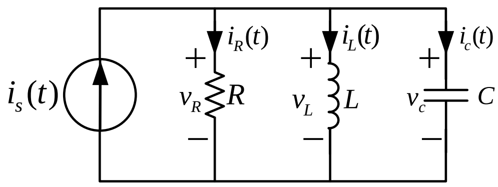


Figure 5: A parallel RLC circuit.

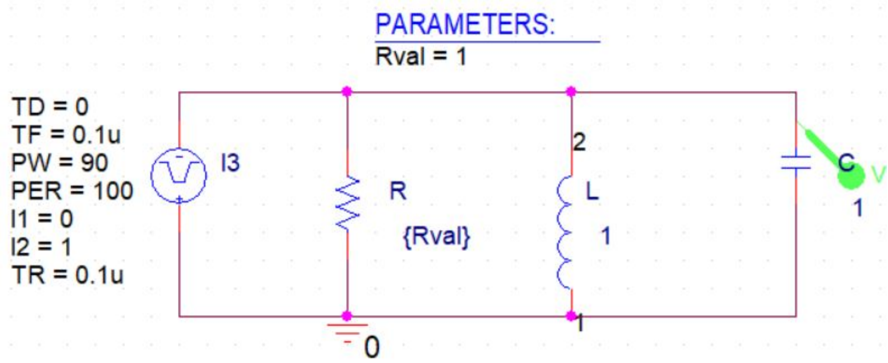


Figure 6: Schematic of the parallel RLC circuit in Capture environment.

Question 5 continued on next page...

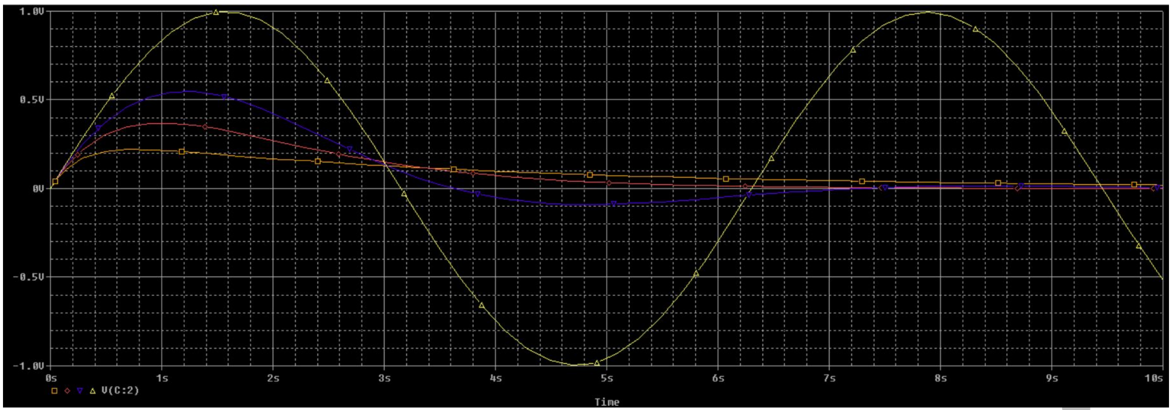


Figure 7: Capacitor voltage step response for $R = 0.25, 0.5, 1, 1000$.

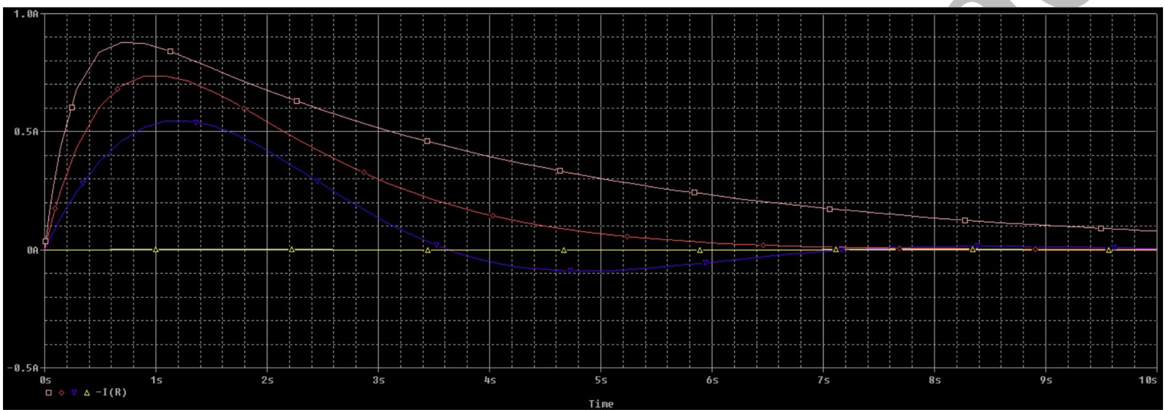


Figure 8: Resistor current step response for $R = 0.25, 0.5, 1, 1000$.

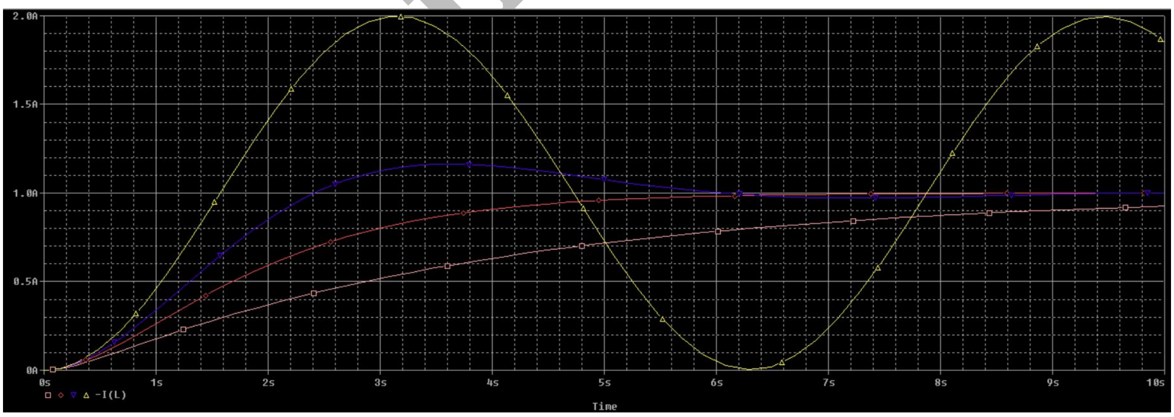


Figure 9: Inductor current step response for $R = 0.25, 0.5, 1, 1000$.

Question 5 continued on next page. . .

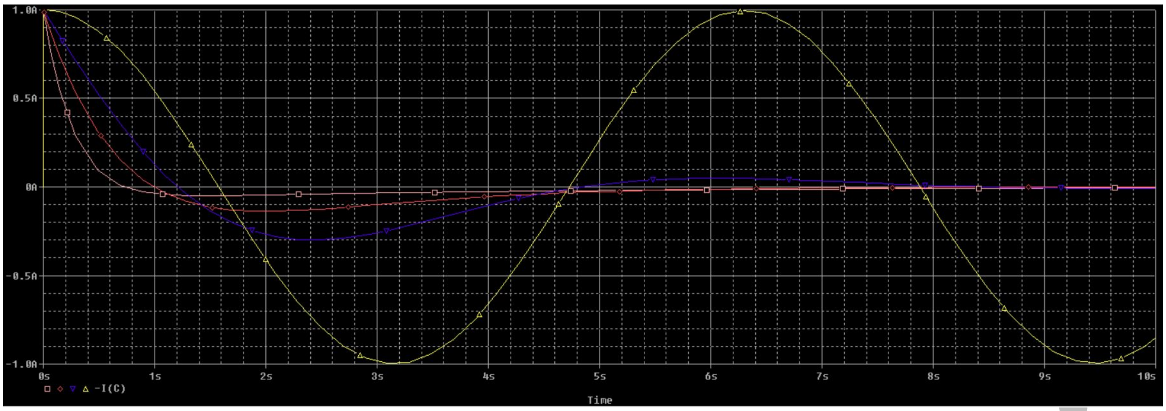


Figure 10: Capacitor current step response for $R = 0.25, 0.5, 1, 1000$.

Fig. 6 shows the schematic of the circuit in PSpice while Figs. 7-10 show the step responses of $v_c(t)$, $i_R(t)$, $i_L(t)$, and $i_c(t)$ for $R = 0.25, 0.5, 1, 1000$. The step responses have overdamped, critically damped, underdamped, and lossless form for $R = 0.25, 0.5, 1, 1000$, respectively.

BONUS QUESTIONS

Question 6

Return your answers by filling the LaTeXtemplate of the assignment.

EXTRA QUESTIONS

Question 7

Feel free to solve the following questions from the book *"Engineering Circuit Analysis"* by W. Hayt, J. Kemmerly, and S. Durbin.

1. **Chapter 9, question 13.**
2. **Chapter 9, question 14.**
3. **Chapter 9, question 20.**
4. **Chapter 9, question 26.**

Question 7 continued on next page...

5. **Chapter 9, question 35.**
6. **Chapter 9, question 37.**
7. **Chapter 9, question 39.**
8. **Chapter 9, question 48.**
9. **Chapter 9, question 50.**
10. **Chapter 9, question 51.**
11. **Chapter 9, question 53.**
12. **Chapter 9, question 61.**
13. **Chapter 9, question 65.**