MATHEMATICAL QUESTIONS

Question 1

Design a circuit with the lowest number of op-amps that implements $v_o(t) = -4v_{s1}(t) + 7v_{s2}(t)$.

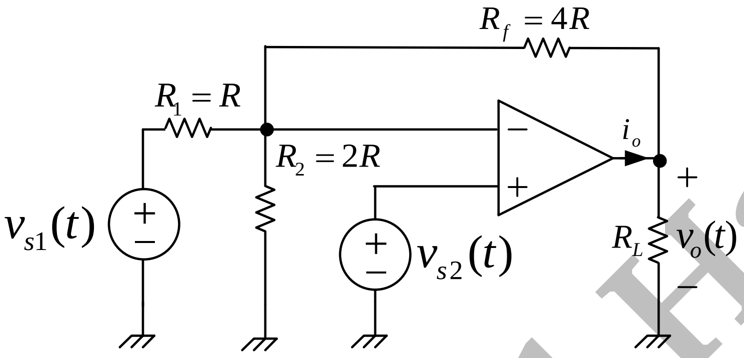

Figure 1: Designed op-amp circuit.

A sample design can be as the circuit shown in Fig. 1, where

$$v_{s2} = v_+ = v_-, \quad \frac{v_- - v_{s1}}{R_1} + \frac{v_- - 0}{R_2} + \frac{v_- - v_o}{R_f} = 0$$

$$v_o = -\frac{R_f}{R_1}v_{s1} + R_f(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_f})v_{s2}$$

$$v_o = -\frac{4R}{R}v_{s1} + 4R(\frac{1}{R} + \frac{1}{2R} + \frac{1}{4R})v_{s2} = -4v_{s1} + 7v_{s2}$$

Question 2

Consider the non-inverting amplifier shown in Fig. 2(a).

Question 2 continued on next page...

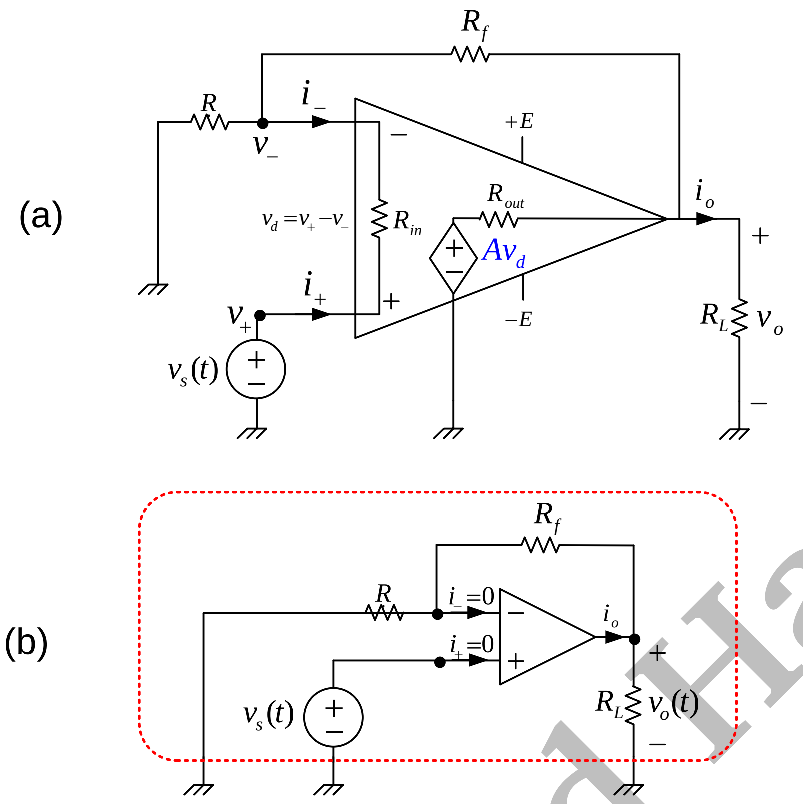

Figure 2: A non-inverting amplifier.

(a) Calculate $G = \frac{v_o}{v_s}$ *in Fig. 2(a).*

Question 2 continued on next page. . .

By writing KCL at node v_-,

$$\frac{v_-}{R} + \frac{v_- - v_s}{R_{in}} + \frac{v_- - v_o}{R_f} = 0$$

and by writing KCL at node v_o,

$$\frac{v_o - v_-}{R_f} + \frac{v_o - A(v_s - v_-))}{R_{out}} + \frac{v_o}{R_L} = 0$$

So,

$$\begin{pmatrix} \frac{1}{R} + \frac{1}{R_{in}} + \frac{1}{R_f} & -\frac{1}{R_f} \\ \frac{A}{R_{out}} - \frac{1}{R_f} & \frac{1}{R_L} + \frac{1}{R_{out}} + \frac{1}{R_f} \end{pmatrix} \begin{pmatrix} v_- \\ v_o \end{pmatrix} = \begin{pmatrix} \frac{v_s}{R_{in}} \\ \frac{A}{R_{out}} v_s \end{pmatrix}$$

By Cramer's rule,

$$v_o = \frac{\begin{vmatrix} \frac{1}{R} + \frac{1}{R_{in}} + \frac{1}{R_f} & \frac{1}{R_{in}} \\ \frac{A}{R_{out}} - \frac{1}{R_f} & \frac{1}{R_{out}} \end{vmatrix}}{\begin{vmatrix} \frac{1}{R} + \frac{1}{R_{in}} + \frac{1}{R_f} & -\frac{1}{R_f} \\ \frac{A}{R_{out}} - \frac{1}{R_f} & \frac{1}{R_L} + \frac{1}{R_{out}} + \frac{1}{R_f} \end{vmatrix}} v_s \Rightarrow$$

$$G = \frac{R_L(R_f R_{in} A + R R_{in} A + R R_{out})}{R_f R_{in} R_{out} + R_f R_{in} R_L + R_{in} R_{out} R_L + R R_{in} R_{out} + R_L R R_{in} + R R_f R_{out} + R R_f R_L + R R_{out} R_L + A R R_{in} R_L}$$

(b) Under which conditions the calculated $G = \frac{v_o}{v_s}$ equals the gain of ideal non-inverting amplifier in Fig. 2(b)?

The gain of ideal non-inverting amplifier can be obtained by writing KCL at node v_-.

$$\frac{v_s}{R} + \frac{v_s - v_o}{R_f} = 0 \Rightarrow\Rightarrow \frac{v_o}{v_s} = 1 + \frac{R_f}{R}$$

If we assume $A \to \infty$, then the gain of part (a) will be as follows.

$$\lim_{A\to\infty} G = \frac{v_s R_L(R_f R_{in} A + R R_{in} A)}{A R R_{in} R_L} = \frac{R_f}{R} + 1$$

So, the desired condition is that $A \to \infty$.

(c) Do the currents crossing the red closed surface of Fig. 2(b) constitute a KCL?

No, because operational amplifiers are active elements and need power supplies. These power supplies also require currents which cross the red closed surface and should be considered in the KCL.

Question 3

Consider the circuit shown in Fig. 3, where V_{ref} is provided by a regulated voltage source. Show that the circuit can act like a current source and find the constant current I_s flowing to the resistive load R_L.

Question 3 continued on next page...

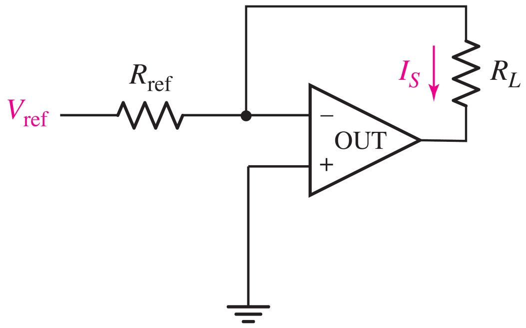

Figure 3: An op amp-based current source.

The voltage at the inputs of the Op Amp is 0. A KCL at the inverting leg of the Op Amp results in $\frac{V_{ref}}{R_{ref}} = I_S$. Clearly, the constant current I_S does not depend on R_L and flows through the load resistor R_L, regardless of its value. Practically, the load current remains constant as long as voltage or current saturation constraints hold.

Question 4

Apply a unit-step function, $x(t) = u(t)$, as the input to a system whose impulse response is $h(t) = u(t) - 2u(t-1) + u(t-2)$, and determine the corresponding output $y(t) = x(t) * h(t)$.

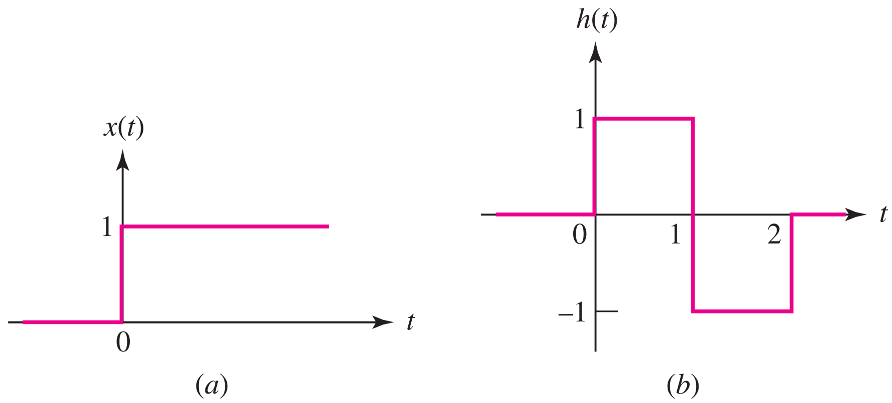

Figure 4: Sketches of (a) the input signal and (b) the unit impulse response for a linear system.

Question 4 continued on next page...

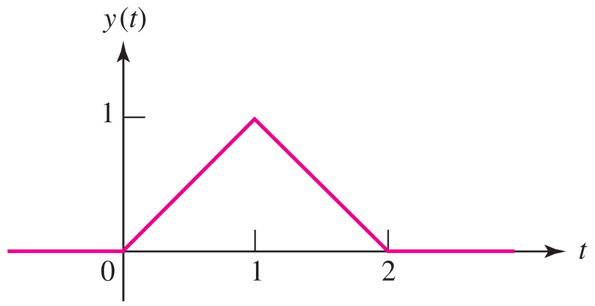

Figure 5: The result of convolving $x(t)$ and $h(t)$ as shown in Fig. 4.

We use the analytical method. Note that $u(t) * u(t) = r(t)$ and $f(t) * \delta(t - t_0) = f(t - t_0)$. We have

$$y(t) = x(t)*h(t) = u(t)*(u(t)-2u(t-1)+u(t-2)) = u(t)*u(t)-2u(t)*u(t-1)+u(t)*u(t-2)$$

$$y(t) = r(t) - 2u(t) * u(t) * \delta(t-1) + u(t) * u(t) * \delta(t-2) = r(t) - 2r(t-1) + r(t-2)$$

. $y(t)$ is plotted in Fig. 5.

Question 5

Find the convolution of the two exponential signals $e^{-\alpha t}u(t)$ and $e^{-\beta t}u(t)$. Feel free to use graphical, analytical, or any other method you know.

Assuming $\alpha \neq \beta$,

$$e^{-\alpha t}u(t) * e^{-\beta t}u(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau = e^{-\beta t}u(t)\int_0^t e^{(\beta-\alpha)\tau}d\tau$$

$$= u(t)e^{-\beta t}\frac{1}{\beta-\alpha}e^{(\beta-\alpha)\tau}\Big|_0^t = u(t)\frac{1}{\beta-\alpha}e^{-\beta t}(e^{(\beta-\alpha)t}-1) = \frac{1}{\beta-\alpha}(e^{-\alpha t}-e^{-\beta t})u(t)$$

. For $\alpha = \beta$,

$$e^{-\alpha t}u(t) * e^{-\alpha t}u(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\alpha(t-\tau)}u(t-\tau)d\tau = e^{-\alpha t}u(t)\int_0^t d\tau = te^{-\alpha t}u(t)$$

Question 6

Consider a series RL circuit driven with the voltage source $v(t)$, where the loop current $i(t)$ should be calculated.

Question 6 continued on next page...

(a) *Find the zero-input response if the initial current is $i(0) = I_0$.*

$$Li'(t) + Ri(t) = 0, \quad i(0) = I_0$$

, which is a homogeneous first-order equation with the solution

$$i(t) = I_0 e^{-\frac{R}{L}t}$$

(b) *Find the step response.*

$$Li'(t) + Ri(t) = u(t), \quad i(0) = 0$$

, which is a non-homogeneous first-order equation with the solution

$$s(t) = i(t) = \frac{1}{R}(1 - e^{-\frac{R}{L}t})u(t)$$

(c) *Find the impulse response.*

$$h(t) = s'(t) = \frac{1}{R}(1 - e^{-\frac{R}{L}t})\delta(t) + \frac{1}{L}e^{-\frac{R}{L}t}u(t) = \frac{1}{L}e^{-\frac{R}{L}t}u(t)$$

(d) *Find the zero-state response if $v(t) = V_0 e^{-t}u(t)$.*

$$i(t) = V_0 e^{-t}u(t) * \frac{1}{L}e^{-\frac{R}{L}t}u(t) = \frac{V_0}{L}e^{-t}u(t) * e^{-\frac{R}{L}t}u(t)$$

. If $R \neq L$,

$$i(t) = \frac{V_0}{L}\frac{1}{\frac{R}{L} - 1}(e^{-t} - e^{-\frac{R}{L}t})u(t) = \frac{V_0}{R - L}(e^{-t} - e^{-\frac{R}{L}t})u(t)$$

. When $R = L$,

$$i(t) = \frac{V_0}{L}te^{-t}u(t)$$

.

(e) *Find the complete response if $v(t) = V_0 e^{-t}u(t)$ and $i(0) = I_0$.*

If $R \neq L$,

$$i(t) = I_0 e^{-\frac{R}{L}t} + \frac{V_0}{R - L}(e^{-t} - e^{-\frac{R}{L}t})u(t)$$

. When $R = L$,

$$i(t) = I_0 e^{-t} + \frac{V_0}{L}te^{-t}u(t)$$

.

SOFTWARE QUESTIONS

Question 7

Simulate a non-inverting amplifier with the gain 2 and an inverting amplifier with the gain -2 in PSpice. Use LM324 for the op amp. Apply a suitable periodic voltage to each amplifier and investigate the corresponding output. Increase the frequency and amplitude of the input and observe the results.

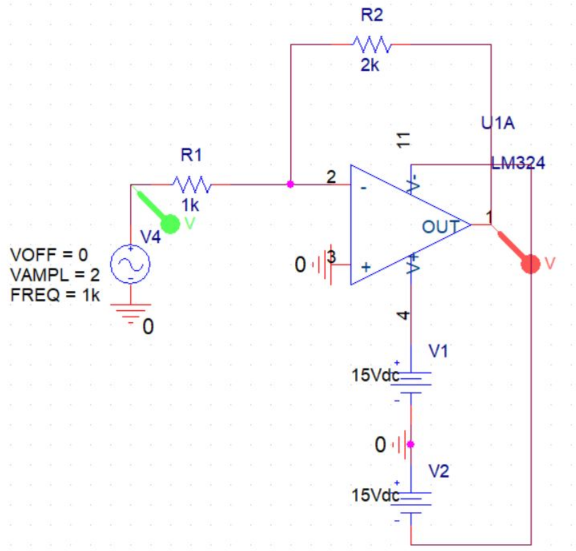

Figure 6: Schematic of the inverting amplifier with gain 2.

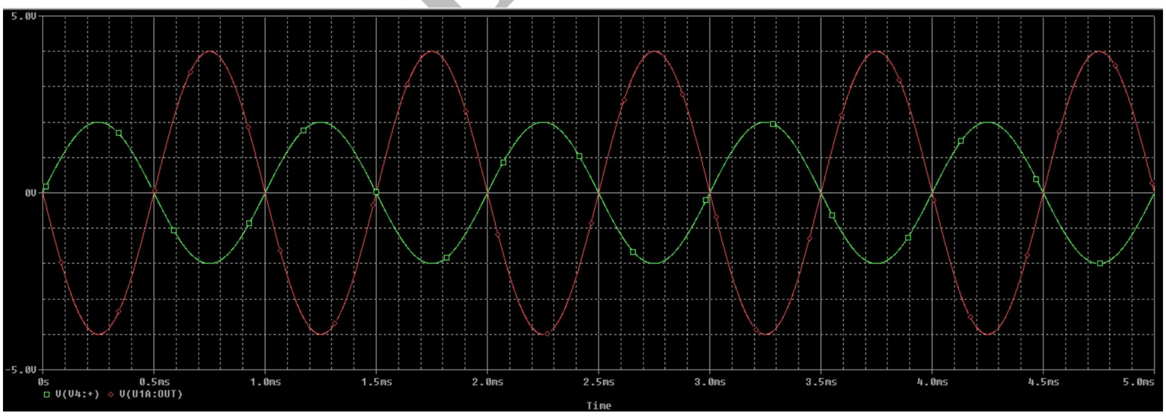

Figure 7: Normal operation of the inverting amplifier when the input amplitude is 2 V, the input frequency is 1 kHz, and the supply voltages are ± 15.

Question 7 continued on next page...

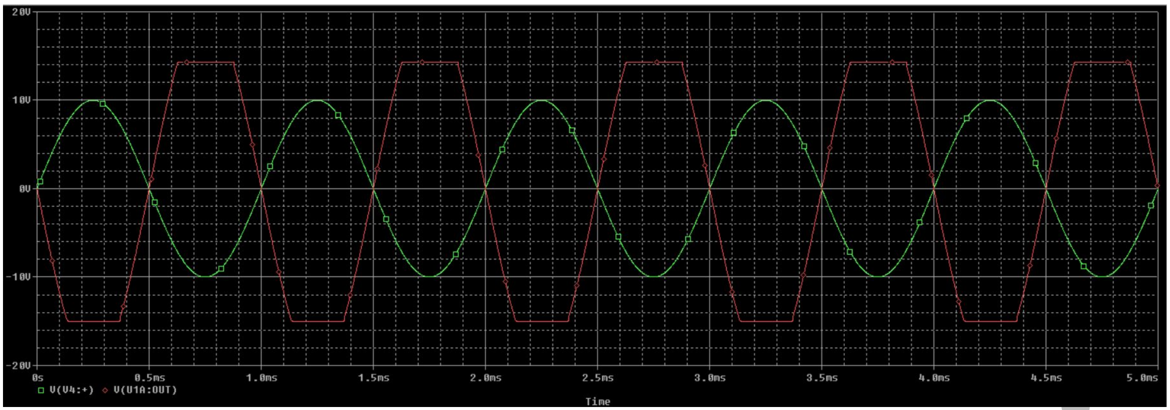

Figure 8: Output saturation for the inverting amplifier when the input amplitude is 10 V, the input frequency is 1 kHz, and the supply voltages are ± 15.

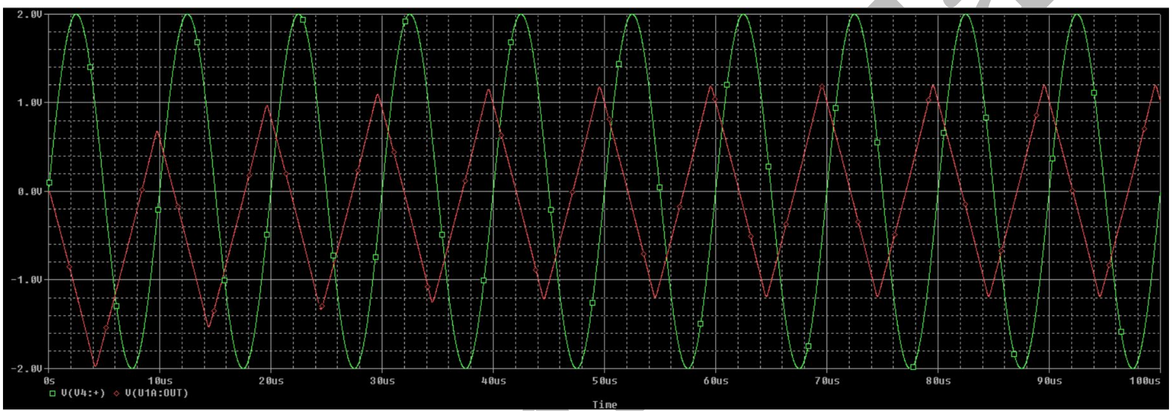

Figure 9: Output slew rate for the inverting amplifier when the input amplitude is 2 V, the input frequency is 100 kHz, and the supply voltages are ± 15.

Question 7 continued on next page...

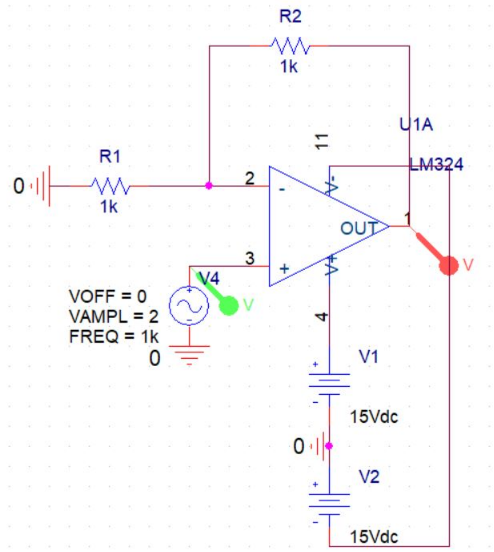

Figure 10: Schematic of the non-inverting amplifier with gain 2.

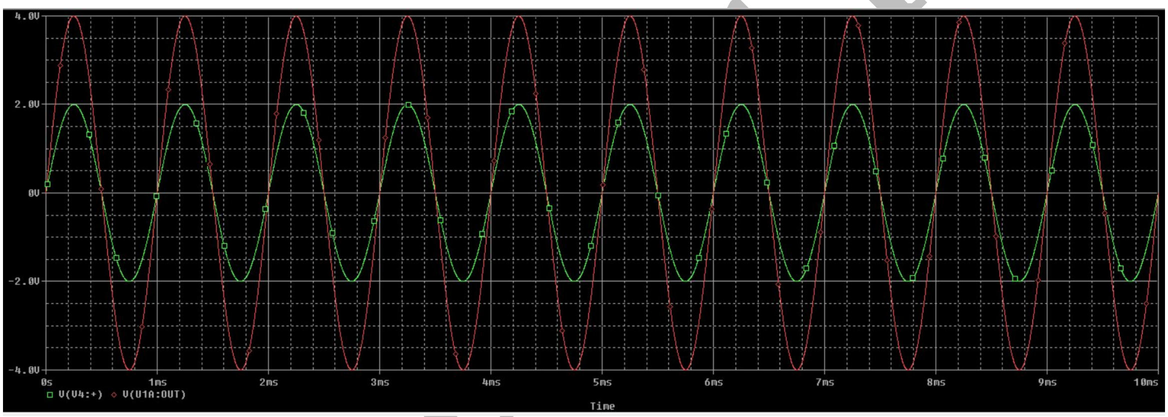

Figure 11: Normal operation of the non-inverting amplifier when the input amplitude is 2 V, the input frequency is 1 kHz, and the supply voltages are ± 15.

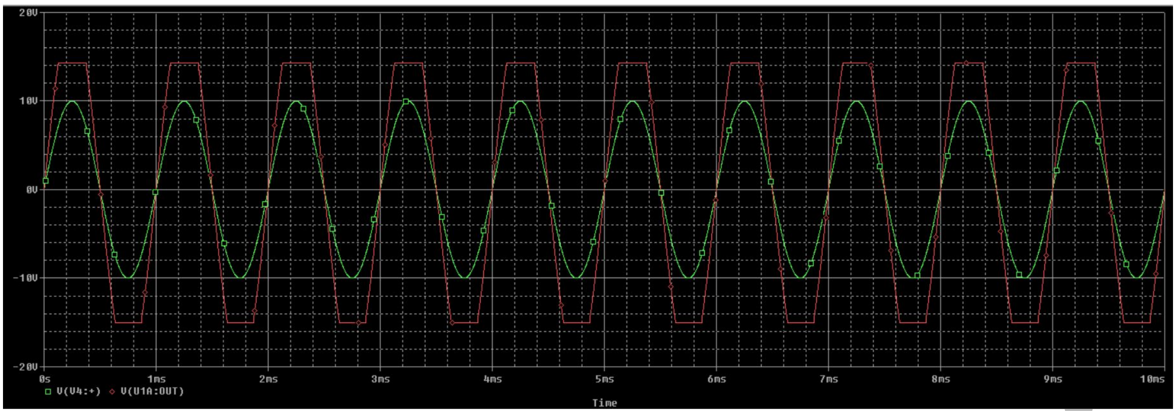

Figure 12: Output saturation for the non-inverting amplifier when the input amplitude is 10 V, the input frequency is 1 kHz, and the supply voltages are ± 15.

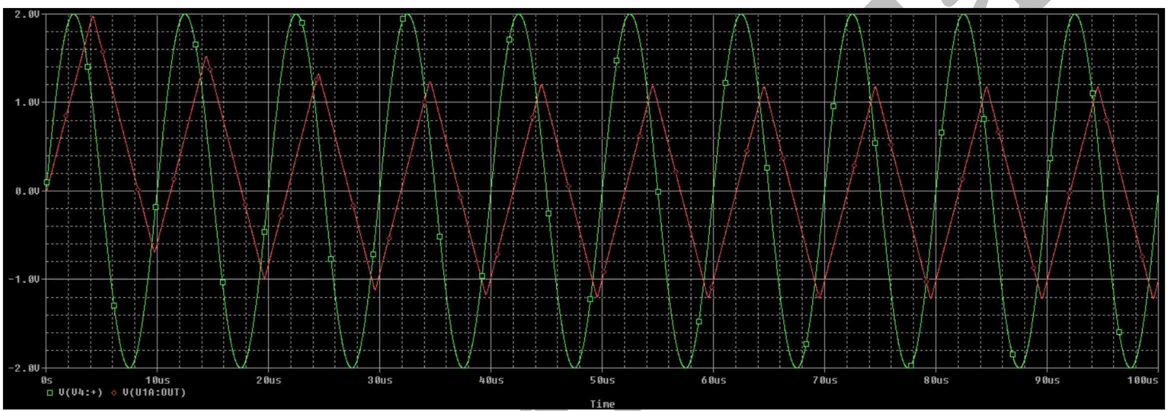

Figure 13: Output slew rate for the non-inverting amplifier when the input amplitude is 2 V, the input frequency is 100 kHz, and the supply voltages are ± 15.

The schematic of the inverting amplifier is shown in Fig. 6. Fig. 7 shows the normal operation of the inverting amplifier while Figs. 8-9 demonstrate how voltage saturation and slew rate affects the normal operation of the amplifier. Figs. 10-13 shows the results for the non-inverting amplifier.

BONUS QUESTIONS

Question 8

Repeat Question 6 if $v(t) = V_0 \cos(\omega t + \theta) u(t)$**.**

The zero-input, impulse, and step responses do not change by altering the input voltage. When the input is $v(t) = V_0 \cos(\omega t + \theta)u(t)$, the zero-state response equals

$$v(t) * h(t) = V_0 \cos(\omega t + \theta)u(t) * \frac{1}{L}e^{-\frac{R}{L}t}u(t) = \frac{V_0}{L}[\frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2}]u(t) * e^{-\frac{R}{L}t}u(t)$$

$$= \frac{V_0}{2L}\left[e^{j\theta}e^{j\omega t}u(t) * e^{-\frac{R}{L}t}u(t) + e^{-j\theta}e^{-j\omega t}u(t) * e^{-\frac{R}{L}t}u(t)\right]$$

$$= \frac{V_0}{2L}\left[e^{j\theta}\frac{1}{\frac{R}{L} + j\omega}(e^{j\omega t} - e^{-\frac{R}{L}t})u(t) + e^{-j\theta}\frac{1}{\frac{R}{L} - j\omega}(e^{-j\omega t} - e^{-\frac{R}{L}t})u(t)\right]$$

$$= \frac{V_0}{2L}u(t)[\frac{1}{\frac{R}{L} + j\omega}e^{j(\omega t+\theta)} + \frac{1}{\frac{R}{L} - j\omega}e^{-j(\omega t+\theta)}] - \frac{V_0}{2L}u(t)e^{-\frac{R}{L}t}[\frac{1}{\frac{R}{L} + j\omega}e^{j\theta} + \frac{1}{\frac{R}{L} - j\omega}e^{-j\theta}]$$

$$= \frac{V_0}{2L}u(t) \times 2\text{Re}\{\frac{1}{\frac{R}{L} + j\omega}e^{j(\omega t+\theta)}\} - \frac{V_0}{2L}u(t)e^{-\frac{R}{L}t} \times 2\text{Re}\{\frac{1}{\frac{R}{L} + j\omega}e^{j\theta}\}$$

$$= \frac{V_0}{L}u(t)\times\text{Re}\{\frac{1}{\sqrt{\frac{R^2}{L^2} + \omega^2}}e^{j(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R}))}\} - \frac{V_0}{L}u(t)e^{-\frac{R}{L}t}\times\text{Re}\{\frac{1}{\sqrt{\frac{R^2}{L^2} + \omega^2}}e^{j(\theta-\tan^{-1}(\frac{\omega L}{R}))}\}$$

$$= \frac{V_0}{L}u(t)\frac{1}{\sqrt{\frac{R^2}{L^2} + \omega^2}}\cos(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R})) - \frac{V_0}{L}u(t)e^{-\frac{R}{L}t}\frac{1}{\sqrt{\frac{R^2}{L^2} + \omega^2}}\cos(\theta-\tan^{-1}(\frac{\omega L}{R}))$$

$$= \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\cos(\omega t + \theta - \tan^{-1}(\frac{\omega L}{R}))u(t) - \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}e^{-\frac{R}{L}t}\cos(\theta - \tan^{-1}(\frac{\omega L}{R}))u(t)$$

Finally, the complete solution equals

$$i(t) = I_0 e^{-\frac{R}{L}t} +$$

$$\frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\cos(\omega t + \theta - \tan^{-1}(\frac{\omega L}{R}))u(t) - \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}e^{-\frac{R}{L}t}\cos(\theta - \tan^{-1}(\frac{\omega L}{R}))u(t)$$

. When the transients die, the complete response is equal to

$$i(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\cos(\omega t + \theta - \tan^{-1}(\frac{\omega L}{R}))u(t)$$

, which is exactly the steady state sinusoidal current. In a better presentation,

$$i(t) = |\frac{V_0 e^{j\theta}}{R + j\omega L}|\cos(\omega t + \angle[\frac{V_0 e^{j\theta}}{R + j\omega L}])u(t)$$

, where $\frac{V_0 e^{j\theta}}{R+j\omega L}$ is the phasor of the current.

Question 9

Return your answers by filling the LaTeXtemplate of the assignment.

EXTRA QUESTIONS

Question 10

Feel free to solve the following questions from the book *"Engineering Circuit Analysis"* by W. Hayt, J. Kemmerly, and S. Durbin.

1. **Chapter 6, question 12.**
2. **Chapter 6, question 13.**
3. **Chapter 6, question 14.**
4. **Chapter 6, question 17.**
5. **Chapter 6, question 20.**
6. **Chapter 6, question 21.**
7. **Chapter 6, question 23.**
8. **Chapter 6, question 28.**
9. **Chapter 6, question 29.**
10. **Chapter 6, question 34.**
11. **Chapter 6, question 38.**
12. **Chapter 6, question 40.**
13. **Chapter 6, question 45.**