Summary

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Overview

Lumped Circuits

Maxwell and Kirchhoff Equations

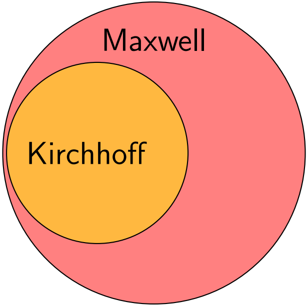


Figure: Maxwell and Kirchhoff equations.

- Maxwell's equation: Sophisticated vector quantities $\vec{E}, \vec{H}, \vec{D}, \vec{B}$
- Kirchhoff's equations: Simplified scalar quantities v, i, q, ϕ
- Lumped condition: max{circuit dimension}$\ll$min{circuit wavelength}

Example (Lump condition)

Intel Core i7-4702HQ processor with the package size 37.5mm $\times$ 32mm $\times$ 1.6mm and the max turo frequency 3.2 GHz is not a lumped circuit since its maximum dimension $d \approx \sqrt{37.5^2 + 32^2 + 1.6^2} = 49.32$ mm is in the order of minimum operating wavelength $\lambda \approx 3 \times 10^{11}/(3.2 \times 10^9) = 93.72$ mm.

Example (Lump condition)

The power transmission system is a lumped circuit over Tehran city since the maximum transmission distance $d \approx 50$ km is much less than the operating wavelength $\lambda \approx 3 \times 10^5/50 = 6000$ km.

Circuit Element

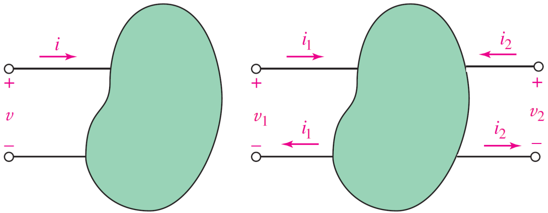


Figure: Passive sign convention in one-port and two-port circuit elements.

- Circuit element: an entity with voltage and current ports.
- One-port element: an element with two connection terminals.
- Passive sign convention: the current flows to the plus terminal.
- Absorbed power: assuming passive sign convention, $p = vi$.

Circuit Laws

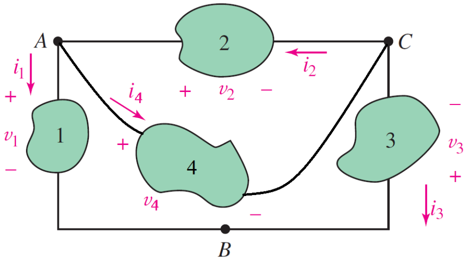
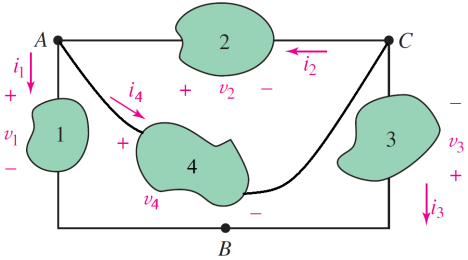


Figure: Kirchhoff's circuit laws for a sample circuit.

- Circuit: an interconnection of elements under an arbitrary topology.
- KCL: for the entering (exiting) currents at each node, $\sum_k i_k = 0$.
- KVL: for the aligned voltages around each closed path, $\sum_k v_k = 0$.
- Tellegen: for all branches, $\sum_k v_k i_k = 0$.

Example (Circuit laws)

In the shown circuit, KCL at node A gives $i_1 + i_4 - i_2 = 0$ and KVL around loop ABC yields $v_1 + v_3 - v_2 = 0$. Elements 1 and 3 absorb the power $p_1 = v_1 i_1$ and $p_4 = -v_3 i_3$, respectively. Further, according to Tellegen's theorem, $v_1 i_1 - v_2 i_2 - v_3 i_3 + v_4 i_4 = 0$.



Circuit Elements

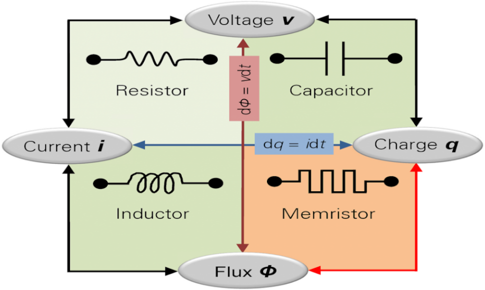


Figure: Basic one-port circuit elements.

- Characteristic curve: $f(y, x, t) = 0, \;\; x, y \in \{v, i, \phi, q\}$.
- Linear element: $f(y, x, t) = 0$ is an explicit linear function.
- Time-invariant element: $f(y, x) = 0$ is independent of t.

Basic One-port Elements

Element	LTI	LTV	NTI	NTV
Resistor	$v = Ri$	$v = R(t)i$	$f(v, i) = 0$	$f(v, i, t) = 0$
Capacitor	$q = Cv$	$q = C(t)v$	$f(q, v) = 0$	$f(q, v, t) = 0$
Inductor	$\phi = Li$	$\phi = L(t)i$	$f(\phi, v) = 0$	$f(\phi, v, t) = 0$

Table: Basic one-port circuit elements. L, N, TI, and TV stand for Linear, Nonlinear, Time-Invariant, Time-Variant, respectively.

1. x-controlled element: $f(y, x, t) = 0 \Rightarrow y = g(x, t)$.
2. x-controlled element: $f(y, x) = 0 \Rightarrow y = g(x)$.
3. Voltage-flux relation: $v = d\phi/dt$.
4. Current-charge relation: $i = dq/dt$.
5. Absorbed power: $p = vi$.
6. Absorbed energy over interval $[t_0, t]$: $w(t_0, t) = \int_{t_0}^{t} pdt'$.
7. Passive element: $\forall [t_0, t], W(t_0, t) \geq 0$.
8. Active element: $\exists [t_0, t], W(t_0, t) < 0$.

Example (Diode)

A diode with the following typical characteristic curve is an NTI voltage-controlled (current-controlled) passive resistor.

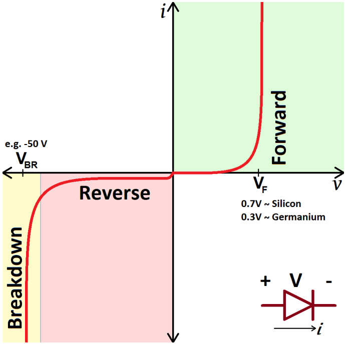


Figure: Typical characteristic curve of a diode.

Element	Characteristic Equation	Voltage Equation	Current Equation
Resistor	$v(t) = Ri(t)$	$v(t) = Ri(t)$	$i(t) = \frac{v(t)}{R}$
Capacitor	$q(t) = Cv(t)$	$v(t) = v(t_0) + \frac{\int_{t_0}^{t} i(t')dt'}{C}$	$i(t) = C\frac{dv(t)}{dt}$
Inductor	$\phi(t) = Li(t)$	$v(t) = L\frac{di(t)}{dt}$	$i(t) = i(t_0) + \frac{\int_{t_0}^{t} v(t')dt'}{L}$

Table: Basic LTI circuit elements.

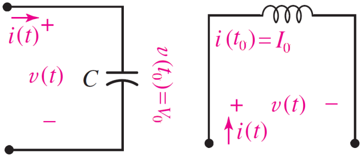


Figure: For complete description of capacitors and inductors, an initial condition is required.

Basic LTI Elements

Element	Characteristic Equation	Energy	Passivity
Resistor	$v(t) = Ri(t)$	$\mathcal{E}_H(t) = R\int_0^t i^2(t')dt'$	$R \geq 0$
Capacitor	$q(t) = Cv(t)$	$\mathcal{E}_E(t) = \frac{1}{2}Cv^2(t)$	$C \geq 0$
Inductor	$\phi(t) = Li(t)$	$\mathcal{E}_M(t) = \frac{1}{2}Li^2(t)$	$L \geq 0$

Table: Energy for basic LTI circuit elements. The initial energy at the reference time t_0 is assumed to be zero.

1. Resistors: the absorbed energy is dissipated as heat energy.
2. Capacitors: the absorbed energy is stored as electrical energy.
3. Inductors: the absorbed energy is stored as magnetic energy.

Basic Active Elements

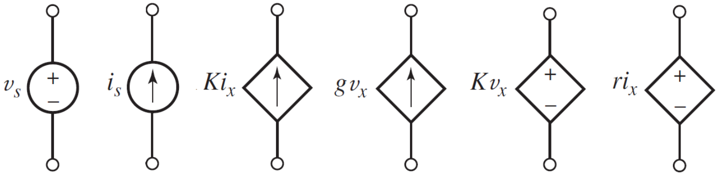


Figure: Basic active circuit elements. From left to right, independent voltage source, independent current source, LTI dependent current-controlled current source, LTI dependent voltage-controlled current source, LTI dependent voltage-controlled voltage source, and LTI dependent current-controlled voltage source.

1. Sources: a subset of (nonlinear) resistors.
2. Dependent sources: a subset of two-port elements.
3. LTI dependent sources: a subset of LTI elements.

Example (Initial condition modeling)

Initial conditions can be modeled by independent sources.

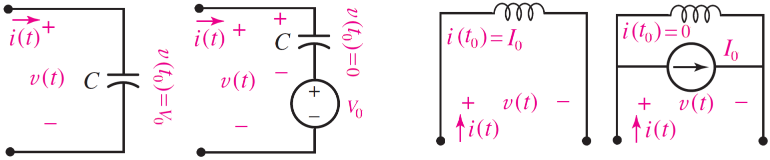


Figure: For complete description of capacitors and inductors, an initial condition is required. Initial conditions can be replaced with independent sources.

Example (Short and open circuit)

A voltage source set to zero acts like a short circuit (zero resistor) while a current source set to zero acts like an open circuit (infinite resistor).

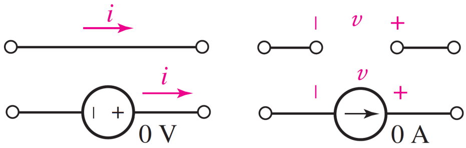


Figure: Zero-voltage and zero-current independent sources.

Parallel and Series Connections

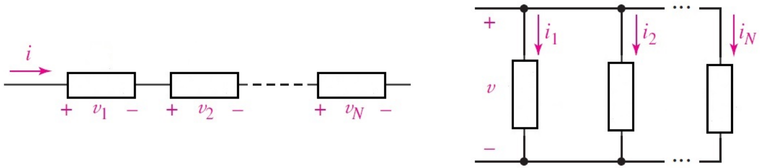


Figure: Parallel and series connection. Series (parallel) elements have the same current (voltage).

Element	Series Connection	Parallel Connection
Resistor	$R = \sum_i R_i$	$G = \sum_i G_i$
Capacitor	$S = \sum_i S_i$	$C = \sum_i C_i$
Inductor	$L = \sum_i L_i$	$\Gamma = \sum_i \Gamma_i$

Table: Parallel and series connection of basic linear elements. R, G, C, S, L, and Γ denote resistance, conductance, capacitance, elastance, inductance, and reciprocal inductance, respectively. The initial conditions are assumed to be zero.

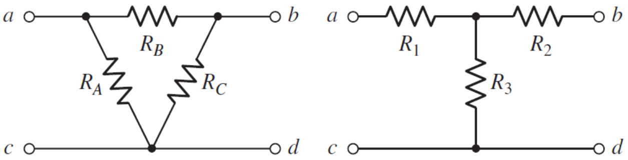


Figure: Resistive Δ (triangle, $\prod$) and Y (star, T) networks. If the two networks are equivalent, then the port voltages and currents must be equal.

$$R_A = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2} \qquad R_1 = \frac{R_AR_B}{R_A + R_B + R_C}$$

$$R_B = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_3} \qquad R_2 = \frac{R_BR_C}{R_A + R_B + R_C}$$

$$R_C = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1} \qquad R_3 = \frac{R_CR_A}{R_A + R_B + R_C}$$

Ideal Operational Amplifier

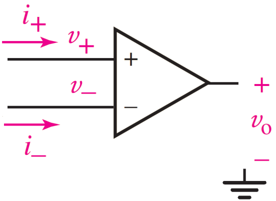


Figure: An ideal operational amplifier in which $i_- = 0$, $i_+ = 0$, and $v_- = v_+$.

1. No current at each input terminal.
2. No voltage difference between the input terminals.
3. Negative feedback for stability.
4. A member of LTI elements.

Circuit Analysis

Circuit Analysis

Definition (Circuit Variables)

Branch currents and branch voltages in a given circuit are called circuit variables.

Definition (Circuit Analysis)

The circuit analysis problem is to determine all or part of the circuit variables for a circuit.

1. Basic circuit analysis procedures: nodal and mesh analysis
2. Nodal analysis: KCL-based analysis.
3. Mesh analysis: KVL-based analysis.

Nodal Analysis

Nodal analysis procedures:

1. Count the number of nodes (N nodes).
2. Designate a reference node (usually, a high-degree node).
3. Label the nodal voltages ($N - 1$ labels).
4. Form a supernode about each voltage source and relate its voltage to nodal voltages.
5. Write a KCL equation for each nonreference node and for each supernode that does not contain the reference node. Use element equations to express the currents in terms of nodal voltages.
6. Express any additional unknowns in terms of appropriate nodal voltages (occurs for dependent sources).
7. Organize the equations.
8. Solve the system of equations for the nodal voltages ($N - 1$ equations).

✓ Handy nodal analysis: appropriate the circuits with a low number of nodes.

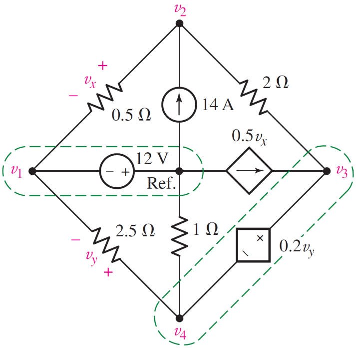
Nodal Analysis

Example (Nodal analysis)

In the circuit below, $v_1 = -12$ V, $v_2 = -4$ V, $v_3 = 0$ V, and $v_4 = -2$ V.

$$\begin{cases} v_1 = -12 \\ v_3 - v_4 = 0.2v_y \\ \dfrac{v_1 - v_2}{0.5} + \dfrac{v_3 - v_2}{2} + 14 = 0 \\ \dfrac{v_1 - v_4}{2.5} + \dfrac{-v_4}{1} + \dfrac{v_2 - v_3}{2} + 0.5v_x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} v_1 = -12 \\ v_3 - v_4 = 0.2v_4 - 0.2v_1 \\ \dfrac{v_1 - v_2}{0.5} + \dfrac{v_3 - v_2}{2} + 14 = 0 \\ \dfrac{v_1 - v_4}{2.5} + \dfrac{-v_4}{1} + \dfrac{v_2 - v_3}{2} + 0.5(v_2 - v_1) = 0 \end{cases}$$



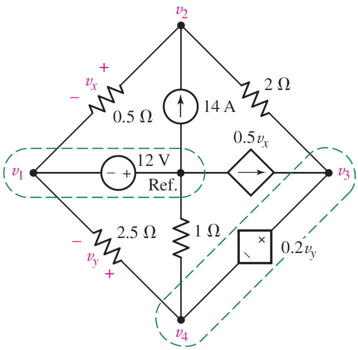
Example (Nodal analysis (cont.))

In the circuit below, $v_1 = -12$ V, $v_2 = -4$ V, $v_3 = 0$ V, and $v_4 = -2$ V.

$$\Rightarrow \begin{cases} -2v_1 + 2.5v_2 - 0.5v_3 = 14 \\ 0.1v_1 - v_2 + 0.5v_3 + 1.4v_4 = 0 \\ v_1 = -12 \\ 0.2v_1 + v_3 - 1.2v_4 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} -2 & 2.5 & -0.5 & 0 \\ 0.1 & -1 & 0.5 & 1.4 \\ 1 & 0 & 0 & 0 \\ 0.2 & 0 & 1 & -1.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ -12 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 = \frac{\begin{vmatrix} -2 & 14 & -0.5 & 0 \\ 0.1 & 0 & 0.5 & 1.4 \\ 1 & -12 & 0 & 0 \\ 0 & 0 & 1 & -1.2 \end{vmatrix}}{\begin{vmatrix} -2 & 2.5 & -0.5 & 0 \\ 0.1 & -1 & 0.5 & 1.4 \\ 1 & 0 & 0 & 0 \\ 0.2 & 0 & 1 & -1.2 \end{vmatrix}} = -4$$



Mesh Analysis

Mesh analysis procedures:

1. Make sure that the circuit is planar.
2. Count the number of meshes (M meshes).
3. Label the mesh currents (M labels).
4. Form a supermesh to enclose the meshes shares a current source and relate its current to mesh currents.
5. Write a KVL equation around each mesh and supermesh. Use element equations to express the voltages in terms of mesh currents.
6. Express any additional unknowns in terms of appropriate mesh currents (occurs for dependent sources).
7. Organize the equations.
8. Solve the system of equations for the mesh currents (M equations).

✓ Handy mesh analysis: appropriate the for the planar circuits with a low number of meshes.

Mesh Analysis

Example (Mesh analysis)

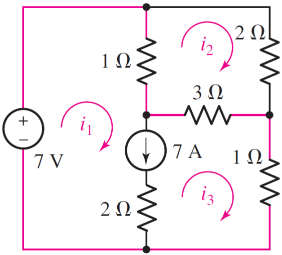
In the circuit below, $i_1 = 9$ A, $i_2 = 2.5$ A, and $i_3 = 2$ A.

$$\begin{cases} i_1 - i_3 = 7 \\ (i_2 - i_1) + 2i_2 + 3(i_2 - i_3) = 0 \\ (i_1 - i_2) + 3(i_3 - i_2) + (i_3) - 7 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} i_1 \quad\quad - \quad i_3 = 7 \\ -i_1 + 6i_2 - 3i_3 = 0 \\ i_1 - 4i_2 + 4i_3 = 7 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ -1 & 6 & -3 \\ 1 & -4 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$$

$$\Rightarrow i_2 = \frac{\begin{vmatrix} 1 & 7 & -1 \\ -1 & 0 & -3 \\ 1 & 7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -1 \\ -1 & 6 & -3 \\ 1 & -4 & 4 \end{vmatrix}} = 2.5$$



Linear and Time-invariant Circuits

Input and Response

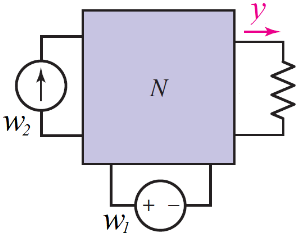


Figure: Inputs w_1, w_2 and response y in a multi-input general circuit.

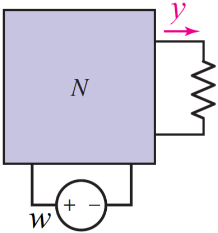


Figure: Input w and response y in a single-input general circuit.

- Each input corresponds to an independent source.
- Each response corresponds to a desired circuit variable.

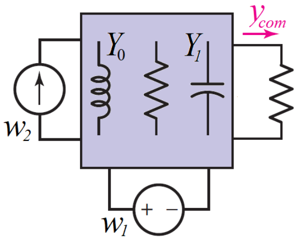


Figure: Complete response y_{com}.

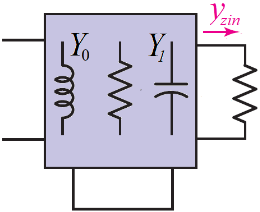


Figure: Zero-input response (natural) y_{zin}.

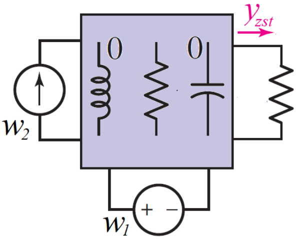


Figure: Zero-state response (forced) y_{zst}.

Circuit Classification

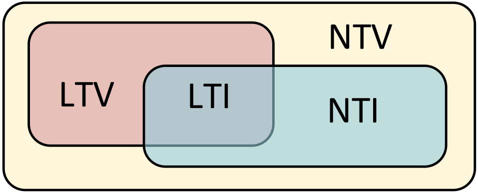


Figure: Common classification of circuits.

- Linear circuit: A circuit with only linear elements or independent sources.
- Time-invariant circuit: A circuit with only time-invariant elements or independent sources.
- LTI circuit: A circuit with only LTI elements or independent sources.

Linear Circuits

Theorem (Linear Circuits)

For linear circuits

- $y_{com} = y_{zin} + y_{zst}$.
- y_{zst} *is a linear function (superposition) of the inputs* $\mathrm{w} = [w_1, w_2, \cdots]$.
- y_{zin} *is a linear function (superposition) of the initial state* $\mathrm{Y} = [Y_0, Y_1, \cdots]$.

LTI Circuits

Theorem (LTI Circuits)

For each input-response pair in an LTI circuits,

- *The complete response satisfies a linear differential equation with constant coefficients.*
- *The zero-state response to an arbitrary input* $w(t)u(t)$ *is* $y_{zst}(t) = [w(t)u(t)] * h(t) = u(u)\int_0^t w(u)h(t-u)du$*, where* $h(t)$ *is the causal impulse response.*
- *If* $y_{zst}(t)$ *is the zero-state response to the input* $w(t)$*, the zero-state response to the input* $w(t-t_0)$ *is* $y_{zst}(t-t_0)$*.*
- *The impulse and unit step responses relate together via* $h(t) = \frac{ds(t)}{dt}$*.*

Constant-coefficient Linear Differential Equations

Theorem (Homogeneous Response)

The homogeneous response of the constant-coefficient linear differential equation

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = 0, \quad y^{(i)}(0) = Y_i, i = 0, 1, \cdots, n-1$$

is of the form

$$y(t) = \sum_{k=1}^{n} A_k e^{s_k t}, t \geq 0$$

, where $s_k, k = 1, \cdots, n$ are distinct roots of the characteristic equation $\sum_{k=0}^{n} a_k s^k = 0$. If a root has multiplicity, the corresponding exponential terms should be replaced by $e^{s_k t}, te^{s_k t}, t^2 e^{s_k t}, \cdots$. The constants A_k are obtained by substituting the initial conditions to the response.

Constant-coefficient Linear Differential Equations

Theorem (Impulse Response)

The impulse response $h(t)$ of the constant-coefficient linear differential equation

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{l=0}^{m} b_l w^{(l)}(t), \quad y^{(i)}(0) = 0, i = 0, 1, \cdots, n-1$$

is of the form

$$h(t) = \begin{cases} u(t) \sum_{k=1}^{n} A_k e^{s_k t} & , \quad n > m \\ u(t) \sum_{k=1}^{n} A_k e^{s_k t} + \sum_{k=n-m}^{0} A_k \delta^{(i)}(t) & , \quad n \leq m \end{cases}$$

, where $s_k, k = 1, \cdots, n$ are distinct roots of the characteristic equation $\sum_{k=0}^{n} a_k s^k = 0$. If a root has multiplicity, the corresponding exponential terms should be replaced by $e^{s_k t}, te^{s_k t}, t^2 e^{s_k t}, \cdots$. The constants A_k are obtained by substituting $y(t) = h(t)$ and $w(t) = \delta(t)$ into the differential equation and equating its both sides.

Example (First-order circuit)

The complete response of a first-order circuit relates to the time constant τ.

$$i(t) = i_{zin}(t) + i_{zst}(t) = i_{zin}(t) + i_{zst1}(t) + i_{zst2}(t), t \geq 0$$

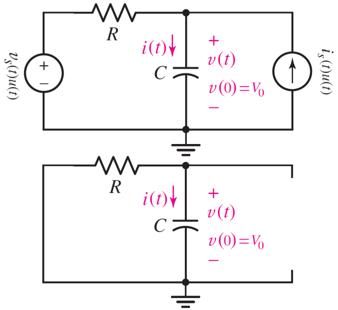
$$Ri_{zin}(t) + V_0 + \frac{1}{C}\int_0^t i_{zin}(u)du = 0, \quad i_{zin}(0) = -\frac{V_0}{R}$$

$$i'_{zin}(t) + \frac{1}{\tau}i_{zin}(t) = 0, \quad i_{zin}(0) = -\frac{V_0}{R}, \tau = RC$$

$$s + \frac{1}{\tau} = 0 \Rightarrow s = -\frac{1}{\tau} \Rightarrow i_{zin}(t) = Ae^{-\frac{t}{\tau}}$$

$$i_{zin}(0) = A = -\frac{V_0}{R}$$

$$i_{zin}(t) = -\frac{V_0}{R}e^{-\frac{t}{\tau}}, t \geq 0$$



Example (First-order circuit (cont.))

The complete response of a first-order circuit relates to the time constant τ.

$$Rh_1(t) + \frac{1}{C}\int_0^t h_1(u)du - \delta(t) = 0, \quad h_1(0) = 0$$

$$h_1'(t) + \frac{1}{\tau}h_1(t) = \frac{1}{R}\delta'(t), \quad h_1(0) = 0, \tau = RC$$

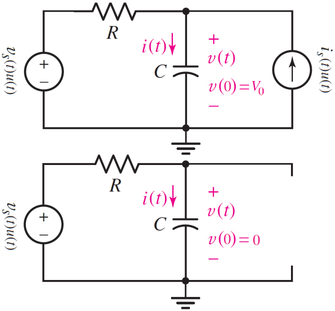
$$s + \frac{1}{\tau} = 0 \Rightarrow s = -\frac{1}{\tau} \Rightarrow h_1(t) = A_1 e^{-\frac{t}{\tau}}u(t) + A_0\delta(t)$$

$$-\frac{A_1}{\tau}e^{-\frac{t}{\tau}}u(t) + A_1\delta(t) + A_0\delta'(t) + \frac{A_1}{\tau}e^{-\frac{t}{\tau}}u(t) + \frac{A_0}{\tau}\delta(t) = \frac{1}{R}\delta'(t)$$

$$A_0 = \frac{1}{R}, A_1 = -\frac{1}{R^2C}$$

$$h_1(t) = -\frac{1}{R^2C}e^{-\frac{t}{\tau}}u(t) + \frac{1}{R}\delta(t)$$

$$i_{zst1}(t) = h_1(t) * v_s(t) = u(t)\int_0^t v_s(u)h_1(t-u)du$$



Example (First-order circuit (cont.))

The complete response of a first-order circuit relates to the time constant τ.

$$\frac{\frac{1}{C}\int_0^t h_2(u)du}{R} + h_2(t) - \delta(t) = 0, \quad h_2(0) = 0$$

$$h_2'(t) + \frac{1}{\tau}h_2(t) = \delta'(t), \quad h_2(0) = 0, \tau = RC$$

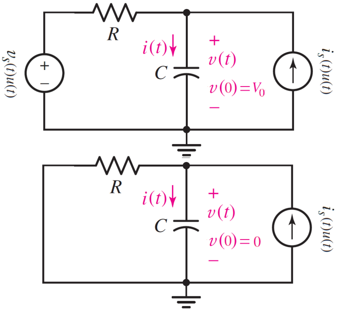
$$s + \frac{1}{\tau} = 0 \Rightarrow s = -\frac{1}{\tau} \Rightarrow h_2(t) = A_1 e^{-\frac{t}{\tau}}u(t) + A_0\delta(t)$$

$$-\frac{A_1}{\tau}e^{-\frac{t}{\tau}}u(t) + A_1\delta(t) + A_0\delta'(t) + \frac{A_1}{\tau}e^{-\frac{t}{\tau}}u(t) + \frac{A_0}{\tau}\delta(t) = \delta'(t)$$

$$A_0 = 1, A_1 = -\frac{1}{RC}$$

$$h_2(t) = -\frac{1}{RC}e^{-\frac{t}{\tau}}u(t) + \delta(t)$$

$$i_{zst2}(t) = h_2(t) * i_s(t) = u(t)\int_0^t i_s(u)h_2(t-u)du$$



Example (Second-order circuit)

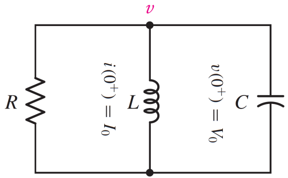
The natural voltage response in a second-order circuit depends on the damping factor α and resonance frequency ω_0 and takes one of possible forms overdamped, critically damped, and underdamped.

$$\begin{cases} \frac{v(t)}{R} + I_0 + \frac{\int_0^t v(u)du}{L} + Cv'(t) = 0 \\ v(0) = V_0, v'(0) = V_1 = \frac{1}{C}(-\frac{V_0}{R} - I_0) \end{cases}$$

$$v''(t) + 2\alpha v'(t) + \omega_0^2 v(t) = 0, \quad \alpha = \frac{1}{2RC}, \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s^2 + 2\alpha s + \omega_0^2 = 0 \Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}, \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$v(t) = \begin{cases} v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} & , \quad \alpha > \omega_0 \\ v(t) = e^{-\alpha t}(A_1 t + A_2) & , \quad \alpha = \omega_0 \\ v(t) = e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)) & , \quad \alpha < \omega_0 \end{cases}$$



Thevenin and Norton Equivalency

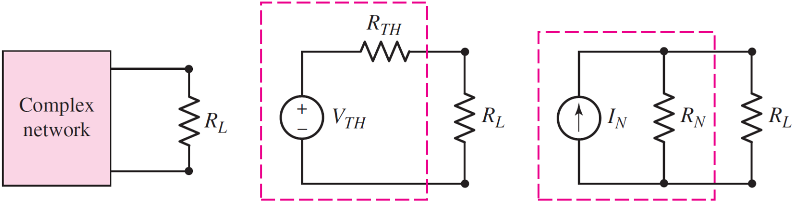


Figure: Thevenin and Norton equivalencies in resistive linear networks, where $R_{TH} = R_N$ and $V_{TH} = R_N I_N$.

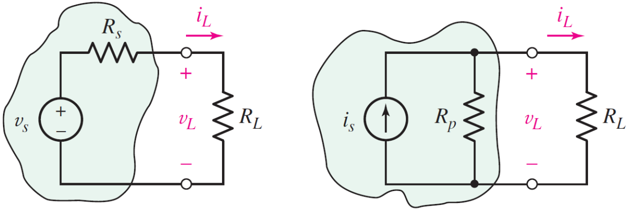


Figure: Source transformation in resistive linear networks, as a special case of Thevenin and Norton equivalencies, where $R_s = R_p$ and $v_s = R_p i_s$.

Dividers and Maximum Power Transfer

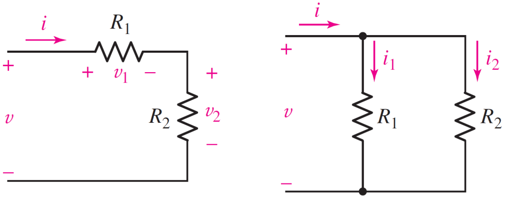


Figure: Rsistive voltage divider, where $v_1 = \frac{R_1}{R_1+R_2} v$ and resistive current dividers, where $i_1 = \frac{R_2}{R_1+R_2} i$.

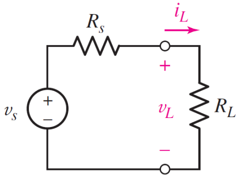


Figure: Maximum power transfer in a resistive network, where $R_s = R_L$.

Sinusoidal Steady-state Analysis

Constant-coefficient Linear Differential Equations

Theorem (Sinusoidal Response)

The sinusoidal response $y(t)$ of the constant-coefficient linear differential equation

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{l=0}^{m} b_l w^{(l)}(t), \quad y^{(i)}(0) = Y_i, i = 0, 1, \cdots, n-1$$

to the input $w(t) = |A|\cos(\omega t + \angle A) = \Re\{Ae^{j\omega t}\}$ is of the form

$$y(t) = y_h(t) + y_p(t) = \sum_{k=1}^{n} A_k e^{s_k t} + |B|\cos(\omega t + \angle B), \quad t \geq 0$$

, where the input phasor $A = |A|e^{j\angle A}$ and $s_k, k = 1, \cdots, n$ are distinct roots of the characteristic equation $\sum_{k=0}^{n} a_k s^k = 0$. If a root has multiplicity, the corresponding exponential terms should be replaced by $e^{s_k t}, te^{s_k t}, t^2 e^{s_k t}, \cdots$. The constants A_k are obtained by substituting the initial conditions into the differential equation while the steady-state response phasor $B = |B|e^{j\angle B}$ is the solution of the equation

$$B/A = H(j\omega) = \sum_{l=0}^{m} b_l (j\omega)^l / \sum_{i=0}^{n} a_i (j\omega)^i$$

, where $H(j\omega)$ is called frequency response or transfer function.

Constant-coefficient Linear Differential Equations

Theorem (Steady-state Sinusoidal Response)

If all the roots of the characteristic equation $\sum_{k=0}^{n} a_k s^k = 0$ *corresponding to the differential equation*

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{l=0}^{m} b_l w^{(l)}(t), \quad y^{(i)}(0) = Y_i, i = 0, 1, \cdots, n-1$$

are in the open left-hand complex plane, the steady-state sinusoidal response $y(t)$ *to the input* $w(t) = |A|\cos(\omega t + \angle A) = \Re\{Ae^{j\omega t}\}$ *is of the form*

$$y(t) = y_p(t) = |B|\cos(\omega t + \angle B), \quad t \geq 0$$

, where the input phasor $A = |A|e^{j\angle A}$*. The steady-state response phasor* $B = |B|e^{j\angle B}$ *is the solution of the equation*

$$B/A = H(j\omega) = \sum_{l=0}^{m} b_l (j\omega)^l / \sum_{i=0}^{n} a_i (j\omega)^i$$

, where $H(j\omega)$ *is called frequency response or transfer function.*

Sinusoidal Steady-state of LTI Circuits

Definition (Natural Frequencies of LTI Circuits)

Natural frequencies are the roots of the characteristic function of the constant-coefficient linear differential equation describing a desired input-response relationship in an LTI system.

Theorem (Sinusoidal Steady-state of LTI Circuits)

If the natural frequencies of an LTI circuit are in the open left-hand complex plane, then, irrespective of the initial state, as time proceeds, the circuit approaches a sinusoidal response, which can be obtained from phasor analysis.

- Nodal and mesh analysis can be used in phasor analysis.
- Superposition can be used for phasor analysis of a multi-input linear circuits whose sinusoidal inputs have the same frequency.
- Thevenin and Norton equivalencies, source transformation, voltage and current division structures, and maximum power transfer condition can be extended to phasor analysis of linear circuits.

Impedance and Admittance

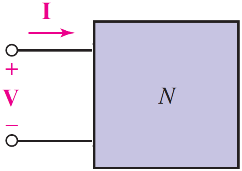


Figure: Impedance $Z = R + jX = \frac{V}{I}$ and admittance $Y = G + jB = \frac{I}{V} = \frac{1}{Z}$ for a one-port network. R, X, G, and B stand for resistance, reactance, conductance, and susceptance.

Element	**Impedance** $Z = \frac{V}{I}$	**Admittance** $Y = \frac{I}{V}$
Resistor	R	G
Capacitor	$\frac{1}{j\omega C}$	$j\omega C$
Inductor	$j\omega L$	$\frac{1}{j\omega L}$

Table: Impedance and admittance for basic LTI one-port circuit elements. Series and parallel combinations as well as delta-why conversion can be used for impedance and admittance.

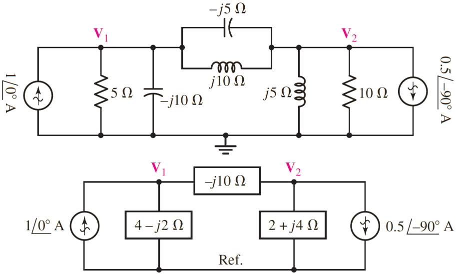
Example (Sinusoidal Steady-state Analysis)

In the circuit below, $V_1 = 1 - j2$ V.

$$V_{11} = (4 - j2)(1\angle 0^\circ)\frac{-j10 + 2 + j4}{4 - j2 - j10 + 2 + j4}$$

$$V_{12} = (4 - j2)(-0.5\angle - 90^\circ)\frac{2 + j4}{2 + j4 - j10 + 4 - j2}$$

$$V_1 = V_{11} + V_{12} = 1 - j2$$



Frequency Response Analysis

Example (Frequency response of series RLC circuit)

For a series RLC circuit with the frequency response $V(j\omega) = H(j\omega)I(j\omega) = I(j\omega)/\left[1/R + j(\omega C - 1/(\omega L))\right]$, the half-power bandwidth of $|V(j\omega)|$ is $B = \omega_0/Q_0$, where $\omega_0 = 1/\sqrt{LC}$ and $Q_0 = R\sqrt{C/L}$ are resonance frequency and quality factor, respectively.

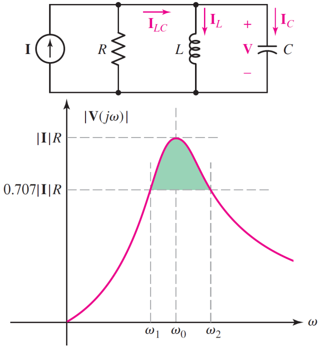
$$V(j\omega) = Z(j\omega)I = \frac{I}{Y(j\omega)} = \frac{I}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}}$$

$$|V(j\omega)| = \frac{|I|}{\sqrt{\frac{1}{R^2} + (\omega C - \frac{1}{\omega L})^2}}$$

$$|V(j\omega_{3db})| = \max\{|V(j\omega)|\}/\sqrt{2} = R|I|/\sqrt{2}$$

$$\omega_{3db} = \omega_{1,2} = \omega_0\left[\sqrt{1 + (\frac{1}{2Q_0})^2} \pm \frac{1}{2Q_0}\right]$$

$$B = |\omega_2 - \omega_1| = \frac{\omega_0}{Q_0}$$



Power in Sinusoidal Steady-state

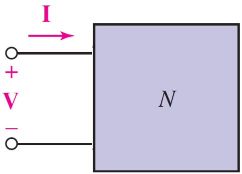


Figure: A one-port LTI network with the voltage $v(t) = |V|\cos(\omega t + \angle V)$ and current $i(t) = |I|\cos(\omega t + \angle I)$, the phasors $V = |V|\angle V$ and $I = |I|\angle I$, the effective phasors $V_e = V/\sqrt{2}$ and $I_e = I/\sqrt{2}$, and the impedance $Z = R + jX$.

- Instantaneous power: $p(t) = \frac{1}{2}|V||I|\left[\cos(\angle V - \angle I) + \cos(2\omega t + \angle V + \angle I)\right]$
- Complex power: $S = \frac{1}{2}VI^* = \frac{1}{2}Z|I|^2 = \frac{1}{2}R|I|^2 + j\frac{1}{2}X|I|^2$
- Average power: $P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p(t')dt' = \frac{1}{2}|V||I|\cos(\angle V - \angle I)$
- Average power: $P = \Re\{S\} = \frac{1}{2}R|I|^2 = \frac{1}{2}|V||I|\cos(\angle V - \angle I)$
- Reactive power: $Q = \Im\{S\} = \frac{1}{2}X|I|^2 = \frac{1}{2}|V||I|\sin(\angle V - \angle I)$

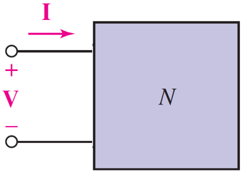


Figure: A one-port LTI network with the voltage $v(t) = |V|\cos(\omega t + \angle V)$ and current $i(t) = |I|\cos(\omega t + \angle I)$, the phasors $V = |V|\angle V$ and $I = |I|\angle I$, the effective phasors $V_e = V/\sqrt{2}$ and $I_e = I/\sqrt{2}$, and the impedance $Z = R + jX$.

- Powe factor: $\mathrm{PF} = \cos(\angle V - \angle I)$
- Apparent (complex) power (VA): $S = V_e I_e^* = Z|I_e|^2 = R|I_e|^2 + jX|I_e|^2$
- Real (active, average) power (W): $P = \Re\{S\} = R|I_e|^2 = |V_e||I_e|\mathrm{PF}$
- Reactive power (VAR): $Q = \Im\{S\} = X|I_e|^2 = |V_e||I_e|\sin(\angle V - \angle I)$

Power in Sinusoidal Steady-state

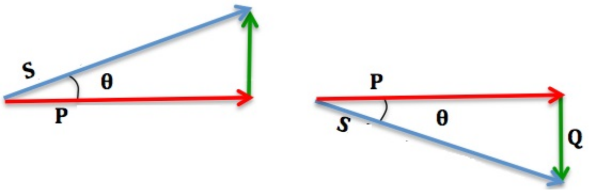


Figure: Power triangle for lagging and leading loads.

- Powe factor: $\mathrm{PF} = \cos(\angle V - \angle I) = \cos(\theta)$
- Resistive load: $\theta = 0 \equiv Q = 0$
- Inductive (lagging) load: $\theta > 0 \equiv Q > 0$
- Capacitive (leading) load: $\theta < 0 \equiv Q < 0$

Example (Sinusoidal Steady-state Power)

The power dissipated by the 10 Ω resistor in the circuit below is $10[79.23\cos(5t - \angle 82.03^\circ) + 811.7\cos(3t - \angle 76.86^\circ)]^2$.

$$I_1 = 2\angle 0^\circ \left[\frac{-j0.4}{10 - j - j0.4}\right] = 79.23\angle -82.03^\circ \text{ mA}$$

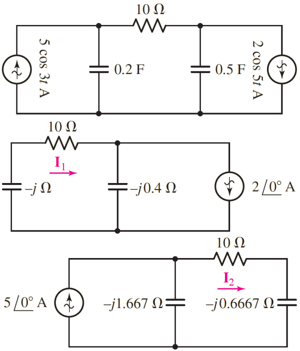
$$i_1(t) = 79.23\cos(5t - 82.03^\circ) \text{ mA}$$

$$I_2 = 5\angle 0^\circ \left[\frac{-j1.667}{10 - j0.6667 - j1.667}\right] = 811.7\angle -76.86^\circ \text{ mA}$$

$$i_2(t) = 811.7\cos(3t - 76.86^\circ) \text{ mA}$$

$$p(t) = 10[i_1(t) + i_2(t)]^2$$

$$P = \frac{1}{2} \times 10 \times 79.23^2 + \frac{1}{2} \times 10 \times 811.7^2$$



Example (Maximum power transfer)

To transfer the maximum power to the load, $Z_{th} = Z_L^*$ in the circuit below.

$$I_L = \frac{V_{th}}{Z_{th} + Z_L} = \frac{V_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}$$

$$V_L = \frac{V_{th} Z_L}{Z_{th} + Z_L} = \frac{V_{th}(R_L + jX_L)}{(R_{th} + R_L) + j(X_{th} + X_L)}$$

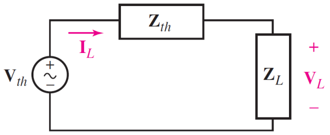
$$P = \Re\{S\} = \Re\{\frac{1}{2} V_L I_L^*\}$$

$$P = \frac{1}{2} \frac{|V_{th}|^2 \sqrt{R_L^2 + X_L^2}}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2} \cos(\tan^{-1}(\frac{X_L}{R_L}))$$

$$\frac{\partial P}{\partial R_{th}} = 0 \Rightarrow R_{th} = R_L$$

$$\frac{\partial P}{\partial X_{th}} = 0 \Rightarrow X_{th} = -X_L$$

$$Z_{th} = R_{th} + jX_{th} = R_L - jX_L = Z_L^*$$



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