Optical Amplifier

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Overview

Preliminaries

Interaction of Photons and Material

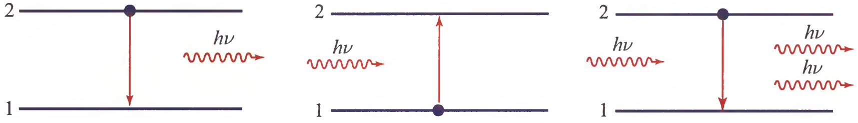


Figure: Three main interactions of a photon with energy $h\nu = E_g = E_2 - E_1$ and atom, spontaneous emission, absorption, and stimulated emission.

- Two-state transitions: Arise from Schrodinger equation
- Fermi's golden rule: Transition rate from one energy state to another due to a weak perturbation
- Two-state approximation: A good approximation for the valance and conduction bands

Interaction of Photons and Material

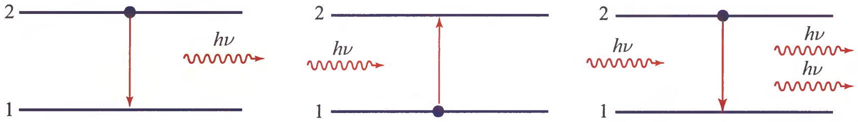


Figure: Three main interactions of a photon with energy $h\nu = E_g = E_2 - E_1$ and atom, spontaneous emission, absorption, and stimulated emission.

- Spontaneous emission probability density: $P_{sp} = \frac{1}{t_{sp}}$
- Spontaneous lifetime: t_{sp}
- Absorption/stimulated emission probability density: $W_i = \phi\sigma(\nu)$
- Photon flux: $\phi = \frac{I}{h\nu}$
- Transition cross section: $\sigma(\nu)$

Interaction of Photons and Material

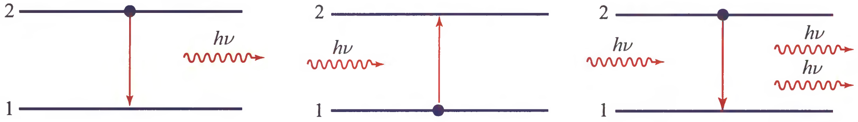


Figure: Three main interactions of a photon with energy $h\nu = E_g = E_2 - E_1$ and atom, spontaneous emission, absorption, and stimulated emission.

- Boltzman occupancy distribution: $P(E_m) \propto \exp(-E_m/kT), \quad m = 1, 2, \cdots$
- Thermal equilibrium: Dominant absorption leads to attenuation
- Population inversion: Dominant emission can lead to amplification

Physical Description of Optical Amplifier

Optical Amplifier

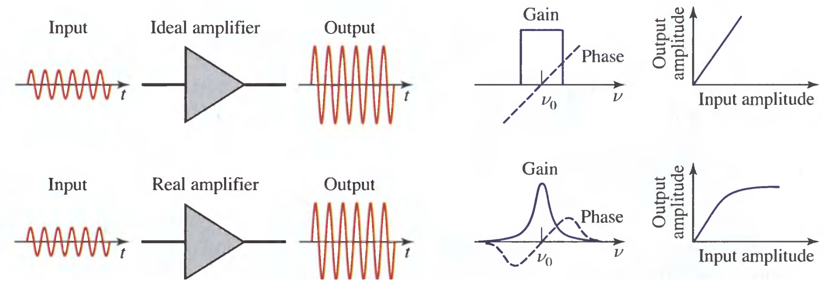


Figure: An ideal amplifier is linear. It increases the amplitude of a signal by a constant gain factor, possibly introducing a linear phase shift. A real amplifier typically has a gain and phase shift that are functions of frequency. For large values of the input, the output signal saturates and the amplifier exhibits nonlinearity.

Optical Amplifier

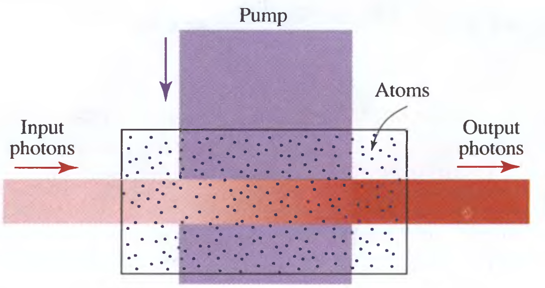


Figure: The laser amplifier, where a pump excites the active medium, producing a population inversion. Photons interact with the atoms. When stimulated emission is more prevalent than absorption, the medium acts as a coherent amplifier.

Optical Amplifier

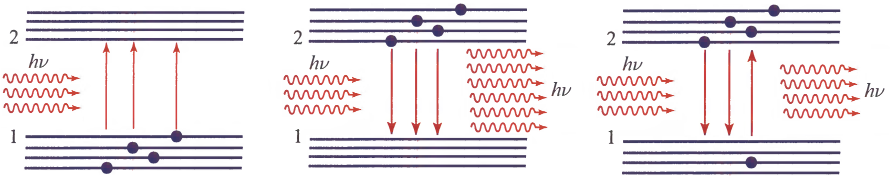


Figure: Absorption and stimulated emission for bands.

- Average density of absorbed photons: $N_1 W_i$
- Average density of clone photons: $N_2 W_i$
- Average density of gained photons: $(N_2 - N_1)W_i = NW_i$
- Equilibrium: $N < 0$
- Transparency: $N = 0$
- Population inversion: $N > 0$

Optical Amplifier

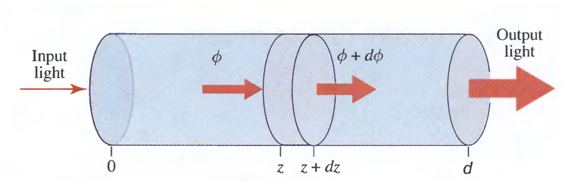


Figure: The photon-flux density entering an incremental cylinder containing excited atoms.

- Photon flux density: $\phi(z)$
- Incremental photon flux density: $d\phi = NW_i dz \Rightarrow \frac{d\phi}{dz} = \phi(z)\gamma(\nu)$
- Unit length gain: $\gamma(\nu) = N\sigma(\nu)$
- Optical intensity: $I(z) = h\nu\phi(z) = h\nu\phi(0)\exp(\gamma(\nu)z) = I(0)\exp(\gamma(\nu)z)$
- Amplification gain: $G(z) = \exp(\gamma(\nu)z)$

Optical Amplifier

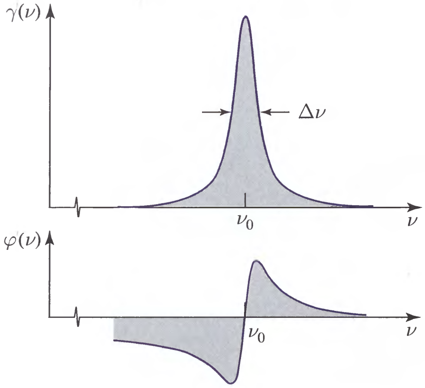


Figure: Lorentzian gain bandwidth.

- Lorentzian gain bandwidth: $\gamma(\nu) = \gamma(\nu_0)\frac{(\Delta\nu/2)^2}{(\nu-\nu_0)^2+(\Delta\nu/2)^2}$
- Lorentzian phase shift: $\phi(\nu) = \gamma(\nu)\frac{\nu-\nu_0}{\Delta\nu}$
- Central frequency gain: $\gamma(\nu_0) = N\frac{\lambda_0^2}{4\pi^2 t_{sp}\Delta\nu}$

Optical Amplifier

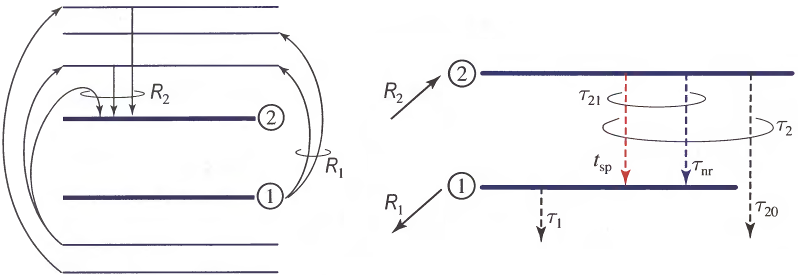


Figure: Energy levels 1 and 2, together with surrounding higher and lower energy levels, in the presence of pumping without amplification. Here, $\tau_2^{-1} = \tau_{21}^{-1} + \tau_{20}^{-1} = t_{sp}^{-1} + \tau_{nr}^{-1} + \tau_{20}^{-1}$.

- Rate equation: $\begin{cases} \frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} \\ \frac{dN_1}{dt} = -R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{\tau_{21}} \end{cases}$
- Steady state condition: $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$
- No-amplification steady-state population difference: $N_0 = R_2\tau_2(1 - \frac{\tau_1}{\tau_{21}}) + R_1\tau_1$

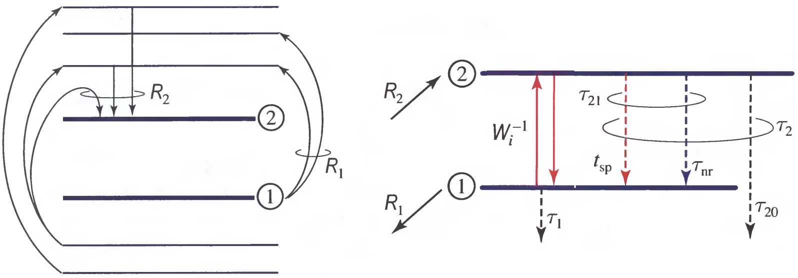


Figure: Energy levels 1 and 2, together with surrounding higher and lower energy levels, in the presence of pumping with amplification. Here, $\tau_2^{-1} = \tau_{21}^{-1} + \tau_{20}^{-1} = t_{sp}^{-1} + \tau_{nr}^{-1} + \tau_{20}^{-1}$.

- Rate equation: $\begin{cases} \frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - N_2 W_i + N_1 W_i \\ \frac{dN_1}{dt} = -R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{\tau_{21}} + N_2 W_i - N_1 W_i \end{cases}$
- Steady state condition: $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$
- Steady-state population difference: $N = \frac{N_0}{1+\tau_s W_i} \leq N_0$
- Characteristic time: $\tau_s = \tau_2 + \tau_1 (1 - \frac{\tau_2}{\tau_{21}}) \geq 0$

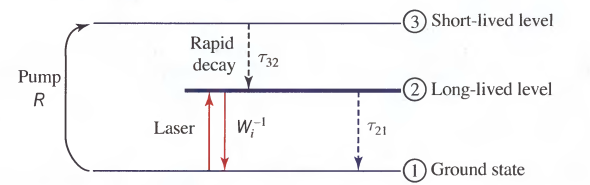


Figure: 3-level pumping schemes.

- Total atomic density: $N_a = N_1 + N_2 + N_3 \approx N_1 + N_2$
- Pumping transition probability density: W
- Pumping rate: $R = (N_1 - N_3)W \approx N_1 W$
- No-amplification steady-state population difference: $N_0 = R_2\tau_2(1 - \frac{\tau_1}{\tau_{21}}) + R_1\tau_1 = \frac{N_a(t_{sp}W - 1)}{1 + t_{sp}W}$
- Steady-state population difference: $N = \frac{N_0}{1 + \tau_s W_i}$
- Characteristic time: $\tau_s = \tau_2 + \tau_1(1 - \frac{\tau_2}{\tau_{21}}) = \frac{2t_{sp}}{1 + t_{sp}W}$

Optical Amplifier

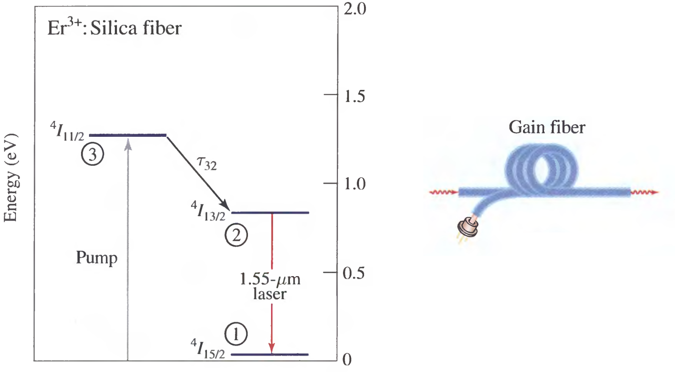


Figure: Erbium-doped fiber amplifier (EDFA) with a pump operating at 980 *nm*.

Statistical Description of Optical Amplifier

BDI Photon Process

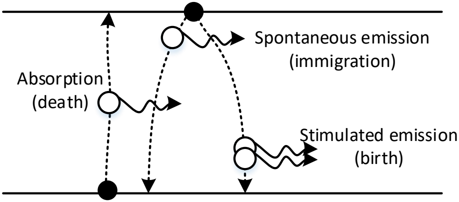


Figure: Birth-death-immigration (BDI) photon process.

- Probability of n photons in $(0, t)$: $p(n, t)$
- Probability of a photon birth due to stimulated emission in $(t, t+\Delta t)$: $an\Delta t \equiv \sigma_e N_2 \phi \Delta t$
- Probability of a photon death due to absorption in $(t, t+\Delta t)$: $bn\Delta t \equiv \sigma_a N_1 \phi \Delta t$
- Probability of a photon immigration due to spontaneous emission in $(t, t+\Delta t)$: $c\Delta t \equiv \frac{N_2}{t_{sp}} \Delta t$

BDI Photon Process

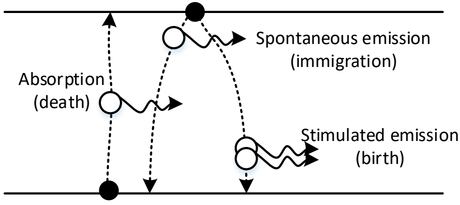


Figure: Birth-death-immigration (BDI) photon process.

- Probability of n photons in $(0, t + \Delta t)$: $p(n, t + \Delta t) = p(n, t)[1 - (an + bn + c)\Delta t] + p(n - 1, t)[a(n - 1) + c]\Delta t + p(n + 1)b(n + 1)\Delta t$
- Forward Kolmogrov equation: $\frac{dp(n,t)}{dt} = \lim_{\Delta t \to 0} \frac{p(n,t+\Delta t) - p(n,t)}{\Delta t} = [(n-1)a + c]p(n-1, t) + [(n+1)b]p(n+1, t) - [n(a+b) + c]p(n, t)$
- Initial photon condition: $p(n, 0) = p_0(n)$

BDI Photon Process

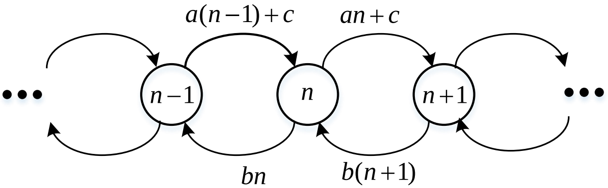


Figure: State transition diagram for birth-death-immigration (BDI) photon process.

- Probability of n photons in $(0, t)$: $p(n,t); p(n,t) = 0, n < 0$
- Probability generating function: $\Phi(z,t) = \sum_{n=-\infty}^{\infty} p(n,t) z^n$
- Forward Kolmogrov equation: $\frac{\partial \Phi(z,t)}{\partial t} = (z-1)\Big[(az-b)\Big]\frac{\partial \Phi(z,t)}{\partial z} + c\Phi(z,t)\Big]$

BDI Photon Process

Example (No stimulated emission and absorption with no incident signal)

The number of photons has Poisson distribution when there is no incident signal, stimulated emission, and absorption.

$$\frac{dp(n,t)}{dt} = cp(n-1,t) - cp(n,t), p(n,0) = \delta[n]$$

$$\frac{\partial\Phi(z,t)}{\partial t} = (z-1)c\Phi(z,t), \Phi(z,0) = 1$$

$$\Phi(z,t) = \Phi_z(z)\Phi_t(t) \Rightarrow \Phi_z(z)\frac{\partial\Phi_t(t)}{\partial t} = (z-1)c\Phi_z(z)\Phi_t(t)$$

$$\Phi_t(t) = e^{c(z-1)t} \Rightarrow \Phi(z,t) = \Phi_z(z)e^{c(z-1)t}, \Phi(z,0) = \Phi_z(z) = 1$$

$$p(n,t) = \frac{(ct)^n}{n!}e^{-ct}$$

BDI Photon Process

Example (No spontaneous emission with single photon initial condition)

With the single photon condition, the amplification occurs if $a > b$.

$$\frac{dp(n,t)}{dt} = -n(a+b)p(n,t) + (n-1)p(n-1,t) + (n+1)p(n+1,t),\ p(n,0) = \delta[n-1]$$

$$\frac{\partial \Phi(z,t)}{\partial t} = (z-1)(az-b)\frac{\partial \Phi(z,t)}{\partial z},\ \Phi(z,0) = z$$

$$\Phi(z,t) = \frac{1+(G-K)(z-1)}{1-K(z-1)},\ G = e^{(a-b)t},\ K = n_{sp}(G-1),\ n_{sp} = \frac{a}{a-b}$$

$$p(n,t) = \begin{cases} \frac{1-G+K}{1+K}, & n = 0 \\ \frac{G}{K}\frac{1}{1+K}\left(\frac{K}{1+K}\right)^n, & n > 0 \end{cases}$$

$$\mathcal{E}\{n\} = \bar{n} = G, \quad \text{Var}\{n\} = G + 2GK - G^2$$

$$\text{SNR} = \frac{\bar{n}^2}{\text{Var}\{n\}} = \frac{G^2}{G+2GK-G^2} = \frac{G}{1+2K-G}$$

BDI Photon Process

Example (BDI model with single photon initial condition)

The photon process can be considered as the sum of two independent stochastic processes.

$$\frac{dp(n,t)}{dt} = [(n-1)a + c]p(n-1,t) + [(n+1)b]p(n+1,t) - [n(a+b)+c]p(n,t),\ p(n,0) = \delta[n-1]$$

$$\frac{\partial \Phi(z,t)}{\partial t} = (z-1)\left[(az-b)\right]\frac{\partial \Phi(z,t)}{\partial z} + c\Phi(z,t)\big],\ \Phi(z,0) = z$$

$$\Phi(z,t) = \frac{1+(G-K)(z-1)}{1-K(z-1)}\left[1-K(z-1)\right]^{-M},\ G = e^{(a-b)t},\ K = n_{sp}(G-1),\ n_{sp} = \frac{a}{a-b},\ M = \frac{c}{a}$$

$$\Phi(z,t) = \Phi_{BD}(z,t)\Phi_I(z,t)$$

$$p_{BD}(n,t) = \begin{cases} \frac{1-G+K}{1+K}, & n=0 \\ \frac{G}{K}\frac{1}{1+K}\left(\frac{K}{1+K}\right)^n, & n>0 \end{cases}$$

$$p_I(n,t) = \binom{n+M-1}{n}\frac{K^n}{(1+K)^{n+M}}$$

BDI Photon Process

Example (BDI model with multiple photon initial condition)

The photon process can be considered as the sum of two independent stochastic processes.

$$\frac{dp(n,t)}{dt} = [(n-1)a + c]p(n-1,t) + [(n+1)b]p(n+1,t) - [n(a+b)+c]p(n,t), p(n,0) = \delta[n-l]$$

$$\frac{\partial\Phi(z,t)}{\partial t} = (z-1)\left[(az-b)\right]\frac{\partial\Phi(z,t)}{\partial z} + c\Phi(z,t)\Big], \Phi(z,0) = z^l$$

$$\Phi(z,t) = \left[\Phi_{BD}(z,t)\right]^l \Phi_I(z,t), \Phi_{BD}(z,t) = \frac{1+(G-K)(z-1)}{1-K(z-1)}, \Phi_I(z,t) = \left[1-K(z-1)\right]^{-M}$$

$$G = e^{(a-b)t}, K = n_{sp}(G-1), n_{sp} = \frac{a}{a-b}, M = \frac{c}{a}$$

BDI Photon Process

Example (BDI model with random photon initial condition)

The photon process can be considered as the sum of two independent stochastic processes.

$$\Phi(z,t) = \sum_{n=-\infty}^{\infty} p(n,t)z^n = \sum_{n=-\infty}^{\infty} z^n \sum_{l=0}^{\infty} p(n,t|l)p(l) = \sum_{l=0}^{\infty} p(l) \sum_{n=-\infty}^{\infty} p(n,t|l)z^n$$

$$= \sum_{l=0}^{\infty} p(l)\left[\Phi_{BD}(z,t)\right]^l \Phi_I(z,t) = \Phi_I(z,t)\sum_{l=0}^{\infty} p(l)\left[\Phi_{BD}(z,t)\right]^l = \Phi_s\left[\Phi_{BD}(z,t)\right]\Phi_I(z,t)$$

$$\Phi_s(z) = \sum_{l=0}^{\infty} p(l)z^l$$

BDI Photon Process

Example (BDI model with Poisson photon initial condition)

The number of photon has Lagurre distribution after amplification if the initial number of photons has Poisson distribution with mean m.

$$\Phi_s(z) = \sum_{l=0}^{\infty} p(l) z^l = e^{m(z-1)}$$

$$\Phi(z,t) = \Phi_s\big[\Phi_{BD}(z,t)\big]\Phi_I(z,t) = e^{m(\Phi_{BD}(z,t)-1)}\Phi_I(z,t)$$

$$= e^{m\left(\frac{1+(G-K)(z-1)}{1-K(z-1)}-1\right)}\big[1-K(z-1)\big]^{-M} = \frac{1}{\big[1-K(z-1)\big]^M} e^{\frac{mG(z-1)}{1-K(z-1)}} \equiv \mathrm{Lag}(mG, K, M-1)$$

$$p(n,t) = \frac{b^n}{(1+b)^{n+c+1}} e^{-\frac{a}{1+b}} L_n^c\Big[-\frac{a}{b(1+b)}\Big] = \frac{K^n}{(1+K)^{n+M}} e^{-\frac{mG}{1+K}} L_n^{M-1}\Big[-\frac{mG}{K(1+K)}\Big]$$

$$L_k^c(x) = \sum_{i=0}^{k} \binom{c+k}{k-i} \frac{(-x)^i}{i!}$$

$$\mathcal{E}\{n\} = \bar{n} = (c+1)b + a = MK + mG$$

$$\mathrm{Var}\{n\} = (c+1)(b+1)b + a(2b+1) = M(K+1)K + mG(2K+1) = mG + 2mGK + M(K+1)K$$

$$\mathrm{SNR} = \frac{(\bar{n}-MK)^2}{\mathrm{Var}\{n\}} = \frac{(mG)^2}{M(K+1)K + mG(2K+1)} \approx \frac{(mG)^2}{mG(2K+1)} = \frac{mG}{2K+1}, \quad m \gg 1$$

Analytical Description of Optical Fiber

ASE Noise

Example (Amplified spontaneous emission noise power)

The amplification process is accompanied by an ASE noise with the power $P_{ASE} = 2n_{sp}h\nu(G-1)B_o$.

$$\text{SNR} = \frac{(\bar{n} - MK)^2}{\text{Var}\{n\}} = \frac{(mG)^2}{M(K+1)K + mG(2K+1)} \approx \frac{(mG)^2}{mG(2K+1)} = \frac{mG}{2K+1}, \quad m \gg 1$$

$$\text{Var}\{n\} \approx 2K+1$$

$$E_{ASE} = (2K+1)h\nu = [2n_{sp}(G-1)+1]h\nu = 2n_{sp}(G-1)h\nu$$

$$P_{ASE} = \frac{E_{ASE}}{T} = 2n_{sp}(G-1)B_o h\nu, \quad G \gg 1$$

Noise Figure

Example (Amplifier noise figure)

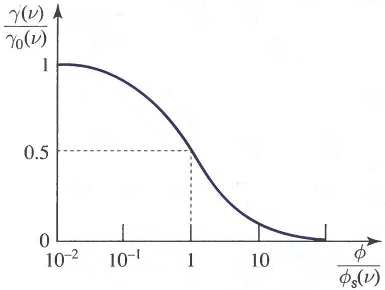
Considering the two polarization and high amplification gain, the noise figure of the amplifier is approximately $F \approx 2n_{sp}$.

$$F = \frac{\text{SNR}_{in}}{\text{SNR}_{out}} \approx \frac{m}{\frac{mG}{2K+1}} = \frac{2n_{sp}(G-1)+1}{G} \approx 2n_{sp}, \quad m, G \gg 1$$

Gain Saturation

Example (Gain saturation)

The dependency of the amplification gain on the input power leads to gain saturation and nonlinearity.

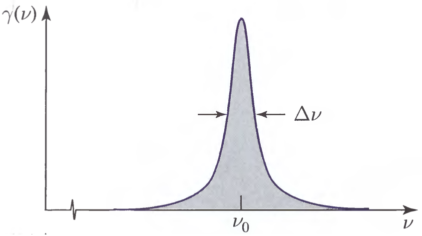


$$N = \frac{N_0}{1+\tau_s W_i} = \frac{N_0}{1+\tau_s\sigma(\nu)\phi} \Rightarrow \gamma(\nu) = N\sigma(\nu) = \sigma(\nu)\frac{N_0}{1+\tau_s\sigma(\nu)\phi} = \frac{\gamma_0(\nu)}{1+\frac{\phi}{\phi_s(\nu)}}$$

Gain Bandwidth

Example (Gain bandwidth)

The amplification gain is a non-flat function of the wavelength.

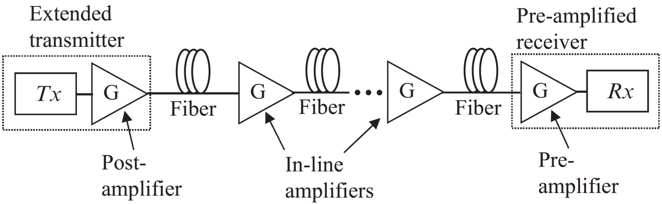


$$\gamma(\nu) = \gamma(\nu_0)\frac{(\Delta\nu/2)^2}{(\nu-\nu_0)^2+(\Delta\nu/2)^2}$$

Amplifier Placement

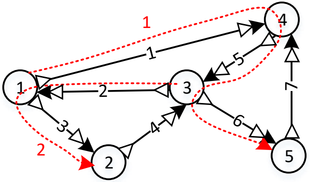
Example (Post, in-line, and pre-amplifiers)

In a typical multi-span point-to-point optical transmission system, post, in-line, and pre-amplifiers are used. The pre-amplifier should have low noise figure, the in-line amplifier should have enough gain and be wideband, and the post-amplifier should have high gian and high saturation power.



Example (Resource allocation with the lowest number of amplifiers)

Assume that a network topology is described by directional graph $G(N, L)$, where each link $l = (b, e) \in L$ begins at node $b \in N$, ends at node $e \in N$. There are R requests, where request $r = (s, d) \in R$ originates from source node $S(r) = s \in N$, terminates at destination node $D(r) = d \in N$. Each link is equipped with a post and a pre-amplifier and can carry at most C requests. The requests can be routed using the following simple resource allocation optimization process by employing the lowest possible number of amplifiers, where $x_{l,r} = 1$ if the request r passes through link l, 0 otherwise and $a_l = 1$ if link l is used and 0, otherwise. Further, K is a large positive real number.



Amplifier Placement

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$$\min_{x_{l,r}, a_l} \quad K \sum_{l} a_l + \sum_{l,r} x_{l,r} \qquad \text{s.t}$$

$$\sum_{l \in L: b=n} x_{l,r} = 1, \sum_{l \in L: e=n} x_{l,r} = 0, \quad \forall r \in R, \forall n \in N : n = S(r)$$

$$\sum_{l \in L: e=n} x_{l,r} = 1, \sum_{l \in L: b=n} x_{l,r} = 0, \quad \forall r \in R, \forall n \in N : n = D(r)$$

$$\sum_{l \in L: e=n} x_{l,r} = \sum_{l \in L: b=n} x_{l,r}, \quad \forall r \in R, \forall n \in N : n \neq S(r), n \neq D(r)$$

$$\sum_{r \in R} x_{l,r} \leq C a_l, \quad \forall l \in L$$

The End