Analog Communication

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Overview

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Preliminaries

Lowpass and Bandpass Signals

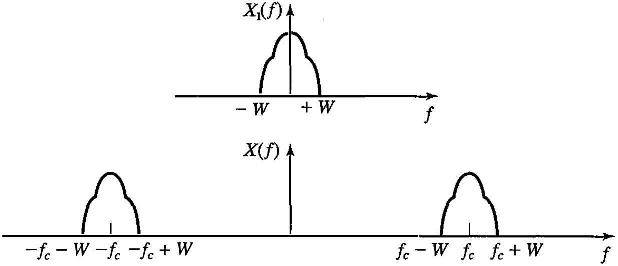


Figure: Bandpass signal and lowpass signals.

- Lowpass signal: A signal whose spectrum is located around the zero frequency.
- Bandpass signal: A signal whose spectrum is far from the zero frequency.

Lowpass and Bandpass Signals

- Lowpass to bandpass conversion: $x(t) = \mathrm{Re}\{x_l(t)e^{j2\pi f_c t}\} = \mathrm{Re}\{A(t)e^{j(2\pi f_c t + \theta(t))}\}$
- Lowpass to bandpass conversion: $x(t) = x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t)$
- Lowpass to bandpass conversion: $X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$
- Bandpass to lowpass conversion: $X_l(f) = 2X(f + f_c)u(f + f_c)$
- Lowpass representation: $x_l(t) = A(t)e^{j\theta(t)} = x_c(t) + jx_s(t)$
- Representation conversion: $x_c(t) = A(t)\cos(\theta(t)), \quad x_s(t) = A(t)\sin(\theta(t))$
- Representation conversion: $A(t) = \sqrt{x_c^2(t) + x_s^2(t)}, \quad \theta(t) = \tan^{-1}(\frac{x_s(t)}{x_c(t)})$
- Envelope component: $A(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$
- Phase component: $\theta(t) = \tan^{-1}\left[\frac{\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)}{x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)}\right]$
- In-phase component: $x_c(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)$
- Quadrature component: $x_s(t) = \hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)$

Lowpass and Bandpass Signals

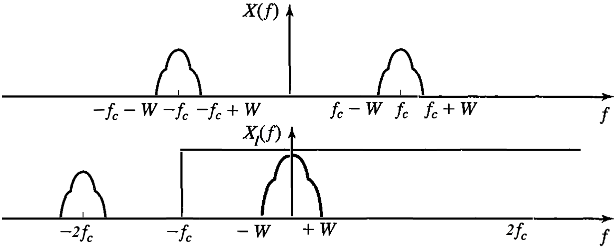


Figure: Spectrum of a bandpass signal and its associated lowpass signal.

- Lowpass to bandpass conversion: $X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$
- Bandpass to lowpass conversion: $X_l(f) = 2X(f + f_c)u(f + f_c)$

Lowpass and Bandpass Random Processes

- Lowpass random process: A process whose correlation function is lowpass.
- Bandpass random process: A process whose correlation function is bandpass.
- Lowpass equivalent process: $X_l(t) = X_c(t) + jX_s(t)$
- In-phase component: $X_c(t) = X(t)\cos(2\pi f_c t) + \hat{X}(t)\sin(2\pi f_c t)$
- Quadrature component: $X_s(t) = \hat{X}(t)\cos(2\pi f_c t) - X(t)\sin(2\pi f_c t)$
- Lowpass equivalent PSD: $S_{X_l}(f) = 4S_X(f + f_c)u(f + f_c)$
- Bandpass PSD: $S_X(f) = \frac{1}{4}\left[S_{X_l}(f - f_c) + S_{X_l}(-f - f_c)\right]$
- Component PSD: $S_{X_c}(f) = S_{X_s}(f) = [S_X(f + f_c) + S_X(f - f_c)] \sqcap (\frac{f}{2f_c})$
- Component cross-PSD: $S_{X_c X_s}(f) = -S_{X_s X_c}(f) = j[S_X(f + f_c) - S_X(f - f_c)] \sqcap (\frac{f}{2f_c})$
- Lowpass equivalent correlation: $R_{X_l}(\tau) = 2(R_X(\tau) + j\widehat{R_X}(\tau))e^{-j2\pi f_c \tau} = 2R_{X_s}(\tau) = 2R_{X_c}(\tau)$

Gaussian Processes

For a real Gaussian random process $X(t)$,

1. $X(t_1)$ has Gaussian distribution.
2. $(X(t_1), X(t_2))$ have two-dimensional jointly Gaussian distribution.
3. $(X(t_1), \cdots, X(t_n))$ have n-dimensional jointly Gaussian distribution.
4. Fourth moment is

$$\mathcal{E}\{X(t_1)X(t_2)X(t_3)X(t_4)\} = \mathcal{E}\{X(t_1)X(t_2)\}\mathcal{E}\{X(t_3)X(t_4)\} \\ + \mathcal{E}\{X(t_1)X(t_3)\}\mathcal{E}\{X(t_2)X(t_4)\} + \mathcal{E}\{X(t_1)X(t_4)\}\mathcal{E}\{X(t_2)X(t_3)\}$$

5. If $X(t)$ is zero-mean, $\mathcal{E}\{X(t_1)\} = 0$ and $\mathcal{E}\{X(t_1)X(t_2)X(t_3)\} = 0$.
6. If $X(t)$ is WSS, $R_{X^2}(\tau) = 2R_X^2(\tau) + R_X^2(0)$.
7. If $X(t)$ is zero-mean, $\mathcal{E}\{X(t_1)\} = \mathcal{E}\{X(t_1)X^2(t_2)\} = 0$.

White and Colored Gaussian Noise

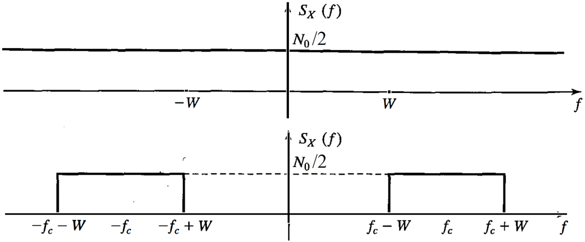


Figure: Spectrum of the white and colored thermal noise.

1. White thermal noise
    - Stationary process
    - Zero-mean process.
    - Gaussian process.
    - White process with a PSD $S_n(f) = \frac{N_0}{2}$.
2. Colored thermal noise
    - Filtered process with PSD $S_X(f) = \frac{N_0}{2}|H(f)|^2$

Bandpass Colored Gaussian Noise

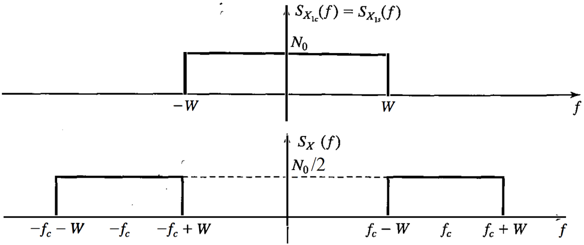


Figure: Spectrum of the colored thermal noise and its associated lowpass components $X_s(t)$ and $X_c(t)$.

1. $X_c(t)$ and $X_s(t)$ are zero-mean, lowpass, jointly WSS, and jointly Gaussian.
2. Processes $X(t)$, $X_c(t)$, and $X_s(t)$ have the identical power P_x.
3. Processes $X_c(t)$ and $X_s(t)$ have the same power spectral density.
4. If $\pm f_c$ are the symmetry axis of the spectrum parts, $X_c(t)$ and $X_s(t)$ are independent processes.

System Model

System Model

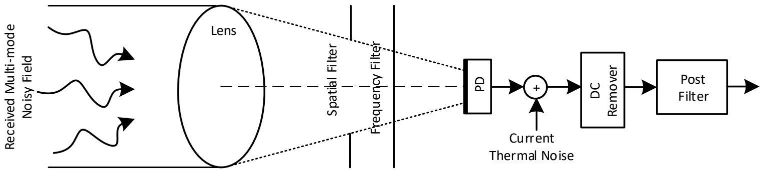


Figure: Optical direct detection (incoherent detection) (analog communication) system.

- Received noisy field: $f_r(t, \boldsymbol{r}) = a(t, \boldsymbol{r}) e^{j2\pi\nu t}$
- Received noisy complex envelope: $a(t, \boldsymbol{r}) = s(t, \boldsymbol{r}) + b(t, \boldsymbol{r})$
- Count intensity process: $n(t) = \alpha \int_A |a(t, \boldsymbol{r})|^2 d\boldsymbol{r}$
- Carrier generation process: $k(0, t)$
- Shot noise process: $i(t) = \sum_{j=1}^{k(0,t)} g_j h(t - t_j), \quad g_j = 1$
- Dark current: I_{dc}
- Thermal noise power spectral density: N_{0c}
- Filter frequency response: $F(\omega)$
- Filtered current: $y(t)$

System Model

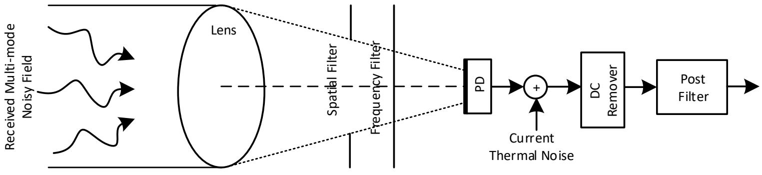


Figure: Optical direct detection (incoherent detection) (analog communication) system.

- Received noisy complex envelope decomposition:

$$a(t,\mathbf{r}) = \sum_{i=0}^{\infty} a_i(t)\Phi_i(\mathbf{r}) = \sum_{i=0}^{\infty} [s_i(t) + b_i(t)]\Phi_i(\mathbf{r})$$

- Count intensity process:

$$n(t) = \alpha \int_A |a(t,\mathbf{r})|^2 d\mathbf{r} = \alpha \int_A |\sum_{i=0}^{\infty} a_i(t)\Phi_i(\mathbf{r})|^2 d\mathbf{r} = \alpha \int_A \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i(t)a_j^*(t)\Phi_i(\mathbf{r})\Phi_j^*(\mathbf{r})d\mathbf{r}$$

$$= \alpha \sum_{i=0} |a_i(t)|^2 \int_A |\Phi_i(\mathbf{r})|^2 d\mathbf{r} + \alpha \sum_{i,j=0,i\neq j}^{\infty} a_i(t)a_j^*(t) \int_A \Phi_i(\mathbf{r})\Phi_j^*(\mathbf{r})d\mathbf{r} = \alpha A \sum_{i=0} |a_i(t)|^2$$

System Model

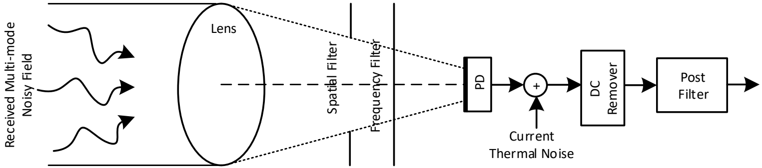


Figure: Optical direct detection (incoherent detection) (analog communication) system.

- Intensity modulation: $|s_i(t)|^2 = I_{s_i}[1 + \beta m(t)]$
- Intensity modulating message: $m(t), \quad \mathcal{E}\{m(t)\} = 0$
- Average field intensity: $I_{s_i} = \mathcal{E}\{|s_i(t)|^2\}$
- Space-division multiplexed intensity modulation: $|s_i(t)|^2 = I_{s_i}[1 + \beta_i m_i(t)]$

System Analysis

Single-mode System

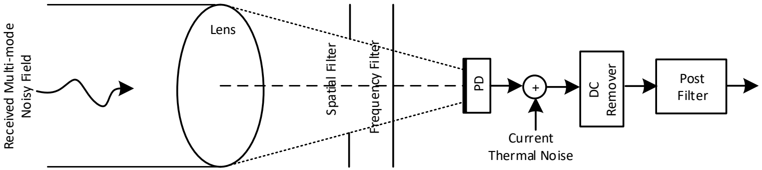


Figure: Single-mode optical direct detection system with stationary received intensity.

- Received noisy complex envelope:
  $a(t, \boldsymbol{r}) = a(t) = s(t) + b(t), \quad \Phi_i(\boldsymbol{r}) = 1, \boldsymbol{r} \in A$
- Intensity modulation: $|s(t)|^2 = I_s[1 + \beta m(t)]$
- Count intensity: $n(t) = \alpha A |a(t)|^2 = \alpha A |s(t) + b(t)|^2$
- Current power spectral density: $S_i(\omega) = |H_T(\omega)|^2 \left[\mathcal{E}_n\{\bar{n}\} + S_n(\omega)\right]$

Single-mode System

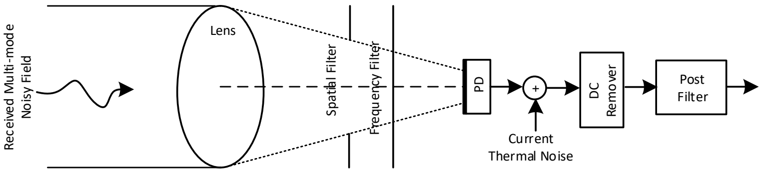


Figure: Single-mode optical direct detection system with stationary received intensity. Signal and background noise processes are assumed independent. The background noise is modeled as a zero-mean colored Gaussian noise. Assuming that the receiver optical bandwidth is assumed B_0, the noise power is $P_{b0} = A\frac{N_0}{A}B_0$, where N_0 is the background noise power spectral density.

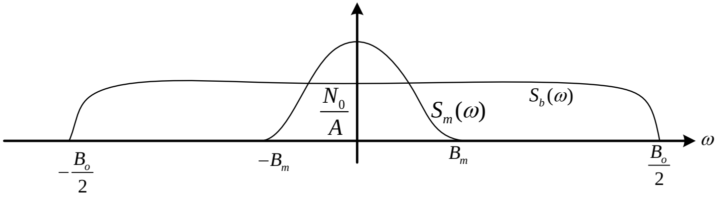


Figure: Power spectral densities of the noise and message,where $B_m \ll B_o$.

Single-mode System

- Expected mean count $\mathcal{E}_n\{\bar{n}\}$:

$$\begin{aligned}
\mathcal{E}_n\{n(t)\} &= \alpha A \mathcal{E}_n\{|s(t)+b(t)|^2\} = \alpha A \mathcal{E}_n\{[s(t)+b(t)][s(t)+b(t)]^*\} \\
&= \alpha A\Big[\mathcal{E}_n\{|s(t)|^2\} + \mathcal{E}_n\{|b(t)|^2\} + \mathcal{E}_n\{s(t)b^*(t)\} + \mathcal{E}_n\{b(t)s^*(t)\}\Big] \\
&= \alpha A \mathcal{E}_n\{|s(t)|^2\} + \alpha A \mathcal{E}_n\{|b(t)|^2\} \\
&= \alpha A R_s(0) + \alpha A R_b(0) \\
&= \alpha A I_s + \alpha A \frac{N_0}{A} B_0 \\
&= \alpha(P_s + P_{b0})
\end{aligned}$$

$$\mathcal{E}_n\{\bar{n}\} = \lim_{T\to\infty} \frac{\int_{-T}^{T} \mathcal{E}_n\{n(t)\}dt}{2T} = \alpha(P_s + P_{b0})$$

Single-mode System

- Correlation $R_n(\tau)$:

$$R_n(\tau) = \mathcal{E}\{n(t)n(t+\tau)\} = (\alpha A)^2 \mathcal{E}\Big\{ \big(s(t)+b(t)\big)\big(s(t)+b(t)\big)^* \big(s(t+\tau)+b(t+\tau)\big)$$

$$\times \big(s(t+\tau)+b(t+\tau)\big)^* \Big\} = (\alpha A)^2 \mathcal{E}\Big\{ \Big[|s(t)|^2 + |b(t)|^2 + s(t)b^*(t) + s^*(t)b(t)\Big]$$

$$\times \Big[|s(t+\tau)|^2 + |b(t+\tau)|^2 + s(t+\tau)b^*(t+\tau) + s^*(t+\tau)b(t+\tau)\Big]\Big\}$$

$$= (\alpha A)^2 \mathcal{E}\Big\{ |s(t)|^2|s(t+\tau)|^2 + |s(t)|^2|b(t+\tau)|^2 + |b(t)|^2|s(t+\tau)|^2 + |b(t)|^2|b(t+\tau)|^2$$

$$+ s(t)s(t+\tau)b^*(t)b^*(t+\tau) + s^*(t)s^*(t+\tau)b(t)b(t+\tau) + s(t)s^*(t+\tau)b^*(t)b(t+\tau)$$

$$+ s^*(t)s(t+\tau)b(t)b^*(t+\tau)\Big\} = \alpha^2 \Big[2AI_s A\frac{N_0}{A}B_o + A^2 R_{|s|^2}(\tau) + A^2 R_{|b|^2}(\tau) + A^2 R_{sb}(\tau)\Big]$$

$$= \alpha^2 \Big[2P_s P_{b0} + A^2 R_{|s|^2}(\tau) + A^2 R_{|b|^2}(\tau) + A^2 R_{sb}(\tau)\Big]$$

- PSD $S_n(\omega)$:

$$S_n(\omega) = \mathcal{F}\{R_n(\tau)\} = \alpha^2 \Big[4\pi P_s P_{b0}\delta(\omega) + A^2 S_{|s|^2}(\omega) + A^2 S_{|b|^2}(\omega) + A^2 S_{sb}(\omega)\Big]$$

Single-mode System

- Correlation $R_{|s|^2}(\tau)$:

$$\begin{aligned} R_{|s|^2}(\tau) &= \mathcal{E}\left\{|s(t)|^2|s(t+\tau)|^2\right\} \\ &= \mathcal{E}\left\{I_s[1+\beta m(t)]I_s[1+\beta m(t+\tau)]\right\} \\ &= I_s^2 + I_s^2\beta\mathcal{E}\left\{m(t)\right\} + I_s^2\beta\mathcal{E}\left\{m(t+\tau)\right\} + I_s^2\beta^2\mathcal{E}\left\{m(t)m(t+\tau)\right\} \\ &= I_s^2[1+\beta^2R_m(\tau)] \end{aligned}$$

- PSD $S_{|s|^2}(\omega)$:

$$A^2S_{|s|^2}(\omega) = A^2\mathcal{F}\{R_{|s|^2}(\tau)\} = 2\pi A^2I_s^2\delta(\omega) + A^2I_s^2\beta^2S_m(\omega) = 2\pi P_s^2\delta(\omega) + P_s^2\beta^2S_m(\omega)$$

Single-mode System

- Correlation $R_{|b|^2}(\tau)$:

$$
\begin{aligned}
R_{|b|^2}(\tau) &= \mathcal{E}\left\{|b(t)|^2|b(t+\tau)|^2\right\} = \mathcal{E}\left\{[b_r^2(t)+b_i^2(t)][b_r^2(t+\tau)+b_i^2(t+\tau)]\right\} \\
&= \mathcal{E}\left\{b_r^2(t)b_r^2(t+\tau)+b_i^2(t)b_i^2(t+\tau)+b_r^2(t)b_i^2(t+\tau)+b_i^2(t)b_r^2(t+\tau)\right\} \\
&= R_{b_r^2}(\tau)+R_{b_i^2}(\tau)+2R_{b_r}(0)R_{b_i}(0) \\
&= 4R_{b_r}^2(\tau)+2R_{b_r}^2(0)+2R_{b_r}^2(0) \\
&= R_b^2(\tau)+R_b^2(0) = R_b^2(\tau)+\frac{N_0^2}{A^2}B_o^2
\end{aligned}
$$

- PSD $S_{|b|^2}(\omega)$:

$$
A^2 S_{|b|^2}(\omega) = A^2\mathcal{F}\{R_{|b|^2}(\tau)\} = 2\pi A^2\frac{N_0^2}{A^2}B_o^2\delta(\omega)+A^2\frac{1}{2\pi}S_b(\omega)*S_b(\omega), \quad S_b(\omega)=\frac{N_0}{A}\sqcap(\frac{\omega}{2\pi B_o})
$$

$$
\begin{aligned}
A^2 S_{|b|^2}(\omega) &= 2\pi P_{b_0}^2\delta(\omega)+A^2\frac{N_0^2}{A^2}B_0\Lambda(\frac{\omega}{2\pi B_o}) \\
&\approx 2\pi P_{b_0}^2\delta(\omega)+N_0^2B_o = 2\pi P_{b_0}^2\delta(\omega)+N_0P_{b_0}, \quad \text{bandwidth of } s(t) \ll B_o
\end{aligned}
$$

Single-mode System

- Correlation $R_{sb}(\tau)$:

$$R_{sb}(\tau) = \mathcal{E}\Big\{s(t)s(t+\tau)b^*(t)b^*(t+\tau) + s^*(t)s^*(t+\tau)b(t)b(t+\tau) + s(t)s^*(t+\tau)b^*(t)b(t+\tau)$$
$$+ s^*(t)s(t+\tau)b(t)b^*(t+\tau)\Big\} = \mathcal{E}\Big\{2\,\mathrm{Re}\{s(t)s(t+\tau)b^*(t)b^*(t+\tau) + s(t)s^*(t+\tau)b^*(t)b(t+\tau)\}\Big\}$$
$$= 2\,\mathrm{Re}\Big\{R_{ss}(\tau)R_{b^*b^*}(\tau) + R_{ss^*}(\tau)R_{b^*b}(\tau)\Big\} = 2\,\mathrm{Re}\Big\{R_{ss^*}(\tau)R_{b^*b}(\tau)\Big\} = 2\,\mathrm{Re}\Big\{R_s(\tau)R_b(\tau)\Big\}$$
$$= 2R_s(\tau)R_b(\tau) \approx 2R_s(0)R_b(\tau) = 2I_sR_b(\tau), \quad \text{bandwidth of } s(t) \ll B_o$$

- PSD $S_{sb}(\omega)$:

$$A^2 S_{sb}(\omega) = A^2\mathcal{F}\{R_{sb}(\tau)\} = 2A^2 I_s \frac{N_0}{A} = 2P_sN_0, \quad |\omega| \leq \frac{B_o}{2}$$

Single-mode System

- Correlation $R_n(\tau)$:

$$R_n(\tau) = \alpha^2 \left[2P_s P_{b0} + A^2 R_{|s|^2}(\tau) + A^2 R_{|b|^2}(\tau) + A^2 R_{sb}(\tau) \right]$$

- PSD $S_n(\omega)$:

$$\begin{aligned} S_n(\omega) &= \alpha^2 \left[4\pi P_s P_{b0} \delta(\omega) + 2\pi P_s^2 \delta(\omega) + P_s^2 \beta^2 S_m(\omega) + 2\pi P_{b_0}^2 \delta(\omega) + N_0 P_{b0} + 2P_s N_0 \right] \\ &= 2\pi\alpha^2 (P_s + P_{b_0})^2 \delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega) + \alpha^2 (N_0 P_{b0} + 2P_s N_0) \end{aligned}$$

- Current power spectral density:

$$\begin{aligned} S_i(\omega) &= |H_T(\omega)|^2 \left[\mathcal{E}_n\{\bar{n}\} + S_n(\omega) \right] \\ &= |H_T(\omega)|^2 \left[\alpha(P_s + P_{b_0}) + 2\pi\alpha^2 (P_s + P_{b_0})^2 \delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega) + \alpha^2 (N_0 P_{b0} + 2P_s N_0) \right] \end{aligned}$$

Single-mode System

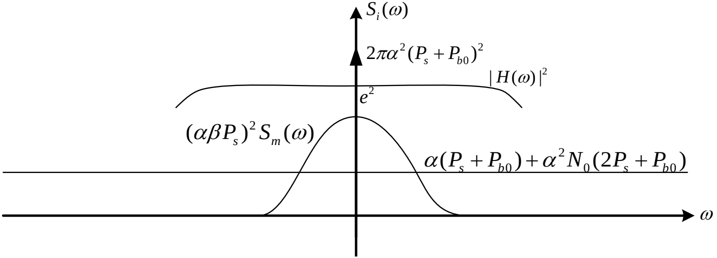


Figure: Current spectrum for a Single-mode optical direct detection system with stationary received intensity.

- Current power spectral density:

$$S_i(\omega) = |H_T(\omega)|^2\left[\alpha(P_s+P_{b_0}) + \alpha^2 N_0(P_{b0}+2P_s) + 2\pi\alpha^2(P_s+P_{b_0})^2\delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega)\right]$$

Single-mode Signal/Multi-mode Detection System

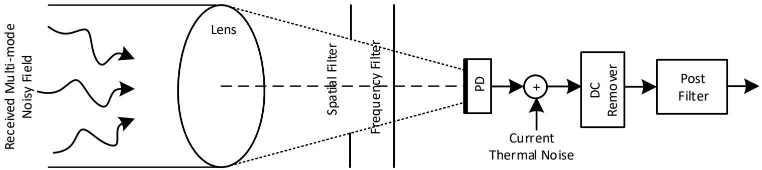


Figure: Multi-mode optical direct detection system with stationary received intensity. The signal is included in the first mode and other modes only contain background noise.

- Received noisy complex envelope: $a(t, \boldsymbol{r}) = s_1(t) + b_1(t) + \sum_{i=2}^{D_s} b_i(t)\Phi_i(\boldsymbol{r})$
- Intensity modulation: $|s_1(t)|^2 = I_s[1 + \beta m(t)]$
- Current power spectral density: $S_i(\omega) = |H_T(\omega)|^2 \left[\mathcal{E}_n\{\bar{n}\} + S_n(\omega)\right]$

Single-mode Signal/Multi-mode Detection System

- Expected mean count $\mathcal{E}_n\{\bar{n}\}$:

$$\begin{aligned}\mathcal{E}_n\{n(t)\} &= \alpha A \mathcal{E}_n\left\{|s(t)+b(t)|^2 + \sum_{i=2}^{D_s} |b_i(t)|^2\right\} \\ &= \alpha A \mathcal{E}_n\left\{|s(t)+b(t)|^2\right\} + \alpha A \sum_{i=2}^{D_s} \mathcal{E}_n\left\{|b_i(t)|^2\right\} \\ &= \alpha(P_s + P_{b0}) + \alpha(D_s - 1)P_{b0}\end{aligned}$$

$$\mathcal{E}_n\{\bar{n}\} = \lim_{T\to\infty} \frac{\int_{-T}^{T} \mathcal{E}_n\{n(t)\}dt}{2T} = \alpha(P_s + D_s P_{b0})$$

Single-mode Signal/Multi-mode Detection System

- Correlation $R_n(\tau)$:

$$
\begin{aligned}
R_n(\tau) &= \mathcal{E}\{n(t+\tau)n(t)\} \\
&= (\alpha A)^2 \mathcal{E}\left\{ \left[|s_1(t+\tau) + b_1(t+\tau)|^2 + \sum_{i=2}^{D_s} |b_i(t+\tau)|^2 \right] \left[|s_1(t) + b_1(t)|^2 + \sum_{i=2}^{D_s} |b_i(t)|^2 \right] \right\} \\
&= (\alpha A)^2 \mathcal{E}\left\{ |s_1(t+\tau) + b_1(t+\tau)|^2 |s_1(t) + b_1(t)|^2 \right\} + (\alpha A)^2 \sum_{i=2}^{D_s} \mathcal{E}\left\{ |b_i(t+\tau)|^2 |b_i(t)|^2 \right\} \\
&+ (\alpha A)^2 \sum_{i=2}^{D_s} \mathcal{E}\left\{ |s_1(t+\tau) + b_1(t+\tau)|^2 |b_i(t)|^2 \right\} + (\alpha A)^2 \sum_{i=2}^{D_s} \mathcal{E}\left\{ |s_1(t) + b_1(t)|^2 |b_i(t+\tau)|^2 \right\} \\
&+ (\alpha A)^2 \sum_{i,j=2, i \neq j}^{D_s} \mathcal{E}\left\{ |b_i(t+\tau)|^2 |b_j(t)|^2 \right\} \\
&= R_{n_1}(\tau) + \sum_{i=2}^{D_s} R_{n_i}(\tau) + 2(D_s - 1)\alpha^2 (P_s + P_{b_0}) P_{b_0} + [(D_s - 1)^2 - (D_s - 1)] \alpha^2 P_{b_0}^2
\end{aligned}
$$

Single-mode Signal/Multi-mode Detection System

- PSD $S_n(\omega)$:

$$S_n(\omega) = 2\pi\alpha^2(P_s + P_{b_0})^2\delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega) + \alpha^2(N_0 P_{b0} + 2P_s N_0) + (D_s - 1)[2\pi\alpha^2 P_{b_0}^2\delta(\omega)$$
$$+ \alpha^2 N_0 P_{b0}] + 2\pi\delta(\omega)\left[2(D_s - 1)\alpha^2(P_s + P_{b_0})P_{b_0} + ((D_s - 1)^2 - (D_s - 1))\alpha^2 P_{b_0}^2\right]$$
$$= 2\pi\alpha^2(P_s + D_s P_{b_0})^2\delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega) + \alpha^2(D_s N_0 P_{b_0} + 2P_s N_0)$$

- Noise power:

$$P_b = D_s P_{b_0} = D_s N_0 B_o = I(\lambda, T) B_o \Omega_{fv} A, \quad I(\lambda, T) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda KT}\right) - 1}$$

- Current power spectral density:

$$S_i(\omega) = |H_T(\omega)|^2\left[\alpha(P_s + P_b) + 2\pi\alpha^2(P_s + P_b)^2\delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega) + \alpha^2(N_0 P_b + 2P_s N_0)\right]$$
$$\alpha N_0 \ll 1 \Rightarrow S_i(\omega) = |H_T(\omega)|^2\left[\alpha(P_s + P_b) + 2\pi\alpha^2(P_s + P_b)^2\delta(\omega) + (\alpha\beta P_s)^2 S_m(\omega)\right]$$

Single-mode Signal/Multi-mode Detection System

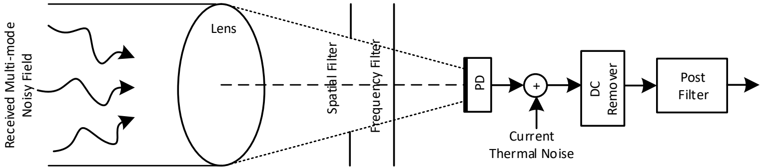


Figure: Multi-mode optical direct detection system with stationary received intensity. The signal is included in the first mode and other modes only contain background noise.

- Current power spectral density with carrier multiplication, dark current, and thermal noise:

$$S_i(\omega) = |H_T(\omega)|^2 \left[\left[\left(\frac{I_{dc}}{e}\right)^2 + (\alpha \bar{g})^2 (P_s + P_b)^2 \right] 2\pi\delta(\omega) + (\alpha \bar{g} \beta P_s)^2 S_m(\omega) + \alpha \bar{g^2}(P_s + P_b) + \frac{I_{dc}}{e} \right] + N_{0c}$$

- Output power spectral density:

$$S_y(\omega) = |F(\omega)|^2 \left[(\alpha e \bar{g} \beta P_s)^2 S_m(\omega) + \alpha e^2 \bar{g^2}(P_s + P_b) + eI_{dc} + N_{0c} \right], B_m \ll B_o, |H(\omega)| = e^2$$

- Electronic SNR:

$$\text{SNR} = \frac{\frac{1}{2\pi} \int_{-2\pi B_m}^{2\pi B_m} (\alpha e \bar{g} \beta P_s)^2 S_m(\omega) d\omega}{\frac{1}{2\pi} \int_{-2\pi B_m}^{2\pi B_m} \left[\alpha e^2 \bar{g^2}(P_s + P_b) + eI_{dc} + N_{0c} \right] d\omega} = \frac{(\alpha e \bar{g} \beta P_s)^2 P_m}{\left[\alpha e^2 \bar{g^2}(P_s + P_b) + eI_{dc} + N_{0c} \right] 2B_m}$$

Single-mode Signal/Multi-mode Detection System

- Electronic SNR:

$$\text{SNR} = \frac{(\alpha e \bar{g} P_s)^2 (\beta^2 P_m)}{\left[\alpha e^2 \overline{g^2} (P_s + P_b) + e I_{dc} + N_{0c}\right] 2B_m}$$

- Shot noise-limited SNR:

$$\text{SNR} \approx \frac{(\alpha P_s)^2 (\beta^2 P_m)}{F \alpha (P_s + P_b) 2B_m}, \quad F = \frac{\overline{g^2}}{\bar{g}^2}, \quad I_{dc} \approx 0, \quad N_{0c} \approx 0$$

- Quantum-limited SNR:

$$\text{SNR}_{QL} \approx \frac{\alpha P_s}{2B_m} = \frac{\eta P_s}{(h\nu) 2B_m}, \quad F = \frac{\overline{g^2}}{\bar{g}^2} = 1, \quad I_{dc} \approx 0, \quad N_{0c} \approx 0, \quad \beta^2 P_m = 1, \quad P_s \gg P_b$$

- Background noise limited SNR:

$$\text{SNR} \approx \frac{\alpha P_s (\beta^2 P_m)}{2FB_m} \frac{P_s}{P_b}, \quad F = \frac{\overline{g^2}}{\bar{g}^2}, \quad P_b \gg P_s, \quad I_{dc} \approx 0, \quad N_{0c} \approx 0$$

Single-mode Signal/Multi-mode Detection System

- Electronic SNR:

$$\text{SNR} = \text{SNR}_{QL}(\beta^2 P_m)\left[F(1+\frac{P_b}{P_s}) + \frac{I_{dc}}{\bar{g}^2 e\alpha P_s F} + \frac{N_{0c}}{(\bar{g}e)^2\alpha P_s F})\right]^{-1}$$

- Electronic SNR:

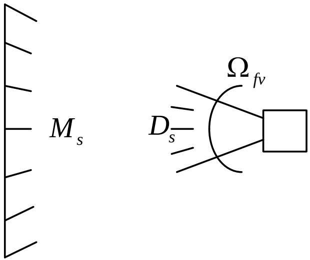
$$\text{SNR} = (\beta^2 P_m)\frac{n_s^2}{[F(n_s+n_b)+n_{dc}+n_c]2B_m}$$

  - Average number of signal electrons per second: $n_s = \alpha P_s$
  - Average number of background noise electrons per second: $n_b = \alpha P_b$
  - Average dark current count rate in counts per second: $n_{dc} = \frac{I_{dc}}{\bar{g}^2 e}$
  - Average circuit noise count rate in counts per second: $n_{dc} = \frac{N_{0c}}{(\bar{g}e)^2}$

Multi-mode Signal/Multi-mode Detection System

Example (Multi-mode direct detection with $\Omega_{fv} \leq \Omega_s$)

A multi-mode incoherent detector has a field of view solid angle Ω_{fv} and receives D_s spatial modes of $M_s \geq D_s$ transmitted signal modes. Each signal mode has signal power P_{s_0}, noise power P_{b_0}, and incoherent modulation $|s_i(t)|^2(1+\beta m(t))$. The corresponding shot-noise limited SNR is $\frac{(\alpha D_s P_{s_0})^2}{2\alpha F B_m D_s (P_{s_0}+P_{b_0})}$.



$$n(t) = \alpha A \sum_{i=1}^{D_s} |s_i(t) + b_i(t)|^2 = \alpha A \sum_{i=1}^{D_s} \left[|s_i(t)|^2 + |b_i(t)|^2 + 2\,\mathrm{Re}\{s_i(t) b_i^*(t)\} \right]$$

Example (Multi-mode direct detection with $\Omega_{fv} \leq \Omega_s$ (cont.))

A multi-mode incoherent detector has a field of view solid angle Ω_{fv} and receives D_s spatial modes of $M_s \geq D_s$ transmitted signal modes. Each signal mode has signal power P_{s_0}, noise power P_{b_0}, and incoherent modulation $|s_i(t)|^2(1+\beta m(t))$. The corresponding shot-noise limited SNR is $\frac{(\alpha D_s P_{s_0})^2}{2\alpha F B_m D_s (P_{s_0}+P_{b_0})}$.

$$\mathcal{E}\{\bar{n}\} = \mathcal{E}\{n(t)\} = \alpha A \sum_{i=1}^{D_s} \mathcal{E}\{|s_i(t)+b_i(t)|^2\} = \alpha \sum_{i=1}^{D_s} (P_{s_0}+P_{b_0}) = \alpha D_s (P_{s_0}+P_{b_0})$$

$$\mathcal{E}\{n(t+\tau)n(t)\} = (\alpha A)^2 \sum_{i,j=1}^{D_s} \mathcal{E}\{|s_i(t+\tau)+b_i(t+\tau)|^2 |s_j(t)+b_j(t)|^2\}$$

$$= (\alpha A)^2 \sum_{i=1}^{D_s} \mathcal{E}\{|s_i(t+\tau)+b_i(t+\tau)|^2 |s_i(t)+b_i(t)|^2\}$$

$$+ (\alpha A)^2 \sum_{i,j=1,i\neq j}^{D_s} \mathcal{E}\{|s_i(t+\tau)+b_i(t+\tau)|^2 |s_j(t)+b_j(t)|^2\}$$

$$= \sum_{i=1}^{D_s} (\alpha\beta)^2 P_{s_i}^2 R_m(\tau) + \sum_{i,j=1,i\neq j}^{D_s} (\alpha\beta)^2 P_{s_i} P_{s_j} R_m(\tau) + \cdots = (\alpha\beta)^2 D_s^2 P_{s_0}^2 R_m(\tau) + \cdots$$

Multi-mode Signal/Multi-mode Detection System

Example (Multi-mode direct detection with $\Omega_{fv} \leq \Omega_s$ (cont.))

A multi-mode incoherent detector has a field of view solid angle Ω_{fv} and receives D_s spatial modes of $M_s \geq D_s$ transmitted signal modes. Each signal mode has signal power P_{s_0}, noise power P_{b_0}, and incoherent modulation $|s_i(t)|^2(1+\beta m(t))$. The corresponding shot-noise limited SNR is $\frac{(\alpha D_s P_{s_0})^2}{2\alpha F B_m D_s (P_{s_0}+P_{b_0})}$.

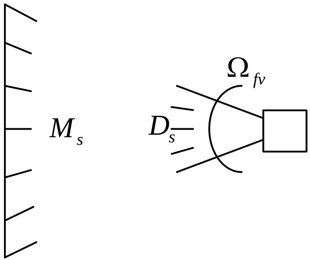
$$S_i(\omega) \approx |F(\omega)|^2 e^2 \left[\alpha D_s \bar{g}^2 (P_{s_0} + P_{b_0}) + (\alpha\beta)^2 D_s^2 \bar{g}^2 P_{s_0}^2 S_m(\omega)\right]$$

$$\text{SNR} = \frac{(\alpha D_s P_{s_0})^2 (\beta^2 P_m)}{F\alpha D_s (P_{s_0} + P_{b_0}) 2B_m} \propto \frac{(\alpha D_s P_{s_0})^2}{2\alpha F B_m D_s (P_{s_0} + P_{b_0})} = D_s \frac{(\alpha P_{s_0})^2}{2\alpha F B_m (P_{s_0} + P_{b_0})}$$

Multi-mode Signal/Multi-mode Detection System

Example (Multi-mode direct detection with $\Omega_{fv} \geq \Omega_s$)

A multi-mode incoherent detector has a field of view solid angle Ω_{fv} and receives D_s spatial modes including $M_s \leq D_s$ transmitted signal modes. Each signal mode has signal power P_{s_0}, noise power P_{b_0}, and incoherent modulation $|s_i(t)|^2(1+\beta m(t))$. The corresponding shot-noise limited SNR is $\frac{(\alpha M_s P_{s_0})^2}{2\alpha F B_m (M_s P_{s_0} + D_s P_{b_0})}$.

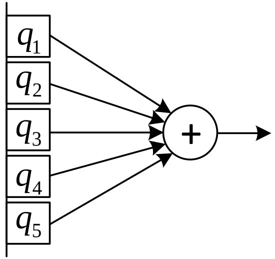


$$\text{SNR} \propto \frac{(\alpha M_s P_{s_0})^2}{2\alpha F B_m (M_s P_{s_0} + D_s P_{b_0})}$$

Multi-mode Signal/Multi-mode Detection System

Example (Max-SNR multi-mode direct detection)

A multi-mode incoherent detector, receiving a linear combination of the spatial modes, achieves the maximum shot noise-limited SNR for the combination coefficients $q_i = C\frac{P_{s_i}}{P_{s_i}+P_{b_0}}$, where P_{s_i} is the signal power is spatial mode i and P_{b_0} denotes the noise power in each spatial mode.



Multi-mode Signal/Multi-mode Detection System

Example (Max-SNR multi-mode direct detection (cont.))

A multi-mode incoherent detector, receiving a linear combination of the spatial modes, achieves the maximum shot noise-limited SNR for the combination coefficients $q_i = C\frac{P_{s_i}}{P_{s_i}+P_{b_0}}$, where P_{s_i} is the signal power is spatial mode i and P_{b_0} denotes the noise power in each spatial mode.

$$\text{SNR} \propto \frac{\left(\sum_i q_i P_{s_i}\right)^2}{2\alpha F B_m \sum_i q_i^2 (P_{s_i} + P_{b_0})}$$

$$\Big(\sum_i q_i P_{s_i}\Big)^2 = \Big(\sum_i q_i \sqrt{P_{s_i} + P_{b_0}} \frac{P_{s_i}}{\sqrt{P_{s_i} + P_{b_0}}}\Big)^2 \le \sum_i \big(q_i \sqrt{P_{s_i} + P_{b_0}}\big)^2 \sum_i \Big(\frac{P_{s_i}}{\sqrt{P_{s_i} + P_{b_0}}}\Big)^2$$

$$\text{SNR} \propto \frac{\left(\sum_i q_i P_{s_i}\right)^2}{2\alpha F B_m \sum_i q_i^2 (P_{s_i} + P_{b_0})} \le \sum_i \frac{(\alpha P_{s_i})^2}{2\alpha F B_m (P_{s_i} + P_{b_0})}$$

$$q_i \sqrt{P_{s_i} + P_{b_0}} = C \frac{P_{s_i}}{\sqrt{P_{s_i} + P_{b_0}}} \Rightarrow q_i = C \frac{P_{s_i}}{P_{s_i} + P_{b_0}}$$

Multi-mode Signal/Multi-mode Detection System

Example (Max-SNR multi-mode direct detection (cont.))

A multi-mode incoherent detector, receiving a linear combination of the spatial modes, achieves the maximum shot noise-limited SNR for the combination coefficients $q_i = C\frac{P_{s_i}}{P_{s_i}+P_{b_0}}$, where P_{s_i} is the signal power is spatial mode i and P_{b_0} denotes the noise power in each spatial mode.

$$P_{s_i} = 0 \Rightarrow q_i = 0$$
$$P_{s_i} \gg P_{b_0} \Rightarrow q_i = 1$$
$$P_{s_i} = P_{s_j} \Rightarrow q_i = q_j$$

The End