MATHEMATICAL QUESTIONS

Question 1

Let $a(t)$ be a random Wiener process satisfying

1. $a(0) = 0$
2. $\mathcal{E}\{a(t)\} = 0$
3. $\mathcal{E}\{a^2(t)\} = \sigma^2 t$
4. $P\{a(t) = a\} = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp(-\frac{a^2}{2\sigma^2 t})$
5. $a(t_3) - a(t_2) \perp\!\!\!\perp a(t_2) - a(t_1), \quad t_1 < t_2 < t_3$

(a) Find the correlation function of the Wiener process $a(t)$.

(b) Find the Karhunen-Loeve expansion of the Wiener process $a(t)$.

Question 2

Find the Karhunen-Loeve expansion of the Brownian bridge process defined as $x(t) = a(t) - ta(1)$ over the interval $[0, 1]$, where $a(t)$ is a Wiener random process.

Question 3

For $\beta > 0$ and $\rho > 0$, we have a stochastic differential equation (SDE) for the stochastic process $z(t)$ as

$$dz(t) = -\beta z(t)dt + \rho\, da(t)$$

, where $a(t)$ is a Wiener random process with $\mathcal{E}\{a^2(t)\} = t$. This SDE can be solved and the corresponding solution is

$$z(t) = z(0)e^{-\beta t} + \rho \int_0^t e^{-\beta(t-s)} da(s)$$

If $z(0) \sim \mathcal{N}(0, c)$ is a normal random variable, where $c = \frac{\rho^2}{2\beta}$, $z(t)$ is called an Ornstein-Uhlenbeck process with parameters β and c.

Question 3 continued on next page. . .

(a) Find the correlation function of the Ornstein-Uhlenbeck process $z(t)$.

(b) Find the Karhunen-Loeve expansion of the Ornstein-Uhlenbeck $z(t)$ over the interval $[0, 1]$.

Question 4

Let $X \sim \mathcal{B}(n, p)$ is a binomial random variable, where n is the number of independent Bernoulli trials and p is the probability of a successful trial.

(a) Show that if $n \to \infty$ while $np = \lambda$ is finite, the random variable X has the Poisson distribution $X \sim \mathcal{P}(\lambda) = \mathcal{P}(np)$.

(b) Show that if $n \to \infty$, the random variable X has the normal distribution $X \sim \mathcal{N}(\mu, \sigma^2) = \mathcal{N}(np, np(1-p))$.

SOFTWARE QUESTIONS

Question 5

Use Python or MATLAB to develop a function that generates sample functions of a Wiener random process.

BONUS QUESTIONS

Question 6

Extend the code in Question 5 to generate sample functions of Wiener, Brownian bridge, and white Gaussian random processes. You might add a new argument to the function that repre-

Question 6 continued on next page...

sents the random process type.

Question 7

Return your answers by filling the LaTeX template of the assignment.