Coherent Light

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Overview

Preliminaries

Special Relativity

- First postulate Einstein's special relativity: The laws of physics are invariant in all inertial frames of reference.
- Second postulate Einstein's special relativity: The speed of light in vacuum is the same for all observers.
- Length contraction: $L = L_0\sqrt{1 - v^2/c^2}$
- Time dilation: $T_0 = T\sqrt{1 - v^2/c^2}$
- Lorentz factor: $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1 + \frac{v^2}{2c^2} \approx 1, v \to 0$
- Rest energy: $E_0 = m_0 c^2$
- Relativistic mass: $m = \gamma m_0$
- Relativistic momentum: $p = \gamma m_0 v$
- Relativistic kinetic energy: $K = (\gamma - 1)m_0 c^2 = \Delta E = E - E_0 = (\gamma - 1)E_0$
- Relativistic energy: $E = \gamma E_0 = \sqrt{p^2 c^2 + (m_0 c^2)^2}$

Matter Wave

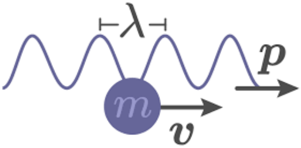

Figure: Wave-particle duality.

- Planck's black body radiation: Waves exhibit particle-like behavior.
- De Broglie's matter wave: Particles exhibit wave-like behavior.
- Planck's photon energy: $E = h\nu$
- Mass-less relativistic energy: $E = pc = h\nu = \frac{hc}{\lambda}$
- Mass-less momentum: $p = \frac{h}{\lambda} = \frac{E}{c}$
- De Broglie's wavelength: $\lambda = \frac{h}{p} = \frac{h}{\gamma m_0 v}$

Free Particle

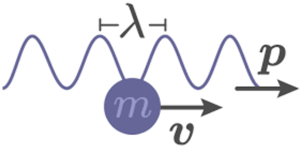

Figure: Non-relativistic free particle wave.

- Free particle De Broglie's wave: $\Psi(x,t) = A\cos(2\pi\nu t - kx), k = \frac{2\pi}{\lambda} = \frac{2\pi mv}{h}$
- Free particle De Broglie's wave at $t = 0$: $\psi(x) = \psi(x,0) = A\cos(kx), k = \frac{2\pi}{\lambda}$
- Free particle kinetic energy: $K = \frac{1}{2}mv^2 = \frac{1}{2m}\frac{h^2k^2}{4\pi^2} = \frac{1}{2m}\hbar^2k^2$
- Free particle differential equation: $\frac{d^2\psi}{dx^2} = -k^2\psi = -\frac{2m}{\hbar^2}K\psi = -\frac{2m}{\hbar^2}(E - V)\psi$
- Time-independent Schrodinger's equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

Schrodinger's Equation

- Wave function; $\Psi(x,t)$
- Potential function; $V(x,t)$
- Schrodinger's equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = j\hbar\frac{\partial\Psi}{\partial t}$
- Location probability: $P(a,b) = \int_a^b |\Psi(x,t)|^2 dx$
- Time-independent potential: $V(x) \Rightarrow \Psi(x,t) = \psi(x)\varrho(t)$
- Time-dependence of wave function potential: $\varrho(t) \propto e^{-j\frac{E}{\hbar}t}$
- Time-independent Schrodinger's equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

Classical Harmonic Oscillator

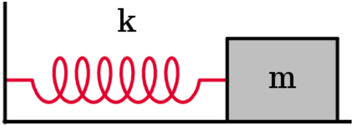

Figure: Mechanical schematic of a harmonic oscillator.

- Newton's second law: $F = mx''$
- Hooke's spring law: $F = -kx$
- Oscillator differential equation: $x'' + \frac{k}{m}x = 0$
- Oscillation frequency: $\omega = \sqrt{\kappa/m}$
- Oscillator position equation: $x(t) = A\cos(\omega t + \phi)$
- Potential energy: $V(x) = \int_0^x -F(u)du = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$
- Kinetic energy: $K(x) = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$
- Kinetic energy: $E(x) = K(x) + V(x) = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2$

Quantum Harmonic Oscillator

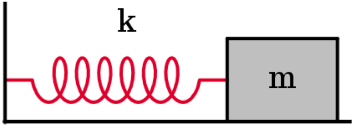

Figure: Mechanical schematic of a harmonic oscillator.

- Potential energy: $V(x) = \frac{1}{2}\kappa x^2$
- Oscillation frequency: $\omega = \sqrt{\kappa/m}$
- Unit-mass assumption: $V(x) = \frac{1}{2}\omega^2 x^2$
- Time-independent Schrodinger's equation: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + V\psi = E\psi$
- Discrete energy levels: $E_n = h\nu(\frac{1}{2} + n), n = 0, 1, \cdots$
- Normalized Hermite-Gaussian wavefunctions:
  $\psi_n(x) = \frac{1}{\sqrt{2^n n!}}(\frac{\omega}{\pi\hbar})^{0.25}\mathbb{H}_n[\sqrt{\frac{\omega}{\hbar}}x]\exp\left(-\frac{\omega x^2}{2\hbar}\right)$
- Normalized time-dependent Hermite-Gaussian wavefunctions:
  $\Psi_n(x,t) = \frac{1}{\sqrt{2^n n!}}(\frac{\omega}{\pi\hbar})^{0.25}\mathbb{H}_n[\sqrt{\frac{\omega}{\hbar}}x]\exp\left(-\frac{\omega x^2}{2\hbar}\right)\exp(-j(n+0.5)\omega t)$

Quantum Harmonic Oscillator

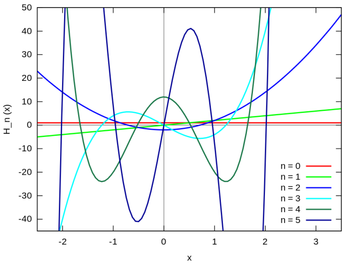

Figure: Hermite polynomials $\mathbb{H}_n(x)$ of order n.

$$\mathbb{H}_0(x) = 1$$
$$\mathbb{H}_1(x) = 2x$$
$$\mathbb{H}_2(x) = 4x^2 - 2$$
$$\mathbb{H}_3(x) = 8x^3 - 12x$$
$$\mathbb{H}_4(x) = 16x^4 - 48x^2 + 12$$
$$\mathbb{H}_5(x) = 32x^5 - 160x^3 + 120x$$
$$\mathbb{H}_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

Quantum Harmonic Oscillator

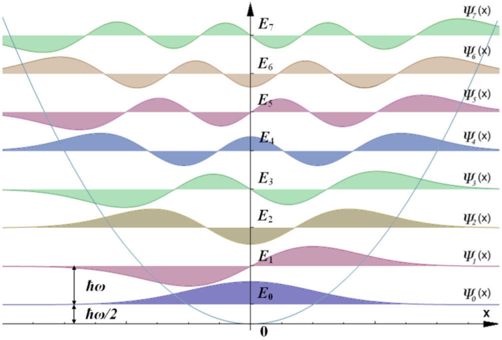

Figure: Wavefunctions quantum harmonic oscillator.

- Normalized Hermite-Gaussian wavefunctions;
  $\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\omega}{\pi\hbar}\right)^{0.25}\mathbb{H}_n\left[\sqrt{\frac{\omega}{\hbar}}x\right]\exp\left(-\frac{\omega x^2}{2\hbar}\right)$

Quantum Harmonic Oscillator

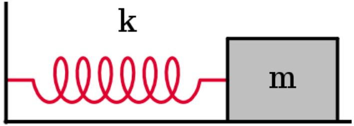

Figure: Mechanical schematic of a harmonic oscillator.

- General wavefunction: $\psi(x) = \sum_n c_n \psi_n(x)$
- General time-dependent wavefunction: $\Psi(x,t) = \sum_n c_n e^{-j(n+0.5)\omega t} \psi_n(x)$
- Probability of carrying n quanta of energy: $p(n) \propto |c_n|^2$
- Probability density of being at position x: $\propto |\psi(x)|^2$
- Probability density of having momentum p: $\propto |\phi(p)|^2, \phi(p) = \frac{1}{\sqrt{h}} F_{\psi}^{-1}(\frac{p}{h})$
- Position root mean square: σ_x
- Momentum root mean square: σ_p
- Heisenberg position-momentum uncertainty: $\sigma_x \sigma_p \geq 0.5\hbar$

Quantum Harmonic Oscillator

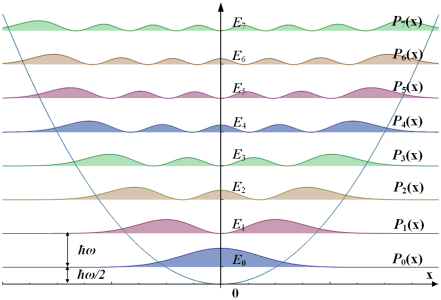

Figure: Position probability density for different energy levels of a harmonic oscillator.

- Probability density of being at position x: $\propto |\psi(x)|^2$

Quantum Harmonic Oscillator

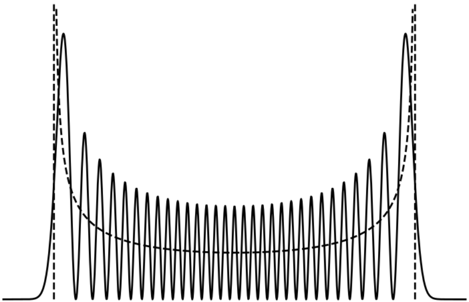

Figure: Position probability density for a high-energy level of a harmonic oscillator.

- Probability density of being at position x: $\propto |\psi(x)|^2$

Quantum Harmonic Oscillator

Example (Coherent light communication)

The amplitude of the classical harmonic oscillator can be selected such that its energy equals to that of a quantum harmonic oscillator in energy state n.

$$\frac{1}{2} m\omega^2 A^2 = (n + 0.5)h\nu \Rightarrow A = \frac{1}{2\pi}\sqrt{(2n+1)\frac{h}{m\nu}}$$

Physical Description of Coherent Light

Monochromatic Plane Wave

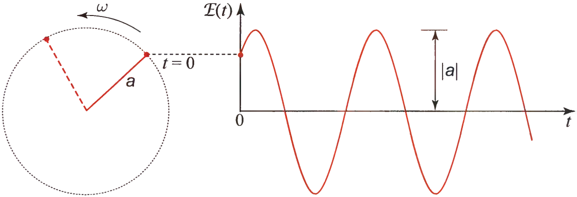

Figure: Phasor of a monochromatic plane wave.

- Monochromatic plane wave: $e(\boldsymbol{r}, t) = \mathrm{Re}\{E(\boldsymbol{r}, t)\}$
- Monochromatic plane wave complex function: $E(\boldsymbol{r}, t) = Ae^{-j\boldsymbol{k}.\boldsymbol{r}}e^{j2\pi\nu t}$
- Classical wave energy: $\frac{1}{2}\epsilon|A|^2 V$
- Quantum wave energy: $h\nu|\boldsymbol{a}|^2$
- Phasor quandrature components: $\boldsymbol{x} = \mathrm{Re}\{\boldsymbol{a}\}, \quad \boldsymbol{p} = \mathrm{Im}\{\boldsymbol{a}\}$

Analogy Between Optical Mode and Harmonic Oscillator

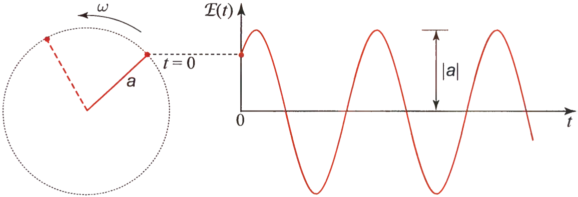

Figure: Phasor of a monochromatic plane wave.

- Energy of an electromagnetic mode: $h\nu|\boldsymbol{a}|^2 = h\nu(\boldsymbol{x}^2 + \boldsymbol{p}^2)$
- Energy of harmonic oscillator: $\frac{1}{2}(p^2 + \omega^2 x^2)$
- Electromagnetic mode location: $\boldsymbol{x} = \frac{\omega x}{\sqrt{2\hbar\omega}}$
- Electromagnetic mode momentum: $\boldsymbol{p} = \frac{p}{\sqrt{2\hbar\omega}}$
- Heisenberg position-momentum uncertainty: $\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{p}} \geq 0.25$

Coherent Light

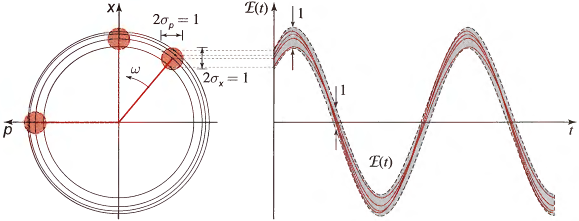

Figure: Phasor of a coherent light. Representative values of $E(t) = \exp(j2\pi\nu t)$ are drawn by choosing several arbitrary points within the uncertainty circle.

- Minimum uncertainty: $\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{p}} = 0.25$
- Equal uncertainty: $\sigma_{\boldsymbol{x}} = \sigma_{\boldsymbol{p}} = 0.5$
- Gaussian momentum wavefuntion: $\phi(\boldsymbol{p}) \propto \exp\left[-(\boldsymbol{p} - \alpha_{\boldsymbol{p}})^2\right]$
- Gaussian location wavefuntion: $\psi(\boldsymbol{x}) \propto \exp\left[-(\boldsymbol{x} - \alpha_{\boldsymbol{x}})^2\right]$
- Gaussian location probability: $|\psi(\boldsymbol{x}, t)|^2 \propto \exp\left[-2(\boldsymbol{x} - \alpha_{\boldsymbol{x}}\cos(\omega t - \theta))^2\right]$
- Probability of carrying n quanta of energy:
  $p(n) = |c_n|^2 = e^{-\bar{n}}\frac{\bar{n}^n}{n!}, \quad \bar{\boldsymbol{n}} = \alpha_{\boldsymbol{x}}^2 + \alpha_{\boldsymbol{p}}^2$

Statistical Description of Coherent Light

Coherent Light

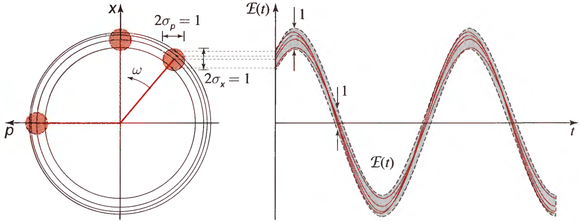

Figure: Phasor of a coherent light.

- Probability of carrying n quanta of energy:
$p(n) = |c_n|^2 = e^{-\bar{n}} \frac{\bar{n}^n}{n!}, \quad \bar{n} = \alpha_{\boldsymbol{x}}^2 + \alpha_{\boldsymbol{p}}^2$
- Probability of having n photons: $p(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$
- Mean photon number: $w = \bar{n} = \frac{1}{h\nu} \int_0^T P dt = \frac{E}{h\nu} = \frac{1}{h\nu} \int_0^T \int_A I(\boldsymbol{r}) dA dt$

Coherent Light

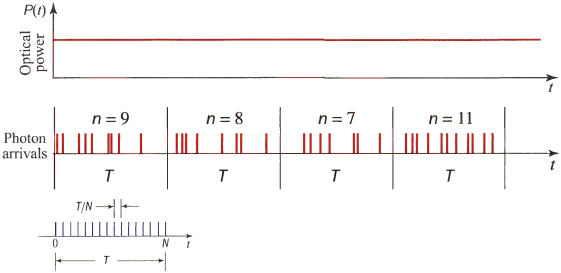

Figure: Constant optical power and the corresponding random photon generation times.

- Independent photon generation time
- Photon generation probability in each sub-interval T/N: $p = \frac{\bar{n}}{N}$
- Photon absence probability in each sub-interval T/N: $1 - p = 1 - \frac{\bar{n}}{N}$
- Probability of n generations in interval T: $p(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$
- Probability of n generations in interval T: $n \to \infty \Rightarrow p(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$

Arbitrary Light

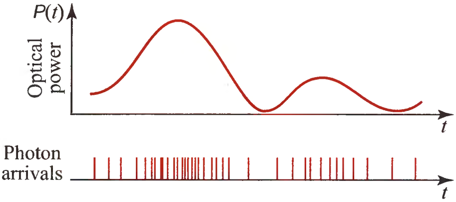

Figure: Time-varying optical power and the corresponding random photon generation times.

- Mean photon number for monochromatic wave:
  $w = \bar{n} = \frac{E}{h\nu} = \frac{1}{h\nu}\int_0^T P(t)dt = \frac{1}{h\nu}\int_0^T \int_A I(\mathbf{r}, t)dAdt$
- Mean photon number for quasi-monochromatic wave:
  $w = \bar{n} \approx \frac{E}{h\bar{\nu}} = \frac{1}{h\bar{\nu}}\int_0^T P(t)dt = \frac{1}{h\bar{\nu}}\int_0^T \int_A I(\mathbf{r}, t)dAdt$
- Random mean photon number pdf: $p(w)$
- Conditional Poisson distribution: $p(n|w) = e^{-w}\frac{w^n}{n!}$
- Mandel's formula: $p(n) = \int_0^\infty e^{-w}\frac{w^n}{n!}p(w)dw$

Arbitrary Light

Example (Thermal light)

For thermal light, the mean photon number is randomly distributed by $p(w) = \frac{1}{\bar{w}} e^{-\frac{w}{\bar{w}}}$.

$$p(n) = \int_0^\infty e^{-w} \frac{w^n}{n!} p(w) dw = \int_0^\infty e^{-w} \frac{w^n}{n!} \frac{1}{\bar{w}} e^{-\frac{w}{\bar{w}}} dw$$

$$= \frac{1}{n!\bar{w}} \int_0^\infty w^n e^{-\frac{\bar{w}+1}{\bar{w}} w} dw = \frac{1}{n!\bar{w}} \frac{n!}{(\frac{\bar{w}+1}{\bar{w}})^n} = \frac{1}{\bar{w}+1} (\frac{\bar{w}}{\bar{w}+1})^n$$

Analytical Description of Coherent Light

Coherent Light

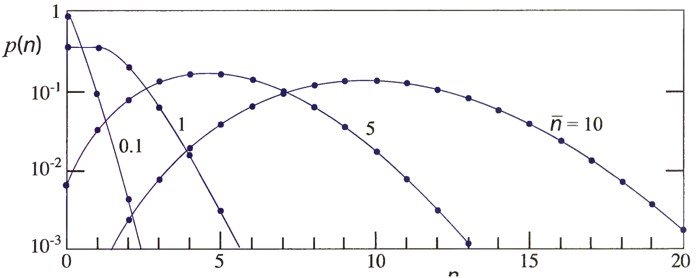

Figure: Poisson distribution $p(n)$ of the photon number n.

- Poisson distribution: $P(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!}, n \in \mathbb{W}$
- Signal to noise ratio: $\text{SNR} = \frac{\text{mean}^2}{\text{variance}} = \frac{\bar{n}^2}{\bar{n}} = \bar{n}$

Coherent Light

Example (Coherent light communication)

In a simple on-off keying digital optical communication system using monochromatic coherent light, where $P(n) = \exp(-\bar{n})\frac{\bar{n}^n}{n!}$, the bit error rate 10^{-9} corresponds to to a mean number of photons $\bar{n} = 20$.

$$P_e = P_1 P_{e|1} + P_0 P_{e|0} = \frac{1}{2}P(0) = \frac{1}{2}\exp(-\bar{n}) \leq 10^{-9} \Rightarrow \bar{n} \geq 20$$

Example (Thermal light communication)

In a simple on-off keying digital optical communication system using monochromatic thermal light, the bit error rate 10^{-9} corresponds to to a mean number of photons $\bar{n} = 5 \times 10^8$.

$$P_e = P_1 P_{e|1} + P_0 P_{e|0} = \frac{1}{2}P(0) = \frac{1}{2}\frac{1}{\bar{n}+1} \leq 10^{-9} \Rightarrow \bar{n} \geq 499999999$$

Arbitrary Light

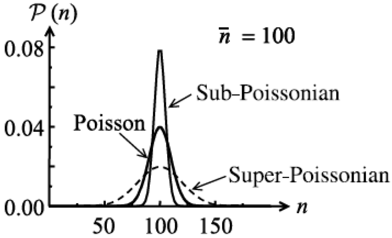

Figure: Comparison of the photon statistics for light with a Poisson distribution, and those for sub-Poissonian and super-Poissonian light. The distributions have been drawn with the same mean photon number. .

- Sub-Poissonian light: $\Delta n < \sqrt{\bar{n}}$
- Poissonian light: $\Delta n = \sqrt{\bar{n}}$
- Super-Poissonian light: $\Delta n > \sqrt{\bar{n}}$

The End