Digital Communication

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Overview

System Model

System Model

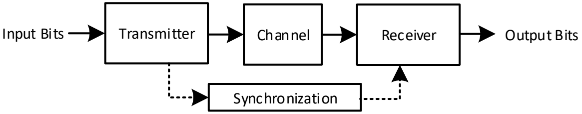

Figure: Digital communication system.

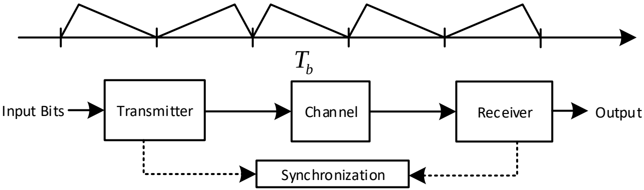

Figure: Binary communication system with the transmission bit rate $R_b = \frac{1}{T_b}$ and two distinguishable waveforms in interval T_b. Each interval T_b corresponds to a transmitted bit.

System Model

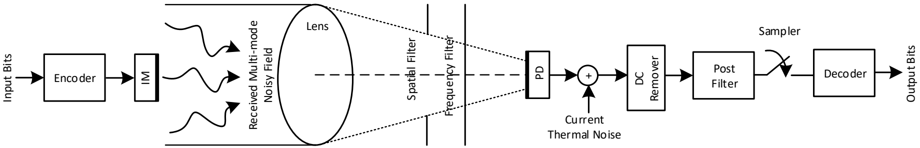

Figure: Binary direct detection system.

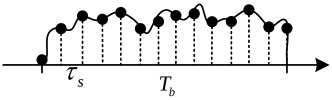

Figure: Equivalent vector representation for each received bit waveform in a binary incoherent system.

System Model

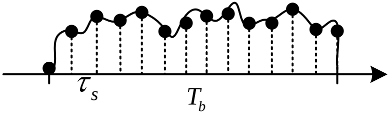

Figure: Equivalent vector representation for each received bit waveform in a binary incoherent system.

- Count intensity: $n(t) = \alpha \int_A |a(t, \boldsymbol{r})|^2 d\boldsymbol{r} = \alpha \int_A I(t, \boldsymbol{r}) d\boldsymbol{r}$
- Shot noise-limited current: $i(t) = \sum_{j=1}^{k(0,t)} h_f(t - t_j) \approx \frac{e}{\tau_s} k(t - \tau_s, t)$
- Sampling rate: $f_s = 2B_n$
- Sampling period: $\tau_s = \frac{1}{2B_n}$
- Sampling time: $t_j = \frac{j}{2B_n}$
- Sampling value: $k_j \sim k(t_j - \tau_s, t_j)$
- Samples vector: $\boldsymbol{k} = (k_1, k_2, \cdots, k_{2B_n T_b})$

System Model

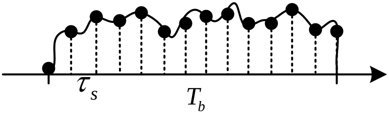

Figure: Equivalent vector representation for each received bit waveform in a binary incoherent system. For shot noise-limited Poisson detection, the count process is approximately Poisson.

- Multi-mode detection: $D = 2B_o\tau_s \gg 1 \equiv 2B_o\frac{1}{B_n} \gg 1 \equiv B_o \gg B_n$ or $D_s \gg 1$
- Poisson detection: $k_j \sim \text{Poisson}(\mu_j + \mu_b), \quad k_j \perp\!\!\!\perp k_{j'}, j \neq j'$
- Signal count energy per sample: $\mu_j = \int_{t_j-\tau_s}^{t_j} n(t)dt$
- Average signal count vector: $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_{2B_nT_b})$
- Background noise count energy per sample: $\mu_b = \alpha N_0 2B_o\tau_s D_s$

System Model

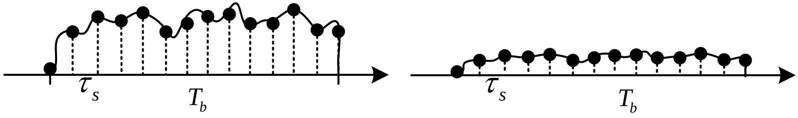

Figure: Equivalent vector representation for zero-bit and one-bit waveforms in a binary incoherent system.

- Maximum A Posteriori (MAP): probability of sending bit i given observation $\boldsymbol{k}$, $P(i|\boldsymbol{k})$
- Maximum Likelihood (ML): probability of observing $\boldsymbol{k}$ given sending bit i, $P(\boldsymbol{k}|i)$
- Equally probable symbols: ML decision $\equiv$ MAP decision
- Likelihood function: $\Lambda_i = P(\boldsymbol{k}|i)$
- Generalized likelihood function: $\Lambda_i = \mathcal{E}_\theta\{P(\boldsymbol{k}|i,\theta)\}$
- ML binary decision rule: $\Lambda_1 \gtrless \Lambda_0$, or $\ln(\Lambda_1) \gtrless \ln(\Lambda_0)$

System Model

- Likelihood function:

$$\Lambda_i = P(\boldsymbol{k}|i) = \prod_{j=1}^{2B_nT_b} \frac{(\mu_{ij}+\mu_b)^{k_j}}{k_j!} e^{-(\mu_{ij}+\mu_b)} = e^{-(K_i+K_b)} \prod_{j=1}^{2B_nT_b} \frac{(\mu_{ij}+\mu_b)^{k_j}}{k_j!}$$

- Signal count energy:

$$K_i = \sum_{j=1}^{2B_nT} \mu_{ij} = \sum_{j=1}^{2B_nT} \int_{t_j-\tau_s}^{t_j} n_i(t)dt = \int_0^{T_b} n_i(t)dt$$

- Background noise count energy:

$$K_b = 2B_nT_b\mu_b = \alpha N_0 2B_o T_b D_s$$

- Likelihood function:

$$\ln(\Lambda_i) = \sum_{j=1}^{2B_nT_b} \ln\left(\frac{(\mu_{ij}+\mu_b)^{k_j}}{k_j!} e^{-(\mu_{ij}+\mu_b)}\right) = \sum_{j=1}^{2B_nT_b} \left[k_j \ln\left(1+\frac{\mu_{ij}}{\mu_b}\right)\right] - K_i + \sum_{j=1}^{2B_nT_b} \left[k_j \ln(\mu_b) - \ln(k_j!)\right] - K_b$$

System Model

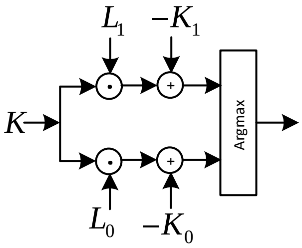

Figure: Discrete detector model for a binary incoherent system.

- Likelihood function: $\ln(\Lambda_i) = \sum_{j=1}^{2B_nT_b}\left[k_j \ln\left(1+\frac{\mu_{ij}}{\mu_b}\right)\right] - K_i$
- Count samples vector: $\boldsymbol{k} = (k_{i1}, k_{i2}, \cdots, k_{i2B_nT_b})$
- Log intensity vector: $\boldsymbol{L}_i = (\ln\left(1+\frac{\mu_{i1}}{\mu_b}\right), \ln\left(1+\frac{\mu_{i2}}{\mu_b}\right), \cdots, \ln\left(1+\frac{\mu_{i2B_nT_b}}{\mu_b}\right))$
- Signal count energy: $K_i = \int_0^{T_b} n_i(t)dt$
- Noise count energy: $K_b = \alpha N_0 2B_o T_b D_s$

System Model

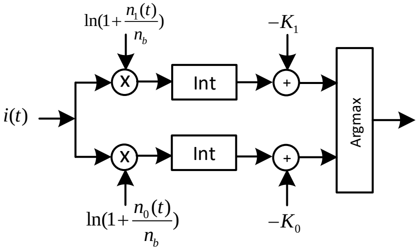

Figure: Continuous detector model for a binary incoherent system.

- Likelihood function:

$$\ln(\Lambda_i) = \sum_{j=1}^{2B_n T_b} \left[k_j \ln\left(1 + \frac{\mu_{ij}}{\mu_b}\right)\right] - K_i \approx \sum_{j=1}^{T_b/\tau_s} \left[k(t_j - \tau_s, t_j) \ln\left(1 + \frac{n_i(t_j)\tau_s}{n_b \tau_s}\right)\right] - K_i$$

$$\ln(\Lambda_i) = \int_0^{T_b} \frac{k(t_j - \tau_s, t_j)}{\tau_s} \ln\left(1 + \frac{n_i(t)}{n_b}\right) dt - K_i = \frac{1}{e\tau_s} \int_0^{T_b} i(t) \ln\left(1 + \frac{n_i(t)}{n_b}\right) dt - K_i$$

- Average noise count rate: $n_b = \mu_b / \tau_s$

OOK

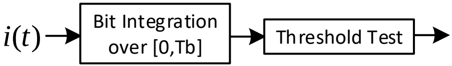

Figure: Continuous detector model for an on-off keying (OOK) incoherent system.

- One-bit count intensity waveform: $n_1(t) = n_s, \quad 0 \leq t \leq T_b$
- Zero-bit count intensity waveform: $n_0(t) = 0, \quad 0 \leq t \leq T_b$
- One-bit log likelihood function: $\ln(\Lambda_1) = \ln\left(1 + \frac{K_s}{K_b}\right) k(0, T_b) - K_s$
- Zero-bit log likelihood function: $\ln(\Lambda_0) = 0$
- Signal count over bit interval: $K_s = n_s T_b$
- Noise count over bit interval: $K_b = \alpha N_0 2 B_o T_b D_s$
- ML decision: $k(0, T_b) \gtrless m_T$
- Decision threshold: $m_T = \frac{K_s}{\ln\left(1 + \frac{K_s}{K_b}\right)}$

Binary PPM

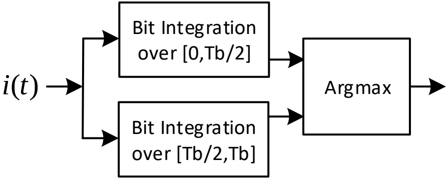

Figure: Continuous detector model for an binary pulse position modulation (PPM) (Manchester Pulsed signal) incoherent system.

- One-bit count intensity waveform: $n_1(t) = n_s, \quad 0 \leq t \leq T_b/2$
- Zero-bit count intensity waveform: $n_0(t) = n_s, \quad T_b/2 \leq t \leq 0$
- One-bit log likelihood function: $\ln(\Lambda_1) = \ln\left(1 + \frac{K_s}{K_b}\right) k(0, T_b/2) - K_s$
- Zero-bit log likelihood function: $\ln(\Lambda_1) = \ln\left(1 + \frac{K_s}{K_b}\right) k(T_b/2, T_b) - K_s$
- Signal count over bit half-interval: $K_s = n_s T_b/2$
- Noise count over bit half-interval: $K_b = \alpha N_0 B_o T_b D_s$
- ML decision: $k(0, T_b/2) \gtrless k(T_b/2, T_b)$

System Analysis

OOK

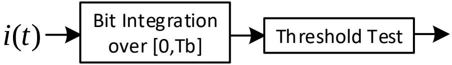

Figure: Continuous detector model for a on-off keying (OOK) incoherent system.

- Binary error probability: $P_e = \frac{1}{2}P(e|1) + \frac{1}{2}P(e|0)$
- OOK error probability:

$$P_e = \frac{1}{2}P(k(0,T_b) < m_T|1) + \frac{1}{2}P(k(0,T_b) > m_T|0) + \frac{1}{4}P(k(0,T_b) = m_T|1) + \frac{1}{4}P(k(0,T_b) = m_T|0)$$

$$= \frac{1}{2}\sum_{k=0}^{\lfloor m_T \rfloor -1} e^{-(K_s+K_b)}\frac{(K_s+K_b)^k}{k!} + \frac{1}{2}\sum_{\lfloor m_T \rfloor +1}^{\infty} e^{-K_b}\frac{K_b^k}{k!} + \frac{1}{4}e^{-(K_s+K_b)}\frac{(K_s+K_b)^{\lfloor m_T \rfloor}}{\lfloor m_T \rfloor!} + \frac{1}{4}e^{-K_b}\frac{K_b^{\lfloor m_T \rfloor}}{\lfloor m_T \rfloor!}$$

$$\approx \frac{1}{2}\sum_{k=0}^{\lfloor m_T \rfloor} e^{-(K_s+K_b)}\frac{(K_s+K_b)^k}{k!} + \frac{1}{2}\sum_{\lfloor m_T \rfloor}^{\infty} e^{-K_b}\frac{K_b^k}{k!}$$

$$= \frac{1}{2}\left[1 + \frac{\Gamma(\lfloor m_T \rfloor, K_b) - \Gamma(\lfloor m_T \rfloor, K_s+K_b)}{\Gamma(\lfloor m_T \rfloor, \infty)}\right] \approx \frac{1}{2}\left[1 + \frac{\Gamma(m_T, K_b) - \Gamma(m_T, K_s+K_b)}{(\lfloor m_T \rfloor - 1)!}\right]$$

- Incomplete gamma function: $\Gamma(n,x) = \int_0^x t^{n-1}e^{-t}dt, \quad \sum_{n=c}^{\infty}\frac{x^n}{n!}e^{-n} = \frac{\Gamma(c,x)}{\Gamma(c,\infty)}$

OOK

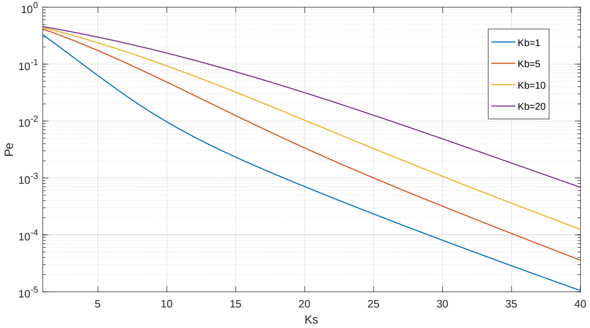

Figure: Error probability curve for on-off keying (OOK) incoherent system.

- OOK error probability: $P_e \approx \frac{1}{2}\left[1 + \frac{\Gamma(m_T, K_b) - \Gamma(m_T, K_s + K_b)}{(\lfloor m_T \rfloor - 1)!}\right]$
- Decision threshold: $m_T = \frac{K_s}{\ln\left(1 + \frac{K_s}{K_b}\right)}$

Binary PPM

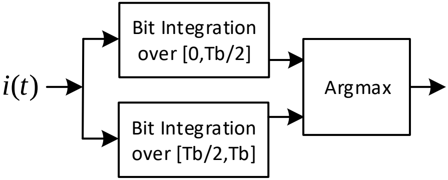

Figure: Continuous detector model for an binary pulse position modulation (PPM) (Manchester Pulsed signal) incoherent system.

- Binary error probability: $P_e = \frac{1}{2}P(e|1) + \frac{1}{2}P(e|0)$
- BPPM error probability:

$$P_e = \frac{1}{2}P(k(0, T_b/2) < k(T_b/2, T_b)|1) + \frac{1}{2}P(k(0, T_b/2) > k(T_b/2, T_b)|0)$$
$$+ \frac{1}{4}P(k(0, T_b/2) = k(T_b/2, T_b)|1) + \frac{1}{4}P(k(0, T_b/2) = k(T_b/2, T_b)|0)$$
$$= \sum_{k_1=0}^{\infty} e^{-(K_s+K_b)} \frac{(K_s + K_b)^{k_1}}{k_1!} \sum_{k_2=k_1+1}^{\infty} e^{-K_b} \frac{K_b^{k_2}}{k_2!} + \frac{1}{2} \sum_{k_1=0}^{\infty} e^{-(K_s+K_b)} \frac{(K_s + K_b)^{k_1}}{k_1!} e^{-K_b} \frac{K_b^{k_1}}{k_1!}$$

Binary PPM

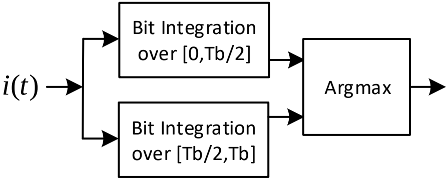

Figure: Continuous detector model for an binary pulse position modulation (PPM) (Manchester Pulsed signal) incoherent system.

- BPPM error probability:

$$
\begin{aligned}
P_e &= \sum_{k_1=0}^{\infty} e^{-(K_s+K_b)} \frac{(K_s+K_b)^{k_1}}{k_1!} \sum_{k_2=k_1+1}^{\infty} e^{-K_b} \frac{K_b^{k_2}}{k_2!} + \frac{1}{2} \sum_{k_1=0}^{\infty} e^{-(K_s+K_b)} \frac{(K_s+K_b)^{k_1}}{k_1!} e^{-K_b} \frac{K_b^{k_1}}{k_1!} \\
&= 1 - Q\left(\sqrt{2(K_s+K_b)}, \sqrt{2(K_b)}\right) + \frac{1}{2} e^{-(K_s+2K_b)} I_0\left(2\sqrt{(K_s+K_b)K_b}\right) \\
&= e^{-(K_s+2K_b)} \left[\frac{1}{2} I_0\left(2\sqrt{(K_s+K_b)K_b}\right) + \sum_{n=1}^{\infty} \left(\frac{K_b}{K_s+K_b}\right)^{\frac{n}{2}} I_n\left(2\sqrt{(K_s+K_b)K_b}\right)\right]
\end{aligned}
$$

- Marcum's Q-function integral form: $Q(a,b) = \int_b^{\infty} x e^{-\frac{x^2+a^2}{2}} I_0(ax)dx$

Binary PPM

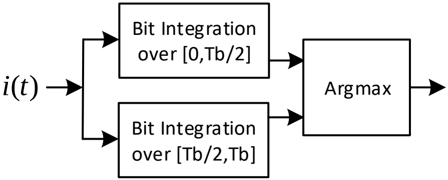

Figure: Continuous detector model for an binary pulse position modulation (PPM) (Manchester Pulsed signal) incoherent system.

- BPPM error probability:

$$P_e = e^{-(K_s+2K_b)}\left[\frac{1}{2}I_0\left(2\sqrt{(K_s+K_b)K_b}\right)+\sum_{n=1}^{\infty}\left(\frac{K_b}{K_s+K_b}\right)^{\frac{n}{2}}I_n\left(2\sqrt{(K_s+K_b)K_b}\right)\right]$$

- Marcum's Q-function series form: $Q(a,b)=\sum_{k=0}^{\infty}\frac{e^{-\frac{a^2}{2}}(\frac{a^2}{2})^k}{k!}\sum_{j=0}^{k}\frac{e^{-\frac{b^2}{2}}(\frac{b^2}{2})^j}{j!}$
- Neumann series expansions: $Q(a,b)=1-e^{-\frac{a^2+b^2}{2}}\sum_{n=1}^{\infty}(\frac{b}{a})^n I_n(ab)$

Binary PPM

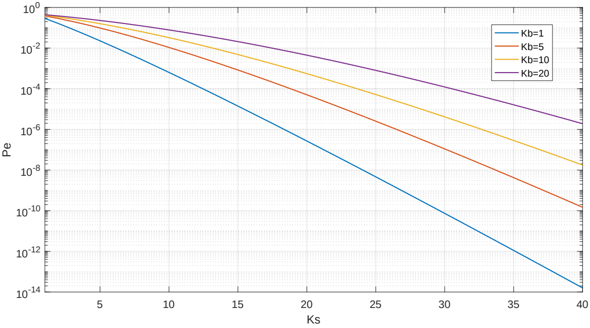

Figure: Error probability curve for binary pulse position modulation (PPM) (Manchester Pulsed signal) incoherent system.

- BPPM error probability:

$$P_e = e^{-(K_s+2K_b)}\left[\frac{1}{2}I_0\left(2\sqrt{(K_s+K_b)K_b}\right)+\sum_{n=1}^{\infty}\left(\frac{K_b}{K_s+K_b}\right)^{\frac{n}{2}}I_n\left(2\sqrt{(K_s+K_b)K_b}\right)\right]$$

Complex Examples

Laguerre Detection

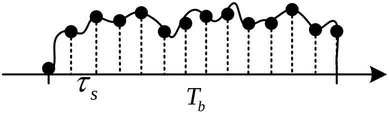

Figure: Equivalent vector representation for each received bit waveform in a binary incoherent system under Laguerre detection.

- Samples vector: $\boldsymbol{k} = (k_1, k_2, \cdots, k_{T_b/\tau_s})$
- Laguerre detection: $k_j \sim \text{Lag}(\mu_j, \mu_b, \alpha N_0 2 B_o \tau_s D_s - 1), \quad k_j \perp\!\!\!\perp k_{j'}, j \neq j'$
- Signal count energy per sample: $\mu_j = \int_{t_j - \tau_s}^{t_j} n(t) dt$
- Average signal count vector: $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_{T_b/\tau_s})$
- Background noise count energy per sample: $\mu_b = \alpha N_0$

Laguerre Detection

- Likelihood function:

$$\Lambda_i = P(\boldsymbol{k}|i) = \prod_{j=1}^{T_b/\tau_s} \frac{\mu_b^{k_j}}{(1+\mu_b)^{\alpha N_0 2 B_o \tau_s D_s + k_j}} \exp\left(-\frac{\mu_{ij}}{1+\mu_b}\right) L_{k_j}^{\alpha N_0 2 B_o \tau_s D_s - 1}\left(-\frac{\mu_{ij}}{\mu_b(1+\mu_b)}\right)$$

$$\equiv \prod_{j=1}^{T_b/\tau_s} \exp\left(-\frac{\mu_{ij}}{1+\mu_b}\right) L_{k_j}^{\alpha N_0 2 B_o \tau_s D_s - 1}\left(-\frac{\mu_{ij}}{\mu_b(1+\mu_b)}\right)$$

- Laguerre approximation:

$$L_0^c(x) = 1, \quad L_1^c(x) = c + 1 - x \Rightarrow L_0^0(x) = 1, \quad L_1^0(x) = 1 - x$$

- Approximated likelihood function:

$$\Lambda_i \approx \prod_{j=1}^{T_b/\tau_s} \exp\left(-\frac{\mu_{ij}}{1+\mu_b}\right)\left(1 + \frac{\mu_{ij}}{\mu_b(1+\mu_b)}\right)^{k_j}, \quad \tau_s \to 0, D \to 1$$

- Approximated likelihood function:

$$\ln(\Lambda_i) \equiv \sum_{j=1}^{T_b/\tau_s} \left[k_j \ln\left(1 + \frac{\mu_{ij}}{\mu_b(1+\mu_b)}\right) - \frac{\mu_{ij}}{1+\mu_b}\right] \equiv \int_0^{T_b} i(t) \ln\left(1 + \frac{n_i(t_j)}{n_b}\right) dt - \frac{K_i}{1+\mu_b}$$

Laguerre Detection

Example (Error probability of binary PPM under Laguerre detection)

If the number of modes and sampling period is low, the error probability can be approximately obtained as follow.

$$k(0, T_b/2) \gtrless k(T_b/2, T_b), \quad K_s = n_s T_b/2, \quad \mu_b = \alpha N_0, \quad D = B_o T_b D_s$$

$$P_e = \frac{1}{2} P(k(0, T_b/2) < k(T_b/2, T_b)|1) + \frac{1}{2} P(k(0, T_b/2) > k(T_b/2, T_b)|0)$$

$$+ \frac{1}{4} P(k(0, T_b/2) = k(T_b/2, T_b)|1) + \frac{1}{4} P(k(0, T_b/2) = k(T_b/2, T_b)|0)$$

$$= \sum_{k_1=0}^{\infty} \mathrm{Lag}(K_s, \mu_b, D-1)[k_1] \sum_{k_2=k_1+1}^{\infty} \mathrm{Lag}(0, \mu_b, D-1)[k_2]$$

$$+ \frac{1}{2} \sum_{k_1=0}^{\infty} \mathrm{Lag}(K_s, \mu_b, D-1)[k_1] \mathrm{Lag}(0, \mu_b, D-1)[k_1]$$

Incoherent Binary FSK

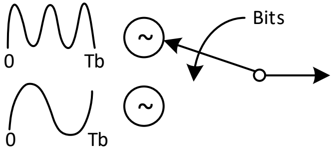

Figure: Incoherent binary frequency shift keying (FSK) transmitter.

- One-bit count intensity waveform: $n_1(t, \theta) = n_s[1 + \sin(\omega_1 t + \theta)], \quad 0 \leq t \leq T_b/2$
- Zero-bit count intensity waveform: $n_0(t, \theta) = n_s[1 + \sin(\omega_0 t + \theta)], \quad T_b/2 \leq t \leq 0$
- Phase distribution: $\theta \sim U[0, 2\pi]$

Incoherent Binary FSK

- Generalized likelihood function:

$$\Lambda_i = \mathcal{E}_\theta\{P(\boldsymbol{k}|i,\theta)\} = \frac{1}{2\pi}\int_0^{2\pi}\prod_{j=1}^{2B_nT_b}\frac{(\mu_{ij}(\theta)+\mu_b)^{k_j}}{k_j!}e^{-(\mu_{ij}(\theta)+\mu_b)}d\theta$$

$$\ln(\Lambda_i) \equiv \sum_{j=1}^{2B_nT_b}\left[k_j \ln\left(1+\frac{\mu_{ij}(\theta)}{\mu_b}\right)\right] - K_i, \quad K_1 = K_0 = n_sT_b$$

- Generalized likelihood function:

$$\Lambda_i \equiv \int_0^{2\pi}\exp\left(\int_0^{T_b} i(t)\ln\left(1+\frac{n_i(t,\theta)}{n_b}\right)dt\right)d\theta$$

- Fourier series expansion:

$$\ln\left(1+\frac{n_i(t,\theta)}{n_b}\right) = \sum_{q=1}^{\infty} a_q \sin(q\omega_i t + \Psi_q + q\theta)$$

Incoherent Binary FSK

- Generalized likelihood function:

$$\Lambda_i \equiv \int_0^{2\pi} \exp\left(\sum_{q=1}^{\infty} X_{qi}\cos(q\theta) + Y_{qi}\sin(q\theta)\right) d\theta = \int_0^{2\pi} \exp\left(\sum_{q=1}^{\infty} \epsilon_{qi}\cos(q\theta + \Phi_{qi})\right) d\theta$$

$$X_{qi} = a_q \int_0^{T_b} i(t)\cos(q\omega_i t + \Psi_q)dt, \quad Y_{qi} = a_q \int_0^{T_b} i(t)\sin(q\omega_i t + \Psi_q)dt$$

$$\epsilon_{qi}^2 = X_{qi}^2 + Y_{qi}^2, \quad \Phi_{qi} = \tan^{-1}(\frac{Y_{qi}}{X_{qi}})$$

- Approximated generalized likelihood function:

$$a_q = \begin{cases} a, & q = 1 \\ 0, & q \geq 2 \end{cases} \Rightarrow \Lambda_i \equiv \int_0^{2\pi} \exp(\epsilon_{1i}\cos(q\theta + \Phi_{qi}))d\theta = 2\pi I_0(\epsilon_{1i})$$

- Approximated ML binary decision rule:

$$\begin{aligned} \Lambda_1 &\gtrless \Lambda_0 \\ I_0(\epsilon_{11}) &\gtrless I_0(\epsilon_{10}) \\ \epsilon_{11} &\gtrless \epsilon_{10} \\ \epsilon_{11}^2 &\gtrless \epsilon_{10}^2 \\ X_{11}^2 + Y_{11}^2 &\gtrless X_{10}^2 + Y_{10}^2 \end{aligned}$$

Incoherent Binary FSK

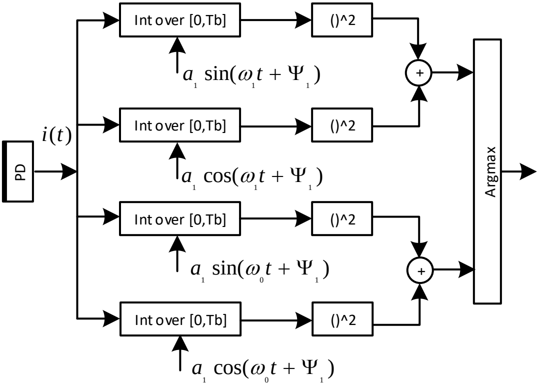

Figure: Incoherent binary frequency shift keying (FSK) receiver.

Block Coded Signaling

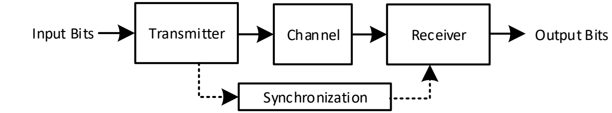

Figure: Digital communication system.

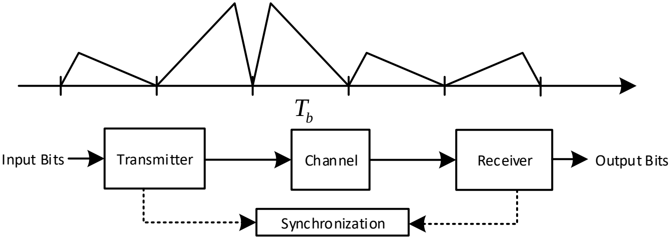

Figure: *M*ary digital communication system with the transmission bit rate $R_b = \frac{\log_2(M)}{T_b}$, where M is the number of distinguishable waveforms in each interval T_b. Each interval T_b corresponds to a word of $\log_2(M)$ transmitted bits.

Block Coded Signaling

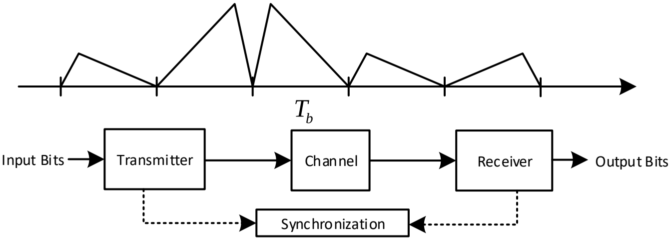

Figure: Mary communication system with the transmission bit rate $R_b = \frac{\log_2(M)}{T_b}$, where M is the number of distinguishable waveforms in each interval T_b. Each interval T_b corresponds to a word of $\log_2(M)$ transmitted bits.

- Word error probability: $P_w = \frac{1}{M}\sum_{q=1}^{M} P(e|q)$
- Union bound: $P(e|q) \leq \sum_{i=1, j\neq q}^{M} P_{q\to j} = \sum_{i=1, j\neq q}^{M} P\left[\Lambda_q \leq \Lambda_j | q\right]$
- Bit error probability: $P_e \approx \frac{M/2}{M-1} P_w$

Mary PPM

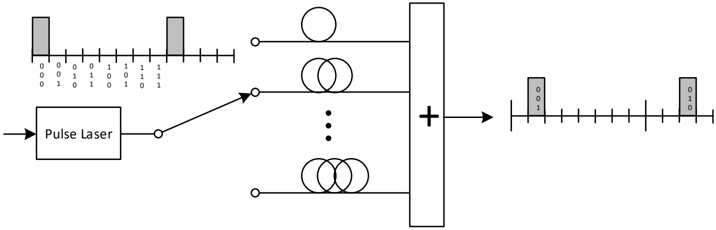

Figure: *M*ary PPM transmitter block diagram.

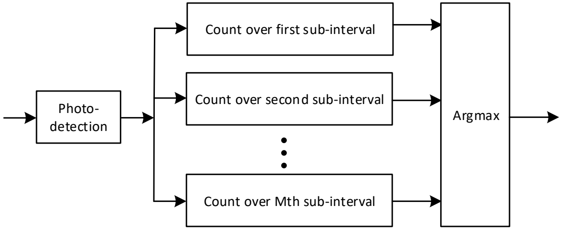

Figure: *M*ary PPM receiver block diagram.

Example (Error probability of Mary PPM under Poisson detection)

Assuming Poisson detection, the bit error probability of Mary PPM can be approximated as follows.

$$k(0,\frac{T_b}{M}) \gtrless k(j\frac{T_b}{M},(j+1)\frac{T_b}{M}), j=1,\cdots,M-1, \quad K_s = n_s\frac{T_b}{M}, \quad K_b = \alpha N_0 2B_o D_s\frac{T_b}{M}$$

$$P(e|1) = 1 - P(c|1), \quad P_w = P(e|1)$$

$$\begin{aligned}P(c|1) &= \sum_{k_1=1}^{\infty} P(k_1, k_2 < k_1, \cdots, k_M < k_1|1) + \frac{1}{M}P(k_1 = \cdots = k_M = 0|1)\\ &+ \frac{1}{2}\sum_{k_1=1}^{\infty} P(k_1, k_2 = k_1, \cdots, k_M < k_1|1) + \cdots + \frac{1}{2}\sum_{k_1=1}^{\infty} P(k_1, k_2 < k_1, \cdots, k_M = k_1|1)\\ &+ \frac{1}{3}\sum_{k_1=1}^{\infty} P(k_1, k_2 = k_1, k_3 = k_1, k_4 < k_1 \cdots, k_M < k_1|1) + \cdots +\\ &\cdots + \cdots + \cdots\\ &+ \frac{1}{M}\sum_{k_1=1}^{\infty} P(k_1, k_2 = k_1, k_3 = k_1, \cdots, k_M = k_1|1)\end{aligned}$$

Example (Error probability of Mary PPM under Poisson detection (cont.))

Assuming Poisson detection, the bit error probability of Mary PPM can be approximated as follows.

$$A = \sum_{l=1}^{k-1} \mathrm{Pos}(l, K_b), \quad B = \mathrm{Pos}(k, K_b)$$

$$\begin{aligned}
P(c|1) &= \frac{e^{-(K_s+MK_b)}}{M} + \sum_{r=0}^{M-1} \frac{1}{r+1} \binom{M-1}{r} \sum_{k=1}^{\infty} \mathrm{Pos}(k, K_s+K_b)[\mathrm{Pos}(k, K_b)]^r \left(\sum_{l=1}^{k-1} \mathrm{Pos}(l, K_b) \right)^{M-1-r} \\
&= \frac{e^{-(K_s+MK_b)}}{M} + \sum_{k=1}^{\infty} \mathrm{Pos}(k, K_s+K_b) \sum_{r=0}^{M-1} \frac{1}{r+1} \binom{M-1}{r} [\mathrm{Pos}(k, K_b)]^r \left(\sum_{l=1}^{k-1} \mathrm{Pos}(l, K_b) \right)^{M-1-r} \\
&= \frac{e^{-(K_s+MK_b)}}{M} + \sum_{k=1}^{\infty} \mathrm{Pos}(k, K_s+K_b) \sum_{r=0}^{M-1} \frac{1}{r+1} \binom{M-1}{r} B^r A^{M-1-r}
\end{aligned}$$

Mary PPM

Example (Error probability of Mary PPM under Poisson detection (cont.))

Assuming Poisson detection, the bit error probability of Mary PPM can be approximated as follows.

$$\sum_{r=0}^{M-1} \frac{1}{r+1} \binom{M-1}{r} B^r A^{M-1-r} = \frac{1}{MB} \sum_{r=0}^{M-1} \binom{M}{r+1} B^{r+1} A^{M-(r+1)} = \frac{1}{MB} \sum_{r=1}^{M} \binom{M}{1} B^{r+1} A^{M-r)}$$

$$= \frac{1}{MB}[(A+B)^M - A^M] = \frac{A^{M-1}}{Ma}[(1+a)^M - 1], \quad a = \frac{B}{A} = \frac{\text{Pos}(k, K_b)}{\sum_{l=1}^{k-1} \text{Pos}(l, K_b)} = \frac{K_b^k}{k! \sum_{l=0}^{k-1} \frac{K_b^l}{l!}}$$

$$P_w = 1 - \frac{e^{-(K_s+MK_b)}}{M} - \sum_{k=1}^{\infty} \text{Pos}(k, K_s + K_b) \left[\sum_{l=0}^{k-1} \text{Pos}(l, K_b) \right]^{M-1} \frac{1}{Ma}[(1+a)^M - 1]$$

$$P_e \approx \frac{M/2}{M-1} \left[1 - \frac{e^{-(K_s+MK_b)}}{M} - \sum_{k=1}^{\infty} \text{Pos}(k, K_s + K_b) \left[\sum_{l=0}^{k-1} \text{Pos}(l, K_b) \right]^{M-1} \frac{1}{Ma}[(1+a)^M - 1] \right]$$

The End