Optical Fields

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Overview

Wave Optics

Optical Waves

- Wave function: $u(\boldsymbol{r}, t)$
- Linear wave equation: $\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad c = \frac{c_0}{n}$
- Optical intensity: $I(\boldsymbol{r}, t) = 2\langle u^2(\boldsymbol{r}, t)\rangle$
- Optical power: $P(t) = \int_A I(\boldsymbol{r}, t) dA$
- Optical energy: $E(t) = \int P(t) dt$

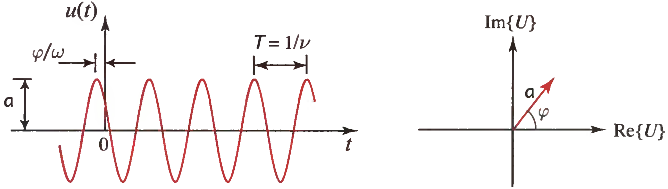


Figure: A monochromatic wave and its complex amplitude.

- Monochromatic wave function: $u(\boldsymbol{r}, t) = |U(\boldsymbol{r})| \cos(2\pi\nu t + \angle U(\boldsymbol{r}))$
- Monochromatic wave function: $u(\boldsymbol{r}, t) = \mathrm{Re}\{U(\boldsymbol{r})e^{j2\pi\nu t}\}$
- Helmholtz's wave equation: $\nabla^2 U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0, \quad k = 2\pi/\lambda = 2\pi\nu/c$
- Optical intensity: $I(\boldsymbol{r}) = |U(\boldsymbol{r})|^2$
- Optical power: $P = \int_A I(\boldsymbol{r}, t) dA$
- Optical energy: $E = \int Pdt$

Elementary Monochromatic Waves

Plane Wave

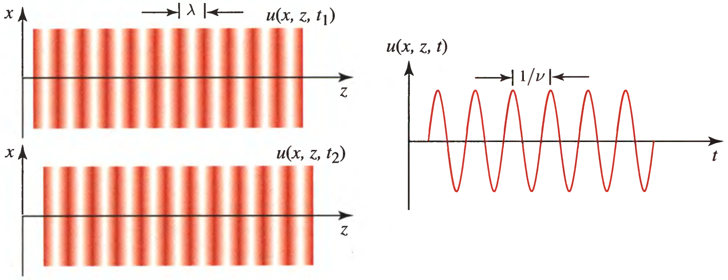


Figure: A plane wave traveling in the z direction.

- Plane wave: $U(\boldsymbol{r}) = A\exp(-j\boldsymbol{k}.\boldsymbol{r}) = A\exp[-j(k_x x + k_y y + k_z z)]$
- Wave vector: $\boldsymbol{k} = (k_x, k_y, k_z), \quad k_x^2 + k_y^2 + k_z^2 = k^2 = (\frac{2\pi\nu}{c})^2$
- Optical intensity: $I(\boldsymbol{r}) = |A|^2$
- z-propagated plane wave: $U(\boldsymbol{r}) = A\exp(-jkz)$

Spherical Wave

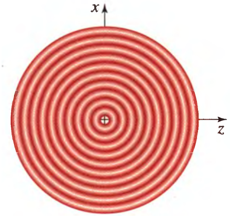
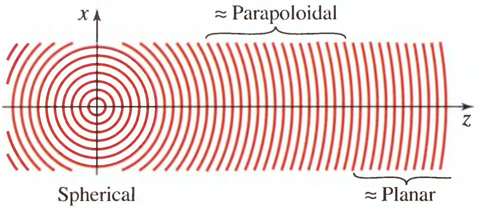


Figure: A spherical wave originating at $\boldsymbol{r}_0 = 0$.

- Spherical wave: $U(\boldsymbol{r}) = \frac{A}{|\boldsymbol{r}-\boldsymbol{r}_0|}\exp(-jk|\boldsymbol{r}-\boldsymbol{r}_0|)$
- Wave number: $k = \frac{2\pi\nu}{c}$
- Optical intensity: $I(\boldsymbol{r}) = \frac{A^2}{|\boldsymbol{r}-\boldsymbol{r}_0|^2}$
- 0-originated spherical wave: $U(\boldsymbol{r}) = \frac{A}{r}\exp(-jkr)$

Example (Fresnel approximation for spherical wave)

If $z \gg \sqrt{x^2+y^2}$, a spherical wave originated at 0 can be approximated by a paraboloidal wave.



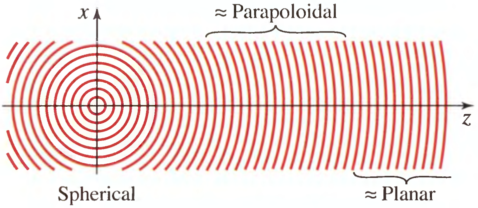
$$r = \sqrt{x^2+y^2+z^2} = z\sqrt{1+\frac{x^2+y^2}{z^2}} \approx z + \frac{x^2+y^2}{2z}, \quad z \gg \sqrt{x^2+y^2}$$

$$U(\boldsymbol{r}) = \frac{A}{r}\exp(-jkr) \approx \frac{A}{z}\exp(-jkz)\exp\left(-jk\frac{x^2+y^2}{2z}\right)$$

Spherical Wave

Example (Fraunhofer approximation for spherical wave)

If $z \ggg \sqrt{x^2 + y^2}$, a spherical wave originated at 0 can be approximated by a plane wave.



$$r = \sqrt{x^2 + y^2 + z^2} = z\sqrt{1 + \frac{x^2 + y^2}{z^2}} \approx z, \quad z \ggg \sqrt{x^2 + y^2}$$

$$U(\boldsymbol{r}) = \frac{A}{r} \exp(-jkr) \approx \frac{A}{z} \exp(-jkz)$$

Paraxial Wave

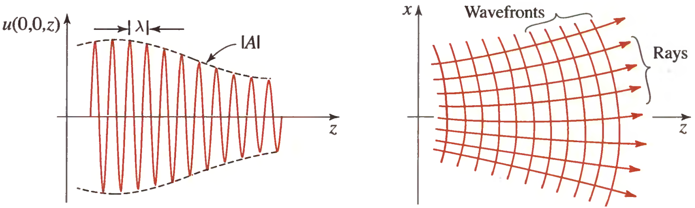


Figure: Wave function and wave front of paraxial wave.

- Paraxial wave: $U(\boldsymbol{r}) = A(\boldsymbol{r})\exp(-jkz)$
- Paraxial condition: $|\frac{\partial A}{\partial z}|\lambda \ll |A|$
- Paraxial Helmholtz's equation: $\nabla_T^2 A - j2k\frac{\partial A}{\partial z} = 0, \quad \nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Gaussian Beam

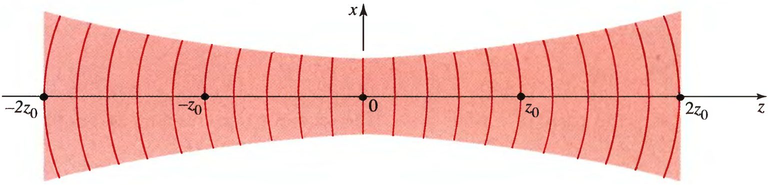
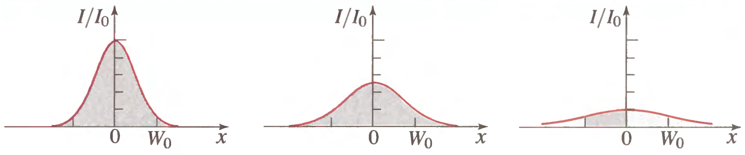


Figure: Wave front of a Gaussian bBeam.

- Gaussian beam: $U(\boldsymbol{r}) = A_0 \frac{W_0}{W(z)} \exp\left[-\frac{\rho^2}{W^2(z)}\right] \exp\left[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right]$
- Radial distance: $\rho = \sqrt{x^2 + y^2}$
- Waist radius: $W_0 = \sqrt{\frac{\lambda z_0}{\pi}}$
- Beam bandwidth: $W(z) = W_0\sqrt{1 + (\frac{z}{z_0})^2}$
- Radius of curvature : $R(z) = z[1 + (\frac{z_0}{z})^2]$
- Gouy phase: $\zeta(z) = \tan^{-1}(\frac{z}{z_0})$

Example (Power of Gaussian beam)

The total power carried by the Gaussian beam is $0.5 I_0 \pi W_0^2$.

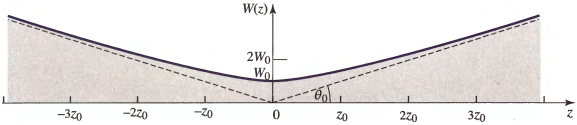


$$I(\rho, z) = I_0 \frac{W_0^2}{W^2(z)} \exp\left[-\frac{2\rho^2}{W^2(z)}\right], \quad I_0 = |A_0|^2$$

$$P_\infty = \int_0^\infty I(\rho, z) 2\pi\rho d\rho = 0.5 I_0 \pi W_0^2 \int_0^\infty e^{-u} du = 0.5 I_0 \pi W_0^2, \quad u = \frac{2\rho^2}{W^2(z)}$$

Example (Bandwdith of Gaussian beam)

The traverse circle with the radius $W(z)$, which is called beam bandwidth, collects 86% of the total power of the Gaussian beam. The bandwidth diverges as a cone of half-angle $\theta_0 = \lambda/(\pi W_0)$.



$$P_{W(z)} = \int_0^{W(z)} I(\rho, z) 2\pi\rho d\rho = 0.5 I_0 \pi W_0^2 \int_0^2 e^{-u} du = (1 - e^{-2}) P_\infty, \quad u = \frac{2\rho^2}{W^2(z)}$$

$$z \to \infty \Rightarrow W(z) = W_0 \sqrt{1 + (\frac{z}{z_0})^2} \approx \frac{W_0}{z_0} z \Rightarrow \theta_0 \approx \frac{\frac{W_0}{z_0} z}{z} = \frac{\lambda}{\pi W_0}$$

Gaussian Beam

Example (Diameter of Gaussian beam)

The traverse circle with the radius $d(z)/2$, where the beam diameter $d(z) = d_0\sqrt{1+(\lambda z/d_0^2)^2}$, $d_0 = 2W_0$, collects 71% of the total power of the Gaussian beam. The diameter diverges as a cone of angle $\theta_b = \lambda/d_0$.

$$z \to \infty \Rightarrow \frac{d^2(z)}{2W^2(z)} \approx \frac{(\lambda z/d_0)^2}{2(W_0 z/z_0)^2} = \frac{(\lambda z_0)^2}{8W_0^4} = \frac{\pi^2}{8}$$

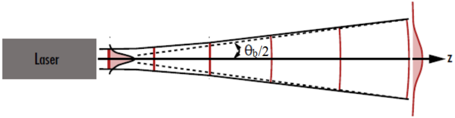
$$P_{0.5d(z)} = \int_0^{0.5d(z)} I(\rho,z)2\pi\rho d\rho = 0.5I_0\pi W_0^2 \int_0^{\frac{d^2(z)}{2W^2(z)}} e^{-u}du \approx (1-e^{-\pi^2/8})P_\infty, \quad u = \frac{2\rho^2}{W^2(z)}$$

$$z \to \infty \Rightarrow d(z) \approx \frac{\lambda z}{d_0} \Rightarrow \theta_b \approx \frac{\frac{\lambda z}{d_0}}{z} = \frac{\lambda}{d_0}$$

Gaussian Beam

Example (Diffraction-limited transmitter beam solid angle)

The power of Gaussian beam is mostly confined to the solid angle $\Omega_b \approx \pi\theta_b^2/4$.



$$\Omega_b = \int_{\theta=0}^{\theta_b/2} \int_{\phi=0}^{2\pi} \sin(\theta) d\phi d\theta = 2\pi[1 - \cos(\theta_b/2)] \approx \pi\theta_b^2/4$$

Example (Transmitter antenna gain)

An optical transmitter with Gaussian beam has the antenna gain $G_t \approx (4d_0/\lambda)^2$.

$$G_t = \frac{4\pi}{\Omega_b} \approx \left(\frac{4d_0}{\lambda}\right)^2$$

Temporal Superposition

Temporal Superposition

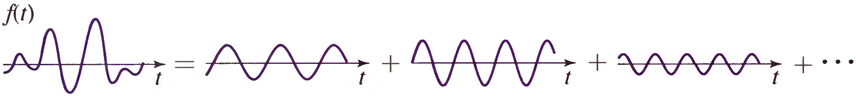


Figure: An arbitrary function $f(t)$ may be analyzed as a sum of temporal harmonic functions of different frequencies and complex amplitudes.

- Fourier transform: $F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-j2\pi\nu t) dt$
- Inverse Fourier transform: $f(t) = \int_{-\infty}^{\infty} F(\nu) \exp(j2\pi\nu t) d\nu$
- Polychromatic optical wave: Superposition of monochromatic optical waves

Polychromatic Optical Waves

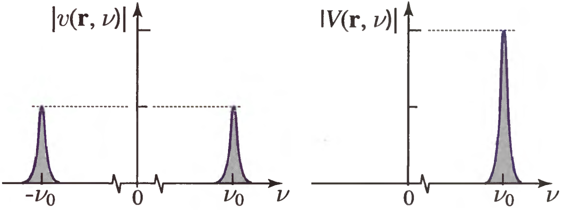


Figure: Fourier transform of a polychromatic wave function and its corresponding complex wave function.

- Fourier analysis: $u(\boldsymbol{r},t) = \int_{-\infty}^{\infty} v(\boldsymbol{r},\nu)\exp(j2\pi\nu t)d\nu$
- Complex wave function: $U(\boldsymbol{r},t) = 2\int_{0}^{\infty} v(\boldsymbol{r},\nu)\exp(j2\pi\nu t)d\nu$
- Real wave function: $u(\boldsymbol{r},t) = \mathrm{Re}\{U(\boldsymbol{r},t)\}, \quad u(\boldsymbol{r},t) \in \mathbb{R}$
- Wave equation: $\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0, \quad c = \frac{c_0}{n}$
- Optical intensity: $I(\boldsymbol{r},t) = \frac{\langle U^2(\boldsymbol{r},t)\rangle}{2} + \frac{\langle {U^*}^2(\boldsymbol{r},t)\rangle}{2} + \langle U(\boldsymbol{r},t)U^*(\boldsymbol{r},t)\rangle$

Quasi-monochromatic Optical Waves

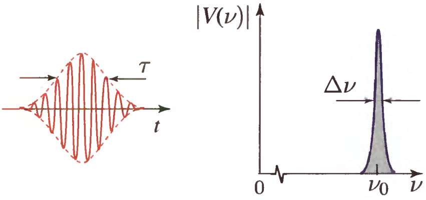


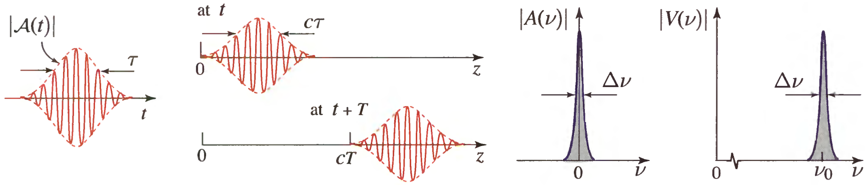
Figure: Fourier transform of the complex wave function of a quasi-monochromatic wave function.

- Quasi-monochromatic condition: $\Delta\nu \ll \nu_0$
- Temporal and spectral width: $\tau\Delta\nu \geq \frac{1}{4\pi}$
- Optical intensity: $I(\boldsymbol{r}, t) = \langle U(\boldsymbol{r}, t)U^*(\boldsymbol{r}, t)\rangle = |U(\boldsymbol{r}, t)|^2$

Quasi-monochromatic Optical Waves

Example (Pulsed plane wave)

The pulse plane wave $u(\boldsymbol{r}, t) = \mathcal{A}(t - z/c)\exp[j2\pi\nu_0(t - z/c)]$, where $\mathcal{A}(t)$ is a slowly time-varying function with the temporal width τ, can represent a wavepacket of fixed extent $c\tau$ traveling in z-direction. For $\tau = 1$ ps, then $\Delta\nu \gtrsim 80$ GHz and the wave is quasi-monochromatic at $\nu_0 = 500$ THz.



Spatial Superposition

Spatial Superposition

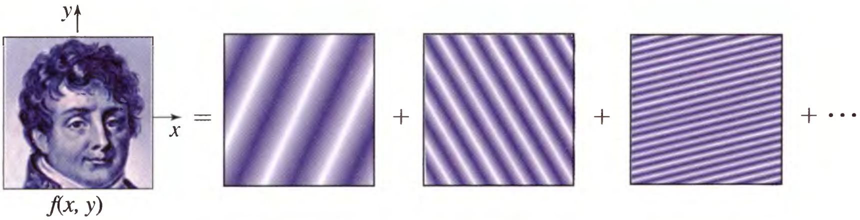


Figure: An arbitrary function $f(x, y)$ may be analyzed as a sum of spatial harmonic functions of different spatial frequencies and complex amplitudes.

- Fourier transform: $F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j2\pi(ux+vy)}dxdy$
- Inverse Fourier transform: $f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{-j2\pi(ux+vy)}dudv$
- Monochromatic complex wave function: Superposition of spatial harmonics (plane waves)

Spatial Superposition

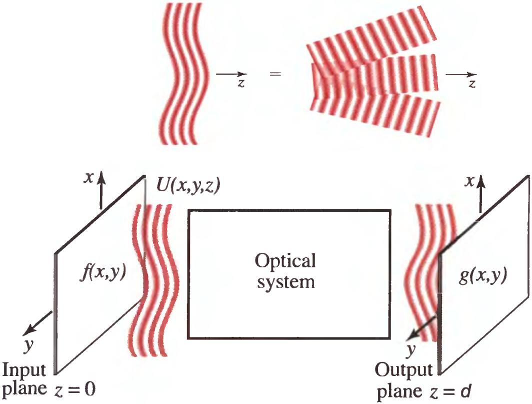


Figure: An arbitrary wave in free space can be analyzed as a superposition of plane waves. The transmission of an arbitrary wave $U(x, y, z)$ through an optical system between an input plane $z = 0$ and an output plane $z = d$. This is regarded as a linear system whose input and output are the functions of $f(x, y) = U(x, y, 0)$ and $g(x, y) = U(x, y, d)$, respectively.

Spatial Harmonic

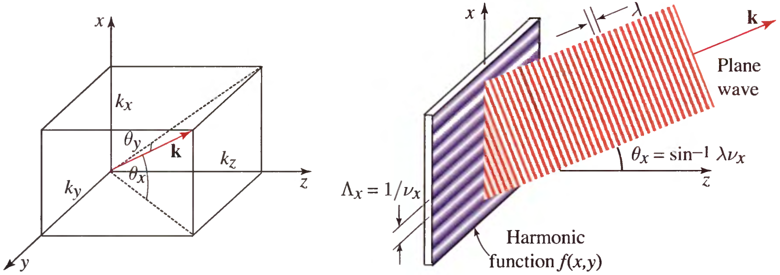


Figure: A harmonic function of spatial frequencies ν_x and ν_y at the plane $z = 0$ is consistent with a plane wave traveling at angles $\theta_x = \sin^{-1}(\lambda\nu_x)$ and $\theta_y = \sin^{-1}(\lambda\nu_y)$.

- Plane wave complex function: $U(x, y, z) = A\exp[-j(k_x x + k_y y + k_z z)]$
- Wave vector angles: $\sin(\theta_x) = k_x/k = \lambda\nu_x, \sin(\theta_y) = k_y/k = \lambda\nu_y$
- Spatial frequencies: $\nu_x = 1/\Lambda_x = k_x/(2\pi), \quad \nu_y = 1/\Lambda_y = k_y/(2\pi)$
- Spatial harmonic function: $f(x, y) = U(x, y, 0) = A\exp[-j2\pi(\nu_x x + \nu_y y)]$
- Plane wave complex function: $U(x, y, z) = f(x, y)\exp\left(-jz\sqrt{k^2 - k_x^2 - k_y^2}\right)$

Spatial Harmonic

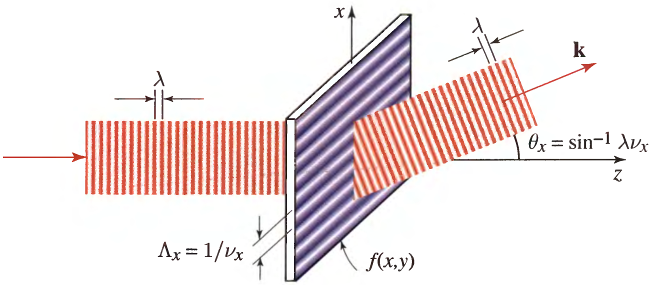


Figure: A thin element whose complex amplitude transmittance is a harmonic function of spatial frequency ν_x.

- Incident plane wave complex function: $1\exp(-jkz)$
- Spatial harmonic function: $f(x,y) = \exp[-j2\pi(\nu_x x + \nu_y y)]$
- Plane wave complex function: $U(x,y,0) = f(x,y)$
- Plane wave complex function: $U(x,y,z) = f(x,y)e^{-j2\pi z\sqrt{\lambda^{-2}-\nu_x^2-\nu_y^2}}$
- Propagation condition: $k_z > 0 \equiv \nu_x^2 + \nu_y^2 < \lambda^{-2}$

Spatial Spectral analysis

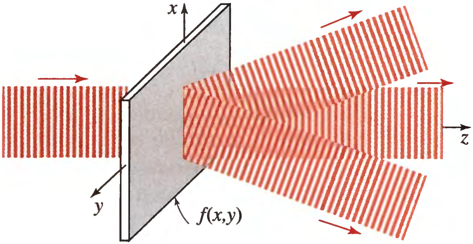


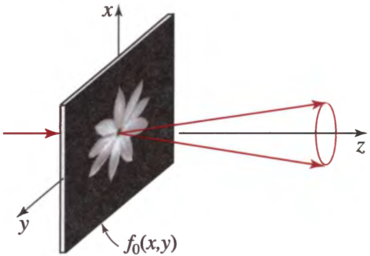
Figure: A thin optical element of amplitude transmittance $f(x, y)$ decomposes an incident plane wave into many plane waves.

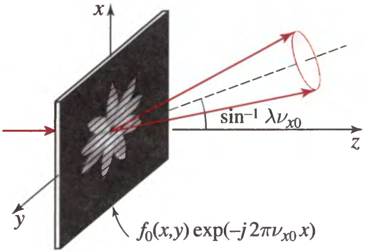
- Optical element transmittance: $f(x, y) = \int\int F(\nu_x, \nu_y) e^{-j2\pi(\nu_x x + \nu_y y)} d\nu_x d\nu_y$
- Transmitted wave: $U(x, y, z) = \int\int F(\nu_x, \nu_y) e^{-j2\pi(\nu_x x + \nu_y y)} e^{-jk_z z} d\nu_x d\nu_y$
- Propagation condition: $k_z > 0 \equiv \nu_x^2 + \nu_y^2 < \lambda^{-2}$

Spatial Spectral analysis

Example (Amplitude modulation and spatial-frequency multiplexing)

A slowly-varying image $f_0(x, y)$ with $\Delta\nu_x \ll \nu_{x_0}$ $\Delta\nu_y \ll \nu_{y_0}$ can be carried on a carrier harmonic function as $f_0(x, y)\exp[-j2\pi(\nu_{x_0}x + \nu_{y_0}y)]$, where the harmonic carries the image on a spatial direction with the angles $\theta_{x_0} = \sin^{-1}(\lambda\nu_{x_0})$ and $\theta_{y_0} = \sin^{-1}(\lambda\nu_{y_0})$. Two images can be spatially multiplexed as $f_1(x, y)\exp[-j2\pi(\nu_{x_1}x + \nu_{y_1}y)] + f_2(x, y)\exp[-j2\pi(\nu_{x_2}x + \nu_{y_2}y)]$.

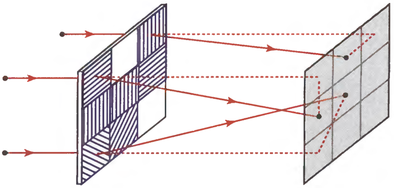


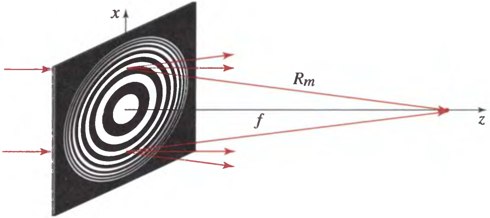


Spatial Spectral analysis

Example (Frequency modulation)

As a frequency-modulating transparency, Fresnel Zone plate with complex amplitude transmittance $f(x,y) = u\left[\cos\left(\pi\frac{x^2+y^2}{\lambda f}\right)\right]$ serves as a spherical lens with multiple focal lengths at $f, f/2, f/3, \cdots$.





$$f(x,y) = \exp[-j2\pi\phi(x,y)] \Rightarrow (\theta_x, \theta_y) = (\sin^{-1}(\lambda\partial\phi/\partial x), (\sin^{-1}(\lambda\partial\phi/\partial y))$$

Free Space Propagation

Free Space Propagation

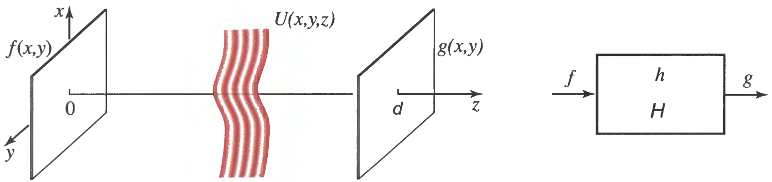


Figure: Propagation of light between two planes.

- Complex envelope of plane-wave components:
  $F(\nu_x, \nu_y) = \int\int f(x, y) e^{j2\pi(\nu_x x + \nu_y y)} dxdy$
- Huygen's diffraction pattern:
  $g(x, y) = \int\int F(\nu_x, \nu_y) H(\nu_x, \nu_y) e^{-j2\pi(\nu_x x + \nu_y y)} d\nu_x d\nu_y$

Huygens's free space transfer functions

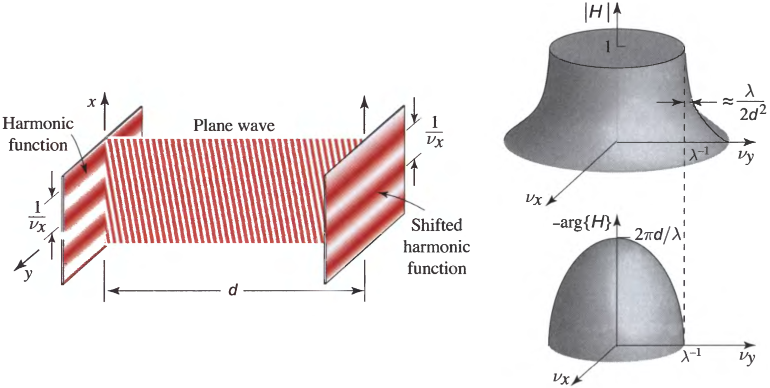
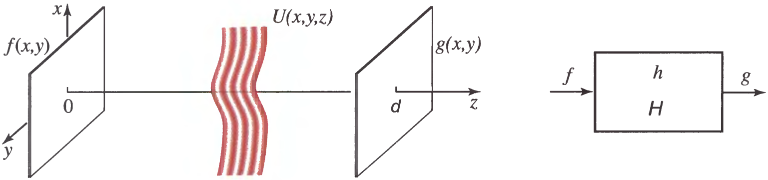


Figure: Free space transfer functions.

- Plane wave complex function: $U(x, y, d) = e^{-j2\pi(\nu_x x + \nu_y y)} e^{-j2\pi d\sqrt{\lambda^{-2} - \nu_x^2 - \nu_y^2}}$
- Free space transfer function: $H(\nu_x, \nu_y) = e^{-j2\pi d\sqrt{\lambda^{-2} - \nu_x^2 - \nu_y^2}}$
- Light propagation spatial bandwidth: λ^{-1} cycles/mm

Huygens's Diffraction Integral



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  $g(x, y) = \int\int F(\nu_x, \nu_y) e^{-j2\pi d\sqrt{\lambda^{-2} - \nu_x^2 - \nu_y^2}} e^{-j2\pi(\nu_x x + \nu_y y)} d\nu_x d\nu_y$

Fresnel's free space transfer functions

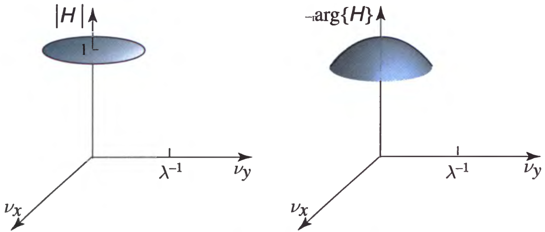
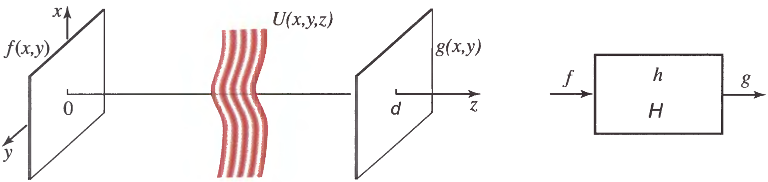


Figure: The transfer function of free-space propagation for low spatial frequencies.

- Free space transfer function: $H(\nu_x, \nu_y) \approx H_0 e^{j\pi\lambda d(\nu_x^2+\nu_y^2)}, \quad H_0 = e^{-jkd}$
- Fresnel's condition: $\nu_x^2 + \nu_y^2 \ll \lambda^{-2}$
- Free space impulse response: $h(x, y) \approx h_0 e^{-jk\frac{x^2+y^2}{2d}}, \quad h_0 = (j/\lambda d)e^{-jkd}$

Fresnel's Diffraction Integral



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  $F(\nu_x, \nu_y) = \int\int f(x,y) e^{j2\pi(\nu_x x + \nu_y y)} dxdy$
- Fresnel's diffraction pattern:
  $g(x,y) \approx H_0 \int\int F(\nu_x, \nu_y) e^{j\pi\lambda d(\nu_x^2 + \nu_y^2)} e^{-j2\pi(\nu_x x + \nu_y y)} d\nu_x d\nu_y$
- Fresnel's diffraction pattern:
  $g(x,y) \approx h_0 \int\int f(u,v) e^{-jk[(x-u)^2 + (y-v)^2]/(2d)} dudv$

Fraunhofer's Diffraction Integral

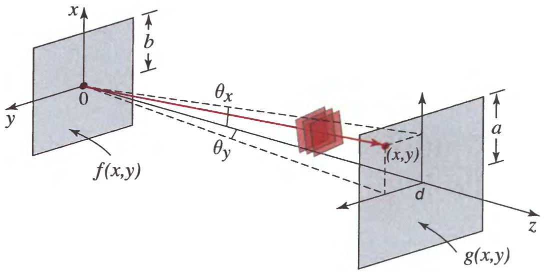


Figure: When the distance d is sufficiently long, the complex amplitude at point (x, y) in the $z = d$ plane is proportional to the complex amplitude of the plane-wave component with angles $\theta_x \approx x/d \approx \lambda\nu_x$ and $\theta_y \approx y/d \approx \lambda\nu_y$, i.e., to the Fourier transform $F(\nu_x, \nu_y)$ of $f(x, y)$, with $\nu_x = x/(\lambda d)$ and $\nu_y = y/(\lambda d)$.

- Fraunhofer's condition: $\nu_x^2 + \nu_y^2 \ll \lambda^{-2}, x^2 + y^2 \ll \lambda d, u^2 + v^2 \ll \lambda d$
- Fraunhofer's diffraction pattern:
  $g(x, y) \approx h_0 \int\int f(u, v)e^{jk(ux+vy)/d} dudv = h_0 F(\frac{x}{\lambda d}, \frac{y}{\lambda d})$

Diffraction Pattern

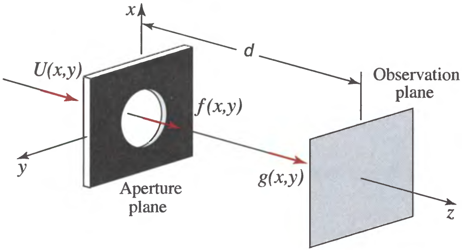


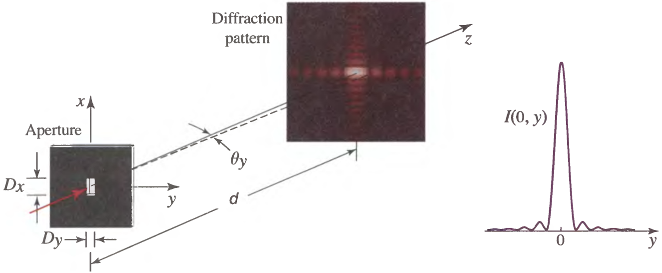
Figure: Fraunhofer fiffraction pattern of an aperture with the transmittance amplitude $p(x, y)$ illuminated by the plane wave $\sqrt{I_i}\exp\{-jkz\}$.

- Aperture transmittance: $p(x, y) = 1$ inside the aperture and 0 otherwise
- Incident wave complex amplitude: $U(x, y)$
- Transmit wave complex amplitude: $f(x, y) = U(x, y)p(x, y)$
- Receive wave complex amplitude: $g(x, y)$
- Receive diffraction pattern: $I(x, y) = |g(x, y)|^2$
- Fraunhofer diffraction pattern: $I(x, y) = \frac{I_i}{(\lambda d)^2}|P(\frac{x}{\lambda d}, \frac{y}{\lambda d})|^2$

Diffraction Pattern

Example (Fraunhofer diffraction from a rectangular aperture)

The Fraunhofer diffraction pattern from a rectangular aperture, of height D_y and width D_x observed at a distance d is $I(x,y) = I_i(D_xD_y/(\lambda d))^2\text{Sinc}^2(D_xx/\lambda d)\text{Sinc}^2(D_yy/\lambda d)$. The central lobe of the pattern has half-angular widths $\theta_x = \lambda/D_x$ and $\theta_y = \lambda/D_y$.

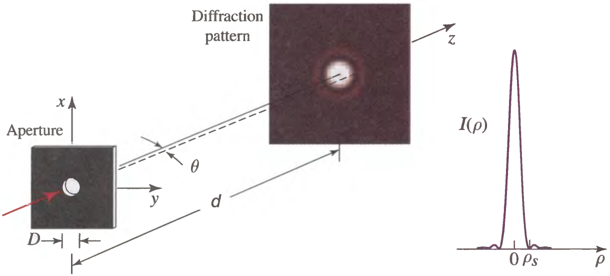


$$I(x,y) = \frac{I_i}{(\lambda d)^2}|P(\frac{x}{\lambda d}, \frac{y}{\lambda d})|^2 = \frac{I_i}{(\lambda d)^2}\left|\int_{-D_x/2}^{D_x/2}\int_{-D_y/2}^{D_y/2} e^{jk(ux+vy)/d}dudv\right|^2$$

Diffraction Pattern

Example (Fraunhofer diffraction from a circular aperture)

The Fraunhofer diffraction pattern from a circular aperture of diameter D observed at a distance d is $I(x,y) = I_i(\pi D^2/(4\lambda d))^2 \mathrm{Jinc}^2(D\rho/(\lambda d))$, where $\rho = \sqrt{x^2+y^2}$ and $\mathrm{Jinc}(x) = 2J_1(\pi x)/(\pi x)$. The central lobe of the pattern has half-angular widths $\theta = 1.22\lambda/D$.

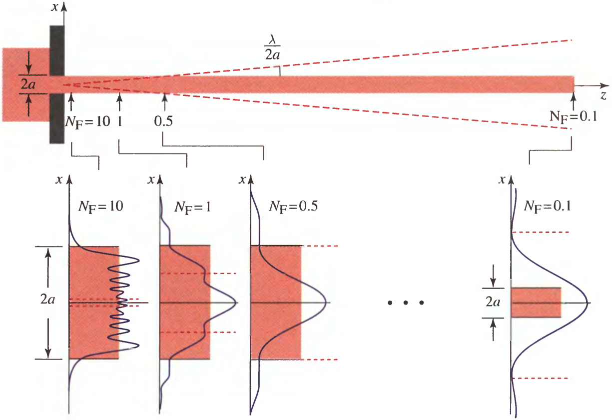


$$I(\rho,\phi) = \frac{I_i}{(\lambda d)^2}\left|P\left(\frac{x}{\lambda d},\frac{y}{\lambda d}\right)\right|^2 = \frac{I_i}{(\lambda d)^2}\left|\int_0^{D/2}\int_0^{2\pi} e^{jk\rho r\frac{\cos(\theta-\phi)}{d}} r dr d\theta\right|^2$$

Diffraction Pattern

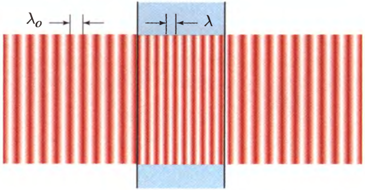
Example (Fresnel diffraction from a circular aperture)

The Fresnel diffraction pattern from a circular aperture depends on Fresnel integrals and approaches the corresponding Fraunhofer pattern as d increases.



Beam Focusing Using Lenses

Thin Transparent Plate Transmittance Function



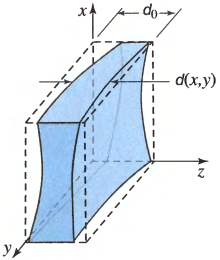


Figure: Transmission of a paraxial wave through a thin transparent plate.

- Incident plane wave: $U(x, y, 0) = A$
- Transmitted plane wave: $U(x, y, d) = A\exp(-jnkd)$
- Plane wave transmittance: $t(x, y) = U(x, y, d)/U(x, y, 0) = \exp(-jnkd)$
- Paraxial wave transmittance: $t(x, y) \approx e^{-jnkd(x,y)}e^{-jk(d_0-d(x,y))}$
- Paraxial wave transmittance: $t(x, y) \approx h_p e^{-j(n-1)kd(x,y)}, \quad h_p = e^{-jkd_0}$

Lens Transmittance Function

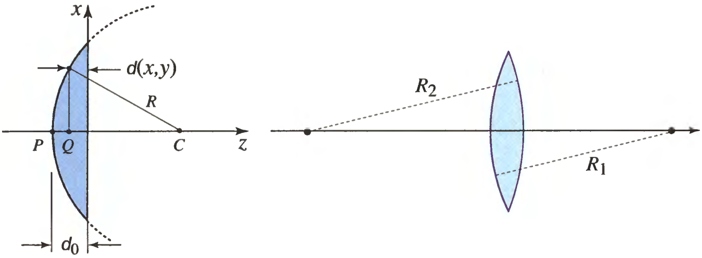


Figure: Planoconvex and double-convex lenses. By definition, $R_1 \geq 0$ and $R_2 \leq 0$.

- Planoconvex lens
  - Thickness function: $d(x,y) = d_0 - [R - \sqrt{R^2 - (x^2 + y^2)}]$
  - Paraxial thickness function: $d(x,y) \approx d_0 - [R - R(1 - \frac{x^2+y^2}{2R^2})] = d_0 - \frac{x^2+y^2}{2R}$
  - Lens focal length: $f = R/(n-1)$
  - Thin lens transmittance: $t(x,y) \approx h_l \exp\left[jk\frac{x^2+y^2}{2f}\right], \quad h_l = \exp(-jnkd_0)$
- Double-convex lens
  - Lens focal length: $f = R_1 R_2/[(n-1)(R_2 - R_1)$
  - Thin lens transmittance: $t(x,y) \approx h_l \exp\left[jk\frac{x^2+y^2}{2f}\right], \quad h_l = \exp(-jnkd_0)$

Lens Focusing Property

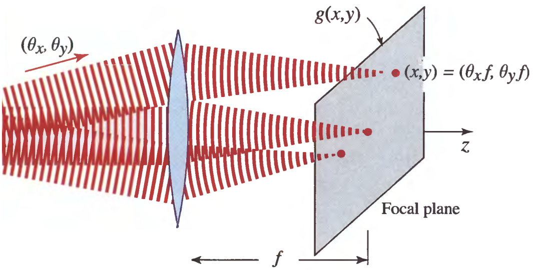


Figure: Lens focusing property at $d = f$.

- Fresnel's approximation: $g(x,y) \approx h_0 \int\int f(u,v)t(u,v)e^{-jk\frac{(x-u)^2+(y-u)^2}{2d}}dudv$
- Fresnel's approximation:
  $g(x,y) \approx h_0 h_l \int\int f(u,v)e^{jk\frac{u^2+v^2}{2f}}e^{-jk\frac{(x-u)^2+(y-v)^2}{2d}}dudv$
- Focal plane: $g(x,y) \approx h_0 h_l e^{-jk\frac{x^2+y^2}{2f}} \int\int f(u,v)e^{jk\frac{ux+yv}{f}}dudv$
- Focusing property: $g(x,y) \approx h_0 h_l e^{-jk\frac{x^2+y^2}{2f}} F(\frac{x}{\lambda f}, \frac{y}{\lambda f}) \propto \frac{1}{\lambda f} F(\frac{x}{\lambda f}, \frac{y}{\lambda f})$

Lens Airy Pattern

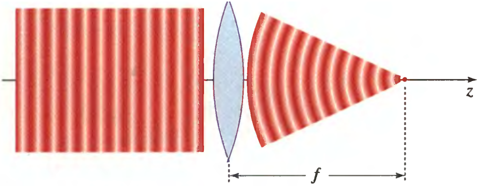


Figure: Airy pattern for an infinite lens for the plane wave traveling towards z-direction.

- Airy pattern: $f_{d_o}(x, y) \propto \lambda f \delta(x, y)$

Lens Airy Pattern

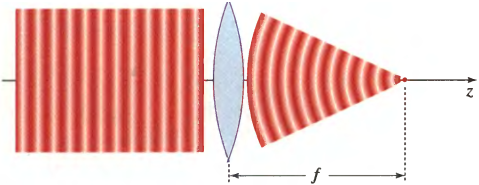


Figure: Airy pattern for a rectangular aperture lens of height D_y and width D_x for the plane wave traveling towards z-direction.

- Airy pattern: $f_{d_o}(x,y) \propto \frac{D_x D_y}{\lambda f}\text{Sinc}(\frac{D_x x}{\lambda f})\text{Sinc}(\frac{D_y y}{\lambda f})$

Lens Airy Pattern

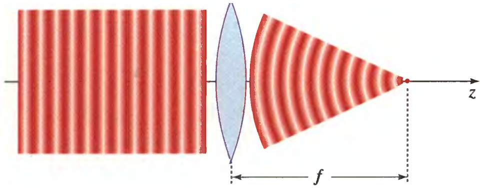


Figure: Airy pattern for a circular lens of diameter D the plane wave traveling towards z-direction.

- Airy pattern: $f_{d_o}(x,y) \propto \frac{\pi D^2}{4\lambda f}\mathrm{Jinc}(\frac{D\sqrt{x^2+y^2}}{\lambda f})$

Lens Focusing Pattern

Example (Focused pattern of a lens illuminated by an arbitrary wave)

If an arbitrary field $f_s(x, y)$ is received at a lens with the pupil function $w(x, y)$, the focused field at the focal plane is $g(x, y) = F_s(x, y) \otimes f_{d_o}(x, y)$, where $F_s(x, y)$ is the Fourier transform of $f_s(x, y)$, $f_{d_o}(x, y)$ is the airy pattern of the lens, and $\otimes$ is the spatial convolution operator.

$$f_1(x, y) \otimes f_2(x, y) = \int\int f_1(u, v) f_1(x - u, y - v) dudv$$

$$f_1(x, y) \otimes f_2(x, y) \Leftrightarrow F_1(\nu_x, \nu_y) f_2(\nu_x, \nu_y)$$

$$f_1(x, y) f_2(x, y) \Leftrightarrow F_1(\nu_x, \nu_y) \otimes f_2(\nu_x, \nu_y)$$

$$f(x, y) = f_s(x, y) w(x, y) \Rightarrow g(x, y) \propto F(x, y) \otimes f_{d_o}(x, y)$$

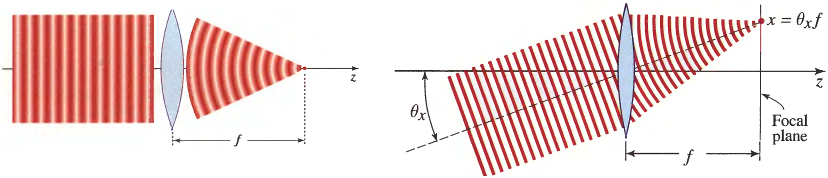


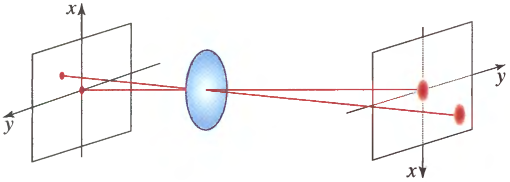
Figure: Focused pattern for a circular lens of diameter D for the plane wave traveling through the angles (θ_x, θ_y).

- Plane wave propagating at (θ_x, θ_y): $g(x, y) \propto \frac{\pi D^2}{4\lambda f} \mathrm{Jinc}\left(\frac{D\sqrt{(x-f\theta_x)^2+(y-f\theta_y)^2}}{\lambda f}\right)$
- Central lobe radius: $\rho_0 = 1.22\frac{f\lambda}{D} \approx \frac{f\lambda}{D}$
- Diffraction-limited angle: $\theta_{dl} = 1.22\frac{\lambda}{D} \approx \frac{\lambda}{D}$
- Lens f-number: $\frac{f}{D} \approx 1 \Rightarrow \theta_{dl} \approx \frac{\lambda}{f}$
- Diffraction-limited solid angle: $\Omega_{dl} \approx \frac{\pi}{4}\theta_{dl}^2 = (\frac{\pi}{4})^2\frac{\lambda^2}{A}$

Diffraction-limited Angle

Example (Spatial degree of freedom)

An optical detector of area A_d can detect around $4A_d/(\pi\lambda^2)$ separable spatial modes. When $A_d = 1$ cm^2 and $\lambda = 1$ μm, around 10^8 separable spatial modes exist.



$$\frac{\Omega_{fv}}{\Omega_{dl}} = \frac{A_d/f^2}{\pi\lambda^2/(4f^2)} = \frac{4A_d}{\pi\lambda^2}$$

Detected Power

Example (Equality of power at aperture and focal planes)

The amount of detected power at the aperture and focal planes are equal.

$$P_0 = \int\int |f(u,v)|^2 dudv = \int\int |F(\nu_x,\nu_y)|^2 d\nu_x d\nu_y$$

$$P_f = \int\int |\frac{1}{\lambda f} F(\frac{x}{\lambda f}, \frac{y}{\lambda f})|^2 dxdy = \int\int |F(\nu_x,\nu_y)|^2 d\nu_x d\nu_y$$

Orthogonal Decomposition

Orthogonal Decomposition

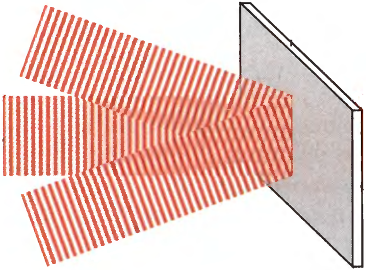


Figure: Orthogonal decomposition of optical field of a finite area.

- Quasi-monochromatic complex wave function: $U(\boldsymbol{r},t) = a(\boldsymbol{r},t)e^{j2\pi\nu t}, \boldsymbol{r} \in A$
- Spatial decomposition: $a(\boldsymbol{r},t) = \sum_{i=1}^{\infty} a_i(t)\Phi_i(\boldsymbol{r})$
- Orthogonality condition: $\int_A \Phi_i(\boldsymbol{r})\Phi_j^*(\boldsymbol{r})dA = 0, \quad i \neq j$
- Decomposition coefficients: $a_i(t) = \frac{\int\int a(\boldsymbol{r},t)\Phi_i^*(\boldsymbol{r})dA}{\int_A \left|\Phi_i(\boldsymbol{r})\right|^2 dA}$

Optical Field Decomposition

Example (Optical field decomposition at the aperture plane)

Plane waves with wavelength λ arriving to a squared aperture area A having arrival angles separated by multiples of $\lambda/\sqrt{A}$ can be used for optical field decomposition over the aperture plane.

$$\Phi_i(\boldsymbol{r}) = e^{-j\boldsymbol{k}_i.\boldsymbol{r}}$$

$$\Rightarrow \int_A \Phi_i(\boldsymbol{r})\Phi_j^*(\boldsymbol{r})dA = \int_{-\sqrt{A}/2}^{+\sqrt{A}/2}\int_{-\sqrt{A}/2}^{+\sqrt{A}/2} e^{-j[x(k_{x_i}-k_{x_j})+y(k_{y_i}-k_{y_j})+0]}dxdy$$

$$= A\frac{\sin\left((k_{x_i}-k_{x_j})\frac{\sqrt{A}}{2}\right)}{(k_{x_i}-k_{x_j})\frac{\sqrt{A}}{2}}\frac{\sin\left((k_{y_i}-k_{y_j})\frac{\sqrt{A}}{2}\right)}{(k_{y_i}-k_{y_j})\frac{\sqrt{A}}{2}}$$

$$= \begin{cases} A, & k_{x_i}-k_{x_j}=0 \text{ and } k_{y_i}-k_{y_j}=0 \\ 0, & k_{x_i}-k_{x_j}=n_1(2\pi/\sqrt{A}) \text{ or } k_{y_i}-k_{y_j}=n_2(2\pi/\sqrt{A}) \end{cases}$$

$$\Rightarrow \begin{cases} \theta_{x_i}-\theta_{x_j} \approx k_{x_i}/k - k_{x_j}/k = n_1\frac{\lambda}{\sqrt{A}} \\ \theta_{y_i}-\theta_{y_j} \approx k_{y_i}/k - k_{y_j}/k = n_2\frac{\lambda}{\sqrt{A}} \end{cases}$$

Optical Field Decomposition

Example (Optical field decomposition at the focal plane)

The focused fields corresponding to the plane waves with wavelength λ arriving to a squared aperture area A having arrival angles separated by multiples of $\lambda/\sqrt{A}$ can be used for optical field decomposition over the focal plane.

$$a(x, y, 0, t) = \sum_{i=1}^{\infty} a_i(t)\Phi_i(x, y)$$

$$a(x, y, f, t) \approx \sum_{i=1}^{\infty} a_i(t)\frac{1}{\lambda f}F_{\Phi_i}(\frac{x}{\lambda f}, \frac{y}{\lambda f})$$

The End