Optical Random Fields

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Overview

Optical Random Fields

Optical Random Fields

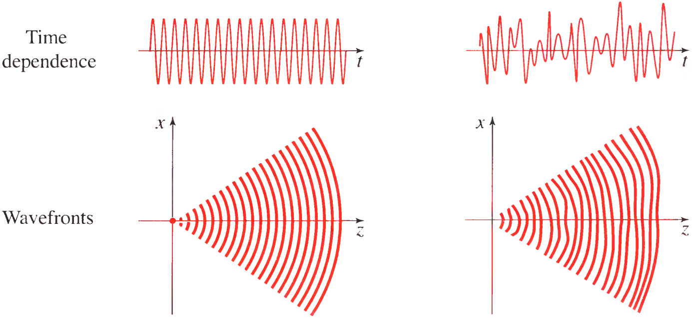


Figure: Coherent and random monochromatic wave.

- Quasi-monochromatic random complex wave function: $U(\boldsymbol{r},t) = a(\boldsymbol{r},t)e^{j2\pi\nu t}$
- Random wave envelope: $a(\boldsymbol{r},t)$
- Instantaneous intensity: $I(\boldsymbol{r},t) = |U(\boldsymbol{r},t)|^2 = |a(\boldsymbol{r},t)|^2$
- Average optical intensity: $I(\boldsymbol{r},t) = \mathcal{E}\{|U(\boldsymbol{r},t)|^2\} = \mathcal{E}\{|a(\boldsymbol{r},t)|^2\}$

Orthogonal Decomposition

Optical Random Field Decomposition

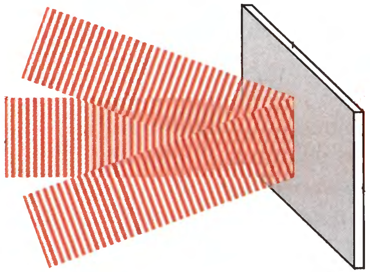


Figure: Orthogonal decomposition of an optical field over a finite area A and a limited interval T.

- Random wave envelope: $a(\boldsymbol{r}, t), \quad \boldsymbol{r} \in A, t \in [0, T]$
- Decomposition: $a(\boldsymbol{r}, t) = \sum_{i=1}^{\infty} a_i \Phi_i(\boldsymbol{r}, t), \quad \boldsymbol{r} \in A, t \in [0, T]$
- Orthogonality condition: $\int_A \int_0^T \Phi_i(\boldsymbol{r}, t) \Phi_j^*(\boldsymbol{r}, t) dt dA = 0, \quad i \neq j$
- Unity condition: $\int_A \int_0^T |\Phi_i(\boldsymbol{r}, t)|^2 dt dA = 1$
- Random modal coefficients: $a_i = \int_A \int_0^T a(\boldsymbol{r}, t) \Phi_i^*(\boldsymbol{r}, t) dt dA$

Optical Random Field Decomposition

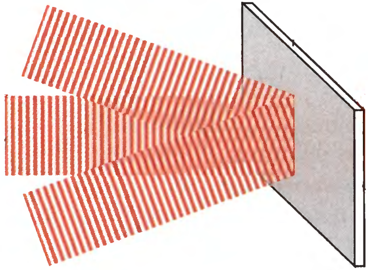


Figure: Orthogonal decomposition of an optical field over a finite area A and a limited interval T.

- Zero-mean envelope: $\mathcal{E}\{a(\boldsymbol{r}, t)\} = 0$
- Space-time coherence function: $R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = \mathcal{E}\{a(\boldsymbol{r}_1, t_1)a^*(\boldsymbol{r}_2, t_2)\}$
- Zero-mean modal coefficient: $\mathcal{E}\{a_i\} = 0$

Optical Random Field Decomposition

Example (Pairwise correlation of modal coefficients)

Pairwise correlation of modal coefficients $\mathcal{E}\{a_i a_j^*\}$ can be expressed in terms of modes $\Phi_i(\boldsymbol{r}, t)$.

$$
\begin{aligned}
\mathcal{E}\{a_i a_j^*\} &= \mathcal{E}\Big\{ \int_A \int_0^T a(\boldsymbol{r}_1, t_1)\Phi_i^*(\boldsymbol{r}_1, t_1) dt_1 dA_1 \int_A \int_0^T a^*(\boldsymbol{r}_2, t_2)\Phi_j(\boldsymbol{r}_2, t_2) dt_2 dA_2 \Big\} \\
&= \mathcal{E}\Big\{ \int_A \int_0^T \int_A \int_0^T a(\boldsymbol{r}_1, t_1)\Phi_i^*(\boldsymbol{r}_1, t_1) a^*(\boldsymbol{r}_2, t_2)\Phi_j(\boldsymbol{r}_2, t_2) dt_1 dA_1 dt_2 dA_2 \Big\} \\
&= \int_A \int_0^T \int_A \int_0^T \mathcal{E}\{a(\boldsymbol{r}_1, t_1) a^*(\boldsymbol{r}_2, t_2)\}\Phi_i^*(\boldsymbol{r}_1, t_1)\Phi_j(\boldsymbol{r}_2, t_2) dt_1 dA_1 dt_2 dA_2 \\
&= \int_A \int_0^T \int_A \int_0^T R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2)\Phi_i^*(\boldsymbol{r}_1, t_1)\Phi_j(\boldsymbol{r}_2, t_2) dt_1 dA_1 dt_2 dA_2 \\
&= \int_A \int_0^T \Big[\int_A \int_0^T R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2)\Phi_j(\boldsymbol{r}_2, t_2) dt_2 dA_2 \Big] \Phi_i^*(\boldsymbol{r}_1, t_1) dt_1 dA_1
\end{aligned}
$$

Karhunen-Loeve Expansion

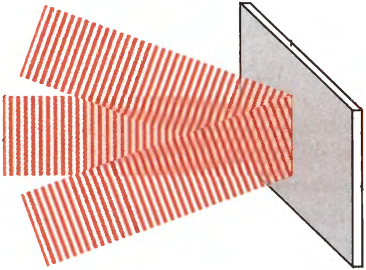


Figure: Karhunen-Loeve expansion of an optical field over a finite area A and a limited interval T.

- Fredholm equation: $\int_A \int_0^T R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2)\Phi_j(\boldsymbol{r}_2, t_2)dt_2 dA_2 = \lambda_j \Phi_j(\boldsymbol{r}_1, t_1)$
- Eigen function: $\Phi_j(\boldsymbol{r}, t)$
- Eigen value: λ_j
- Coefficient correlation: $\mathcal{E}\{a_i a_j^*\} = 0, \quad i \neq j$
- Coefficient power: $\mathcal{E}\{|a_i|^2\} = \lambda_i$
- Coefficient mean: $\mathcal{E}\{a_i\} = 0$

Karhunen-Loeve Expansion

Example (Mercer's formula)

In a Karhunen-Loeve expansion, coherence function can be expressed in terms of the eigen values λ_i and eigen functions $\Phi_i(\boldsymbol{r}, t)$.

$$
\begin{aligned}
&R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = \mathcal{E}\{a(\boldsymbol{r}_1, t_1)a^*(\boldsymbol{r}_2, t_2)\} \\
&= \mathcal{E}\{\sum_{i=1}^{\infty} a_i \Phi_i(\boldsymbol{r}_1, t_1) \sum_{j=1}^{\infty} a_j^* \Phi_j^*(\boldsymbol{r}_2, t_2)\} \\
&= \sum_{i=1}^{\infty} \mathcal{E}\{|a_i|^2\}\Phi_i(\boldsymbol{r}_1, t_1)\Phi_i^*(\boldsymbol{r}_2, t_2) + \sum_{i,j=1, i\neq j}^{\infty} \mathcal{E}\{a_i a_j^*\}\Phi_i(\boldsymbol{r}_1, t_1)\Phi_j^*(\boldsymbol{r}_2, t_2) \\
&= \sum_{i=1}^{\infty} \lambda_i \Phi_i(\boldsymbol{r}_1, t_1)\Phi_i^*(\boldsymbol{r}_2, t_2)
\end{aligned}
$$

Example (Coherence function of Karhunen-Loeve expansion)

In a Karhunen-Loeve expansion, mean field energy equals $\sum_i \lambda_i$.

$$\begin{aligned}
E &= \int_A \int_0^T I(\mathbf{r}, t) dt dA \\
&= \int_A \int_0^T \mathcal{E}\{|U(\mathbf{r}, t)|^2\} dt dA \\
&= \int_A \int_0^T \mathcal{E}\{|a(\mathbf{r}, t)|^2\} dt dA \\
&= \int_A \int_0^T R_f(\mathbf{r}, \mathbf{r}, t, t) dt dA \\
&= \int_A \int_0^T \sum_{i=1}^{\infty} \lambda_i \Phi_i(\mathbf{r}, t) \Phi_i^*(\mathbf{r}, t) dt dA \\
&= \sum_{i=1}^{\infty} \lambda_i \int_A \int_0^T \Phi_i(\mathbf{r}, t) \Phi_i^*(\mathbf{r}, t) dt dA = \sum_{i=1}^{\infty} \lambda_i
\end{aligned}$$

Temporal and Spatial Coherency

Temporal and Spatial Coherency

- Coherence-separable field: $R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = R_t(t_1, t_2) R_s(\boldsymbol{r}_1, \boldsymbol{r}_2)$
- Temporally stationary field: $R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1 - t_2)$
- Spatially homogeneous field: $R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = R_f(\boldsymbol{r}_1 - \boldsymbol{r}_2, t_1, t_2)$
- Completely homogeneous field: $R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = R_f(\boldsymbol{r}_1 - \boldsymbol{r}_2, t_1 - t_2)$
- Spectrally pure field: $R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = R_t(t_1 - t_2) R_s(\boldsymbol{r}_1 - \boldsymbol{r}_2) = R_t(\tau) R_s(\boldsymbol{\rho})$
- Space-coherent field: $R_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = C, \quad \forall \boldsymbol{r}_1, \boldsymbol{r}_2 \in A$
- Space-incoherent field: $R_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = I(\boldsymbol{r}_2)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$

- Temporal correlation function: $R_t(t_1, t_2)$
- Spatial coherence function: $R_s(\boldsymbol{r}_1, \boldsymbol{r}_2)$
- Normalized spatial coherence function: $\tilde{R}_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{R_s(\boldsymbol{r}_1, \boldsymbol{r}_2)}{\sqrt{R_s(\boldsymbol{r}_1, \boldsymbol{r}_1) R_s(\boldsymbol{r}_2, \boldsymbol{r}_2)}}$
- Power spectral density: $S(\nu) = F\big(R_t(\tau)\big)$
- Average optical intensity: $I(\boldsymbol{r}, t) = \mathcal{E}\{|a(\boldsymbol{r}, t)|^2\} = R_f(\boldsymbol{r}, \boldsymbol{r}, t, t)$

Coherence-separable Karhunen-Loeve Expansion

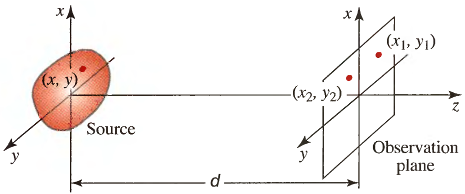
- Fredholm equation: $\int_A \int_0^T R_f(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2)\Phi_j(\boldsymbol{r}_2, t_2)dt_2 dA_2 = \lambda_j \Phi_j(\boldsymbol{r}_2, t_2)$
- Spatial Fredholm equation: $\int_A R_s(\boldsymbol{r}_1, \boldsymbol{r}_2)W_j(\boldsymbol{r}_2)dA_2 = \lambda_{sj} W_j(\boldsymbol{r}_1)$
- Temporal Fredholm equation: $\int_0^T R_t(t_1, t_2)g_j(t_2)dt_2 = \lambda_{tj} g_j(t_1)$
- Eigen function: $\Phi_j(\boldsymbol{r}, t) = W_j(\boldsymbol{r})g_j(t)$
- Eigen value: $\lambda_j = \lambda_{sj}\lambda_{tj}$
- Coherence-separable Decomposition: $a(\boldsymbol{r}, t) = \sum_{i=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} g_i(t) W_j(\boldsymbol{r}_2)$
- Random modal coefficients: $a_{ij} = \int_A \int_0^T a(\boldsymbol{r}, t) g_i^*(t) W_j^*(\boldsymbol{r}_2) dt dA$
- Coefficient power: $\mathcal{E}\{|a_{ij}|^2\} = \lambda_{ti}\lambda_{sj}$
- Mean field energy: $E = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \lambda_{ti}\lambda_{sj}$
- Mean field temporal energy: $E_t = \lambda_{ti}\sum_{j=1}^{\infty} \lambda_{sj}$
- Mean field spatial energy: $E_s = \lambda_{sj}\sum_{i=1}^{\infty} \lambda_{ti}$

Spatial Modes

Spatial Coherence Function

Example (Gain of coherence in free space)

Spatial coherence function at a distance d away can be expressed in terms of the coherence function at the aperture.



$$R_s(\mathbf{r}_1,\mathbf{r}_2)=\mathcal{E}\left\{a(x_1,y_1)a^*(x_2,y_2)\right\}$$

$$=\mathcal{E}\left\{h_0h_0^*\int\int\int\int a(u_1,v_1)a^*(u_2,v_2)e^{-j\frac{k}{2d}[(x_1-u_1)^2+(y_1-v_1)^2]}e^{j\frac{k}{2d}[(x_2-u_2)^2+(y_2-v_2)^2]}du_1dv_1du_2dv_2\right\}$$

$$=\frac{1}{\lambda^2d^2}\int\int\int\int\mathcal{E}\left\{a(u_1,v_1)a^*(u_2,v_2)\right\}e^{-j\frac{k}{2d}[(x_1-u_1)^2+(y_1-v_1)^2]}e^{j\frac{k}{2d}[(x_2-u_2)^2+(y_2-v_2)^2]}du_1dv_1du_2dv_2$$

$$=\frac{1}{\lambda^2d^2}\int_A\int_A R_s(\mathbf{w}_1,\mathbf{w}_2)e^{-j\frac{k}{2d}|\mathbf{r}_1-\mathbf{w}_1|^2}e^{j\frac{k}{2d}|\mathbf{r}_2-\mathbf{w}_2|^2}d\mathbf{w}_1d\mathbf{w}_2$$

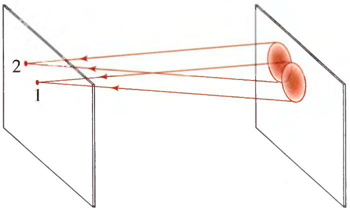


Figure: Coherence function of the incoherent source.

- Incoherent source: $R_s(\boldsymbol{w}_1, \boldsymbol{w}_2) = I(\boldsymbol{w}_2)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2), \quad \boldsymbol{w}_i = (u_i, v_i)$
- Intensity pattern: $\beta(\boldsymbol{r}) = F_I(\frac{x}{\lambda d}, \frac{y}{\lambda d}), \quad \boldsymbol{r}_i = (x_i, y_i)$
- Coherence function: $R_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\exp\left(j\frac{k}{2d}(|\boldsymbol{r}_2|^2 - |\boldsymbol{r}_1|^2)\right)}{\lambda^2 d^2}\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad \boldsymbol{r}_i = (x_i, y_i)$
- Normalized coherence function:
$\tilde{R}_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\exp\left(j\frac{k}{2d}(|\boldsymbol{r}_2|^2 - |\boldsymbol{r}_1|^2)\right)}{\beta(0)}\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad \boldsymbol{r}_i = (x_i, y_i)$

Incoherent Source

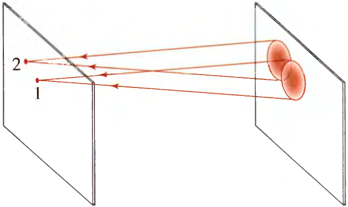
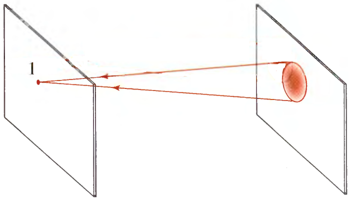


Figure: Coherence function of the incoherent source.

- Normalized coherence function: $\tilde{R}_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\exp\left(j\frac{k}{2d}(|\boldsymbol{r}_2|^2 - |\boldsymbol{r}_1|^2)\right)}{\beta(0)}\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$
- Spatial Fredholm equation: $\int_A \tilde{R}_s(\boldsymbol{r}_1, \boldsymbol{r}_2) W_j(\boldsymbol{r}_2) d\boldsymbol{r}_2 = A\lambda_{sj} W_j(\boldsymbol{r}_1)$
- Changed eigen function: $\hat{W}_j(\boldsymbol{r}) = W_j(\boldsymbol{r}) \exp\left(j\frac{k}{2d}|\boldsymbol{r}|^2\right)$
- Changed spatial Fredholm equation: $\int_A \frac{\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{\beta(0)} \hat{W}_j(\boldsymbol{r}_2) d\boldsymbol{r}_2 = A\lambda_{sj} \hat{W}_j(\boldsymbol{r}_1)$

Example (Spatial modes for an incoherent point source)

An incoherent point source has one spatial mode.



$$R_s(\boldsymbol{w}_1, \boldsymbol{w}_2) = I(\boldsymbol{w}_2)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2) = \beta(0)\delta(\boldsymbol{w}_2)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2)$$

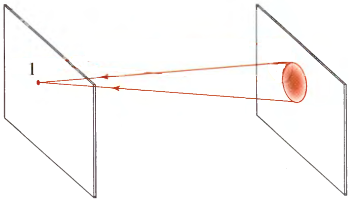
$$I(\boldsymbol{w}) = \beta(0)\delta(\boldsymbol{w}) \Rightarrow \beta(\boldsymbol{w}) = \beta(0)$$

$$\int_A \frac{\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{\beta(0)} \hat{W}_j(\boldsymbol{r}_2) d\boldsymbol{r}_2 = \int_A \tilde{W}_j(\boldsymbol{r}_2) d\boldsymbol{r}_2 = A\lambda_{sj} \hat{W}_j(\boldsymbol{r}_1)$$

$$\hat{W}_1(\boldsymbol{r}) = \frac{1}{\sqrt{A}}, \boldsymbol{r} \in A, \quad \lambda_{s1} = 1$$

Example (Spatial modes for an incoherent point source)

An incoherent point source has one spatial mode.



$$R_s(\boldsymbol{w}_1, \boldsymbol{w}_2) = I(\boldsymbol{w}_2)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2) = \beta(0)\delta(\boldsymbol{w}_2 - \boldsymbol{w}_s)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2)$$

$$I(\boldsymbol{w}) = \beta(0)\delta(\boldsymbol{w}_2 - \boldsymbol{w}_s) \Rightarrow \beta(\boldsymbol{r}) = \beta(0)e^{\frac{k}{d}\boldsymbol{w}_s.(\boldsymbol{r}_1 - \boldsymbol{r}_2)}$$

$$\int_A \frac{\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{\beta(0)} \tilde{W}_j(\boldsymbol{r}_2)d\boldsymbol{r}_2 = \int_A e^{\frac{k}{d}\boldsymbol{w}_s.(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \hat{W}_j(\boldsymbol{r}_2)d\boldsymbol{r}_2 = A\lambda_{sj}\hat{W}_j(\boldsymbol{r}_1)$$

$$\hat{W}_1(\boldsymbol{r}) = \frac{1}{\sqrt{A}} e^{\frac{k}{d}\boldsymbol{w}_s.\boldsymbol{r}}, \boldsymbol{r} \in A, \quad \lambda_{s1} = 1$$

Incoherent Source

Example (Spatial modes for an incoherent source)

Diffraction-limited separated plane waves over a large rectangle aperture of height a and width b can be used for spatial Karhunen-Loeve expansion of the incoherent source.

$$\hat{W}_j(\boldsymbol{r}) = \frac{1}{\sqrt{A}} e^{j2\pi[m\frac{x}{a}+n\frac{y}{b}]}, \boldsymbol{r} \in A$$

$$\int_A \frac{\beta(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{\beta(0)} \tilde{W}_j(\boldsymbol{r}_2) d\boldsymbol{r}_2$$

$$= \frac{1}{\beta(0)} \int_A \int_A I(u_1, v_1) e^{j\frac{k}{d}[u_1(x_1-x_2)+v_1(y_1-y_2)]} du_1 dv_1 \frac{1}{\sqrt{A}} e^{j2\pi[m\frac{x_2}{a}+n\frac{y_2}{b}]} dx_2 dy_2$$

$$= \frac{1}{\sqrt{A}\beta(0)} \int_A I(u_1, v_1) e^{j\frac{k}{d}[u_1x_1+v_1y_1]} du_1 dv_1 \int_A e^{j2\pi[x_2(\frac{m}{a}-\frac{u_1}{\lambda d})+y_2(\frac{m}{b}-\frac{v_1}{\lambda d})]} dx_2 dy_2$$

$$= \frac{1}{\sqrt{A}\beta(0)} \int_A I(u_1, v_1) e^{j\frac{k}{d}[u_1x_1+v_1y_1]} a\mathrm{Sinc}(\frac{a}{\lambda d}(u_1 - \frac{m\lambda d}{a})) b\mathrm{Sinc}(\frac{b}{\lambda d}(v_1 - \frac{n\lambda d}{b})) du_1 dv_1$$

$$\approx \frac{1}{\sqrt{A}\beta(0)} \int_A I(u_1, v_1) e^{j\frac{k}{d}[u_1x_1+v_1y_1]} \pi\delta(\frac{\pi}{\lambda d}(u_1 - \frac{m\lambda d}{a})) \pi\delta(\frac{\pi}{\lambda d}(v_1 - \frac{n\lambda d}{b})) du_1 dv_1$$

$$= \frac{\lambda^2 d^2}{\sqrt{A}\beta(0)} I(\frac{m\lambda d}{a}, \frac{n\lambda d}{b}) e^{j2\pi[\frac{mx_1}{a}+\frac{ny_1}{b}]} = \frac{\lambda^2 d^2}{\beta(0)} I(\frac{m\lambda d}{a}, \frac{n\lambda d}{b}) \hat{W}_j(x_1, y_1) = \lambda_{m,n} \hat{W}_j(x_1, y_1)$$

Incoherent Source

Example (Number of spatial modes)

An incoherent source includes $\lceil AA_0/\lambda^2 d^2 \rceil$ spatial modes, where A_0 and A are the areas of transmitter and receiver and d is the distance between transmitter and receiver.

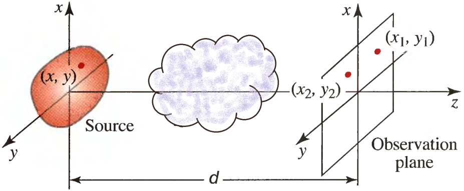
$$\Omega_{fv} = \frac{A_0}{d^2}$$

$$\Omega_{dl} = \frac{\frac{\lambda d}{a} \frac{\lambda d}{b}}{d^2} = \frac{\lambda^2}{A}$$

$$\frac{\Omega_{fv}}{\Omega_{dl}} = \lceil AA_0/\lambda^2 d^2 \rceil$$

Example (Gain of coherence in a random multiplicative environment)

Spatial coherence function at a distance d away can be expressed in terms of the coherence function of the aperture and medium.

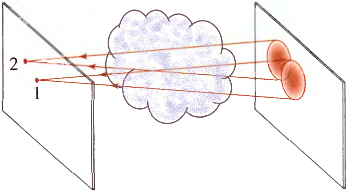


$$
\begin{aligned}
R_s(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \mathcal{E}\left\{a(\boldsymbol{r}_1)a^*(\boldsymbol{r}_2)\right\} \\
&= \mathcal{E}\left\{h_0 h_0^* \int_A \int_A g(\boldsymbol{r}_1, \boldsymbol{w}_1)a(\boldsymbol{w}_1)g^*(\boldsymbol{r}_2, \boldsymbol{w}_2)a^*(\boldsymbol{w}_2)e^{-j\frac{k}{2d}|\boldsymbol{r}_1-\boldsymbol{w}_1|^2}e^{j\frac{k}{2d}|\boldsymbol{r}_2-\boldsymbol{w}_2|^2}d\boldsymbol{w}_1 d\boldsymbol{w}_2\right\} \\
&= h_0 h_0^* \int_A \int_A \mathcal{E}\left\{a(\boldsymbol{w}_1)a^*(\boldsymbol{w}_2)\right\}\mathcal{E}\left\{g(\boldsymbol{r}_1, \boldsymbol{w}_1)g^*(\boldsymbol{r}_2, \boldsymbol{w}_2)\right\}e^{-j\frac{k}{2d}|\boldsymbol{r}_1-\boldsymbol{w}_1|^2}e^{j\frac{k}{2d}|\boldsymbol{r}_2-\boldsymbol{w}_2|^2}d\boldsymbol{w}_1 d\boldsymbol{w}_2 \\
&= \frac{1}{(\lambda d)^2}\int_A \int_A R_s(\boldsymbol{w}_1, \boldsymbol{w}_2)M(\boldsymbol{r}_1, \boldsymbol{w}_1, \boldsymbol{r}_2, \boldsymbol{w}_2)e^{-j\frac{k}{2d}|\boldsymbol{r}_1-\boldsymbol{w}_1|^2}e^{j\frac{k}{2d}|\boldsymbol{r}_2-\boldsymbol{w}_2|^2}d\boldsymbol{w}_1 d\boldsymbol{w}_2
\end{aligned}
$$

Random Multiplicative Environment

Example (Coherence function of an incoherent source through random multiplicative environment)

Spatial coherence function at a distance d away can be expressed in terms of the coherence function of the aperture and medium.



$$R_f(\boldsymbol{w}_1, \boldsymbol{w}_2) = I(\boldsymbol{w}_1)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2)$$

$$R_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{(\lambda d)^2}\int_A\int_A R_s(\boldsymbol{w}_1, \boldsymbol{w}_2) M(\boldsymbol{r}_1, \boldsymbol{w}_1, \boldsymbol{r}_2, \boldsymbol{w}_2) e^{-j\frac{k}{2d}|\boldsymbol{r}_1 - \boldsymbol{w}_1|^2} e^{j\frac{k}{2d}|\boldsymbol{r}_2 - \boldsymbol{w}_2|^2} d\boldsymbol{w}_1 d\boldsymbol{w}_2$$

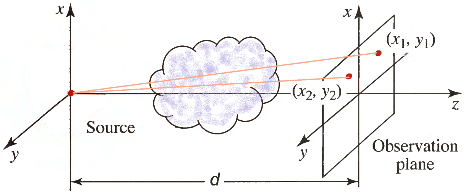
$$= \frac{1}{(\lambda d)^2}\int_A I(\boldsymbol{w}_1) M(\boldsymbol{r}_1, \boldsymbol{w}_1, \boldsymbol{r}_2, \boldsymbol{w}_1) e^{-j\frac{k}{2d}|\boldsymbol{r}_1 - \boldsymbol{w}_1|^2} e^{j\frac{k}{2d}|\boldsymbol{r}_2 - \boldsymbol{w}_1|^2} d\boldsymbol{w}_1$$

$$= \frac{e^{j\frac{k}{2d}[|\boldsymbol{r}_2|^2 - |\boldsymbol{r}_1|^2]}}{(\lambda d)^2}\int_A I(\boldsymbol{w}_1) M(\boldsymbol{r}_1, \boldsymbol{w}_1, \boldsymbol{r}_2, \boldsymbol{w}_1) e^{j\frac{k}{d}\boldsymbol{w}_1 \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)} d\boldsymbol{w}_1$$

Random Multiplicative Environment

Example (Coherence function of an incoherent point source through random multiplicative environment)

Spatial coherence function at a distance d away can be expressed in terms of the coherence function of the aperture and medium.



$$R_s(\boldsymbol{w}_1, \boldsymbol{w}_2) = \delta(\boldsymbol{w}_1 - \boldsymbol{w}_s)\delta(\boldsymbol{w}_1 - \boldsymbol{w}_2)$$

$$R_s(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{e^{j\frac{k}{d}[|\boldsymbol{r}_2|^2 - |\boldsymbol{r}_1|^2]} e^{j\frac{k}{d}\boldsymbol{w}_s.(\boldsymbol{r}_1 - \boldsymbol{r}_2)}}{(\lambda d)^2} M(\boldsymbol{r}_1, \boldsymbol{w}_s, \boldsymbol{r}_2, \boldsymbol{w}_s)$$

Temporal Modes

Lorentzian Source

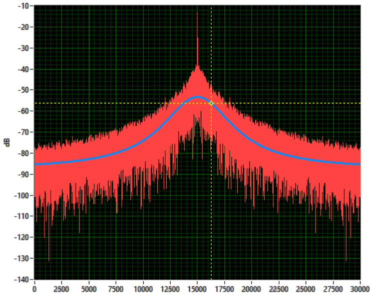


Figure: Power spectral density of a laser as a Lorentzian source.

- Power spectral density: $S(\nu) = \frac{2\alpha P}{4\pi^2\nu^2+\alpha^2}$
- Temporal correlation function: $R_t(\tau) = P\exp(-\alpha|\tau|)$
- Temporal Fredholm equation: $\int_{-T}^{T} Pe^{-\alpha|t-u|}g(u)du = \lambda g(t), \quad |t| \leq T$

Lorentzian Source

Example (Lorentzian source)

Temporal eigen values in the Lorentzian source decomposition are $\lambda_i = \frac{2\alpha P}{b_i^2+\alpha^2}$, where b_is are the roots of $(\tan(bT) + \frac{b}{\alpha})(\tan(bT) - \frac{\alpha}{b}) = 0$.

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f\left(x, b(x)\right) \cdot \frac{d}{dx}b(x) - f\left(x, a(x)\right) \cdot \frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt$$

$$\int_{-T}^{T} Pe^{-\alpha|t-u|}g(u)du = \lambda g(t)$$

$$\int_{-T}^{t} Pe^{-\alpha(t-u)}g(u)du + \int_{t}^{T} Pe^{\alpha(t-u)}g(u)du = \lambda g(t)$$

$$-P\alpha e^{-\alpha t}\int_{-T}^{t} e^{\alpha u}g(u)du + P\alpha e^{\alpha t}\int_{t}^{T} e^{-\alpha u}g(u)du = \lambda g'(t)$$

$$P\alpha^2\int_{-T}^{T} e^{-\alpha|t-u|}g(u)du - 2P\alpha g(t) = \alpha^2\lambda g(t) - 2P\alpha g(t) = \lambda g''(t)$$

$$g''(t) + b^2 g(t) = 0,\, b^2 = -\frac{\alpha^2(\lambda - 2P/\alpha)}{\lambda}, \lambda \neq 0$$

$$g(t) = c_1 e^{jbt} + c_2 e^{-jbt}$$

Lorentzian Source

Example (Lorentzian source (cont.))

Temporal eigen values in the Lorentzian source decomposition are $\lambda_i = \frac{2\alpha P}{b_i^2+\alpha^2}$, where b_is are the roots of $(\tan(bT) + \frac{b}{\alpha})(\tan(bT) - \frac{\alpha}{b}) = 0$.

$$\int_{-T}^{T} Pe^{-\alpha|t-u|} g(u)du = \lambda g(t), \quad g(t) = c_1 e^{jbt} + c_2 e^{-jbt}$$

$$\frac{2P\alpha}{\alpha^2 + b^2} g(t) - Pe^{-\alpha t}\left[c_1 \frac{e^{-T(\alpha+jb)}}{\alpha + jb} + c_2 \frac{e^{-T(\alpha-jb)}}{\alpha - jb}\right] + Pe^{\alpha t}\left[c_1 \frac{e^{-T(\alpha-jb)}}{-\alpha + jb} + c_2 \frac{e^{-T(\alpha+jb)}}{-\alpha - jb}\right] = \lambda g(t)$$

$$- Pe^{-\alpha t}\left[c_1 \frac{e^{-T(\alpha+jb)}}{\alpha + jb} + c_2 \frac{e^{-T(\alpha-jb)}}{\alpha - jb}\right] + Pe^{\alpha t}\left[c_1 \frac{e^{-T(\alpha-jb)}}{-\alpha + jb} + c_2 \frac{e^{-T(\alpha+jb)}}{-\alpha - jb}\right] = 0$$

$$A = \frac{e^{-T(\alpha+jb)}}{\alpha + jb} = e^{-\alpha T} \frac{\alpha\cos(bT) - b\sin(bT) + j[\alpha\sin(bT) + b\cos(bT)]}{\alpha^2 + b^2}$$

$$\Rightarrow \begin{cases} c_1 A + c_2 A^* &= 0 \\ c_2 A + c_1 A^* &= 0 \end{cases} \quad \Rightarrow c_1^2 - c_2^2 = 0 \Rightarrow \begin{cases} c_1 &= c_2 \Rightarrow A + A^* &= 2\,\mathrm{Re}\{A\} = 0 \\ c_1 &= -c_2 \Rightarrow A - A^* &= 2j\,\mathrm{Im}\{A\} = 0 \end{cases}$$

Lorentzian Source

Example (Lorentzian source (cont.))

Temporal eigen values in the Lorentzian source decomposition are $\lambda_i = \frac{2\alpha P}{b_i^2+\alpha^2}$, where b_is are the roots of $(\tan(bT) + \frac{b}{\alpha})(\tan(bT) - \frac{\alpha}{b}) = 0$.

$$\begin{cases} c_1 = c_2 \Rightarrow (\tan(bT) - \frac{\alpha}{b}) = 0 \\ c_1 = -c_2 \Rightarrow (\tan(bT) + \frac{b}{\alpha}) = 0 \end{cases} \Rightarrow (\tan(bT) + \frac{b}{\alpha})(\tan(bT) - \frac{\alpha}{b}) = 0 \Rightarrow \lambda_i = \frac{2\alpha P}{b_i^2 + \alpha^2}$$

$$\begin{cases} c_1 = c_2 \Rightarrow g_i(t) = 2c_1 \cos(b_i t) \\ c_1 = -c_2 \Rightarrow g_i(t) = 2jc_1 \sin(b_i t) \end{cases}, \int_{-T}^{T} |g_i(t)|^2 dt = 1$$

$$\begin{cases} g_i(t) = \dfrac{\cos\left(b_i t\right)}{\sqrt{T}\sqrt{1+\frac{\sin\left(2b_i T\right)}{2b_i T}}}, |t| \leq T, \quad c_1 = c_2 \\ g_i(t) = \dfrac{\sin\left(b_i t\right)}{\sqrt{T}\sqrt{1-\frac{\sin\left(2b_i T\right)}{2b_i T}}}, |t| \leq T, \quad c_1 = -c_2 \end{cases}$$

Lorentzian Source

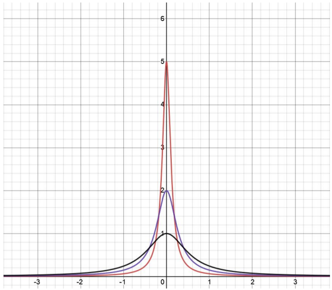


Figure: Lorentzian spectrum for different values of the narrowness factor α.

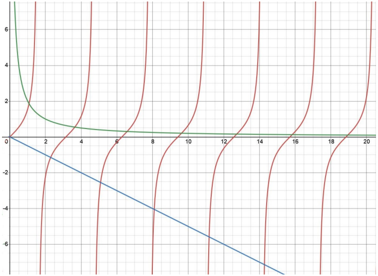


Figure: Angular frequencies at which the Lorentzian spectrum are sampled.

- Average intensity: $R_t(0) = P$
- Narrower spectrum: $\alpha \to 0$
- Low number of effective modes: $\alpha \to 0 \Rightarrow \lambda_i \approx \frac{2\alpha P_1}{b_i^2}$
- High number of effective modes: $\alpha \to \infty \Rightarrow \lambda_i \approx \frac{2P}{\alpha}$

Wiener Random Process

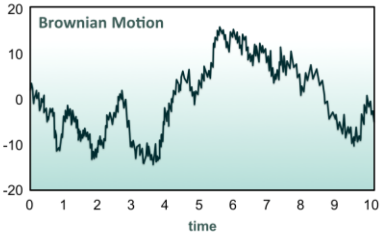


Figure: Wiener random process as a model for Brownian motion.

- Initial value: $a(0) = 0$
- Mean: $\mathcal{E}\{a(t)\} = 0$
- Variance: $\mathcal{E}\{a^2(t)\} = \sigma^2 t$
- PDF: $P\{a(t) = a\} = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{a^2}{2\sigma^2 t}\right)$
- Independency condition: $a(t_3) - a(t_2) \perp\!\!\!\perp a(t_2) - a(t_1), t_1 < t_2 < t_3$
- Correlation function: $R(t_1, t_2) = \sigma^2 \min(t_1, t_2)$
- Temporal Fredholm equation: $\int_0^T \sigma^2 \min(t_1, t_2) g(t_1) dt_1 = \lambda g(t_2)$

Wiener Random Process

Example (Wiener Random Process)

Temporal eigen values in the decomposition of Wiener random process are $\lambda_n = \frac{\sigma^2 T^2}{(n-0.5)^2 \pi^2}, n = 1, 2, \cdots$.

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f\left(x, b(x)\right) \cdot \frac{d}{dx} b(x) - f\left(x, a(x)\right) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt$$

$$\int_0^T \sigma^2 \min(t_1, t_2) g(t_1) dt_1 = \int_0^{t_2} \sigma^2 t_1 g(t_1) dt_1 + \sigma^2 t_2 \int_{t_2}^T g(t_1) dt_1 = \lambda g(t_2)$$

$$\sigma^2 \int_{t_2}^T g(t_1) dt_1 = \lambda g'(t_2)$$

$$g''(t_2) + \frac{\sigma^2}{\lambda} g(t_2) = 0, \quad \lambda \neq 0$$

$$\lambda_n = \frac{\sigma^2 T^2}{(n-0.5)^2 \pi^2}, \quad n = 1, 2, \cdots$$

$$g_n(t) = \sqrt{\frac{2}{T}} \sin\left((n-0.5)\frac{\pi t}{T}\right), \quad 0 \leq t \leq T$$

White Gaussian Noise Process

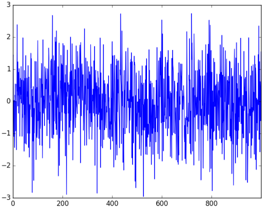


Figure: White Gaussian noise process as the derivative of the Wiener random process.

- Wiener random process: $a(t) = \sum_{n=1}^{\infty} a_n g_n(t) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{T}} \sin\left(\frac{(n-0.5)\pi t}{T}\right)$
- Wiener random process eigen value: $\lambda_n = \mathcal{E}\{a_n^2\} = \frac{\sigma^2 T^2}{(n-0.5)^2 \pi^2}, n = 1, 2, \cdots$
- WGN process: $w(t) = a'(t) = \sum_{n=1}^{\infty} \frac{a_n (n-0.5)\pi t}{T} \left[\sqrt{\frac{2}{T}} \cos\left(\frac{(n-0.5)\pi t}{T}\right)\right]$
- WGN process mode coefficient: $X_n = \frac{a_n (n-0.5)\pi t}{T}, n = 1, 2, \cdots$
- WGN process eigen value: $\lambda_n = \mathcal{E}\{X_n^2\} = \sigma^2, n = 1, 2, \cdots$

White Gaussian Noise Process

Example (Correlation function of white Gaussian noise process)

Correlation function of white Gaussian noise process is $R(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)$.

$$R_a(t_1, t_2) = \sigma^2 \min(t_1, t_2) = \sigma^2 \left[t_1 u(t_2 - t_1) + t_2 u(t_1 - t_2) \right]$$

$$R_w(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_a(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)$$

White Gaussian Noise Process

Example (Mercer's formula for white Gaussian noise process)

Applying Mercer's formula to a white Gaussian noise process yields $\delta(t_1 - t_2) = \sum_{n=1}^{\infty} g_n(t_1)g_n(t_2)$, where $g_n(t)$'s are any set of orthogonal functions.

$$\int_0^T R_w(t_1, t_2)g(t_1)dt_1 = \int_0^T \sigma^2\delta(t_1 - t_2)g(t_1)dt_1 = \sigma^2 g(t_2) = \lambda g(t_2) = \sigma^2 g(t_2), \quad 0 \leq t_2 \leq T$$

$$R_w(t_1, t_2) = \sum_{n=1}^{\infty} \lambda_n g_n(t_1)g_n(t_2), \quad 0 \leq t_1, t_2 \leq T$$

$$\delta(t_1 - t_2) = \sum_{n=1}^{\infty} g_n(t_1)g_n(t_2), \quad 0 \leq t_1, t_2 \leq T$$

White Gaussian Noise Process

Example (Lorentzian laser source polluted by additive white Gaissian noise)

For a Lorentzian laser signal polluted by an independent zero-mean additive white Gaussian noise, the energy in each mode is $\lambda_i = \frac{2\alpha P}{b_i^2+\alpha^2} + \sigma^2$.

$$r(t) = a(t) + n(t) = \sum_{i=1}^{\infty} a_i g_i(t) + \sum_{i=1}^{\infty} n_i g_i(t) = \sum_{i=1}^{\infty} r_i g_i(t)$$

$$r_i = a_i + n_i = \int_{-T}^{T} r(t) g_i^*(t) dt$$

$$\mathcal{E}\{a_i n_i^*\} = \mathcal{E}\{\int_{-T}^{T} a(t_1) g_i^*(t_1) dt_1 \int_{-T}^{T} n^*(t_2) g_i(t_2) dt_2\} = \int_{-T}^{T}\int_{-T}^{T} \mathcal{E}\{a(t_1) n^*(t_1)\} g_i^*(t_1) g_i(t_2) dt_1 dt_2$$

$$\mathcal{E}\{a_i n_i^*\} = \int_{-T}^{T}\int_{-T}^{T} \mathcal{E}\{a(t_1)\}\mathcal{E}\{n^*(t_2)\} g_i^*(t_1) g_i(t_2) dt_1 dt_2 = 0$$

$$\lambda_i = \mathcal{E}\{|r_i|^2\} = \mathcal{E}\{(a_i + n_i)(a_i + n_i)^*\} = \mathcal{E}\{|a_i|^2\} + \mathcal{E}\{|n_i|^2\} + \mathcal{E}\{a_i n_i^*\} + \mathcal{E}\{n_i a_i^*\} = \frac{2\alpha P}{b_i^2 + \alpha^2} + \sigma^2$$

$$\frac{\lambda_{a_i}}{\lambda_{n_i}} = \frac{2\alpha P}{\sigma^2(b_i^2 + \alpha^2)}$$

The End