Photo-Detection

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Overview

Preliminaries

Photoelectric Effect

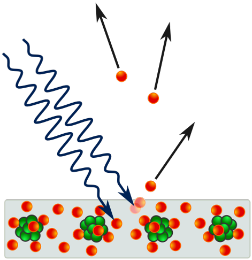

Figure: Photoelectric effect.

- Photoelectric effect: The emission of electrons from a material by light.

Photoelectric Effect

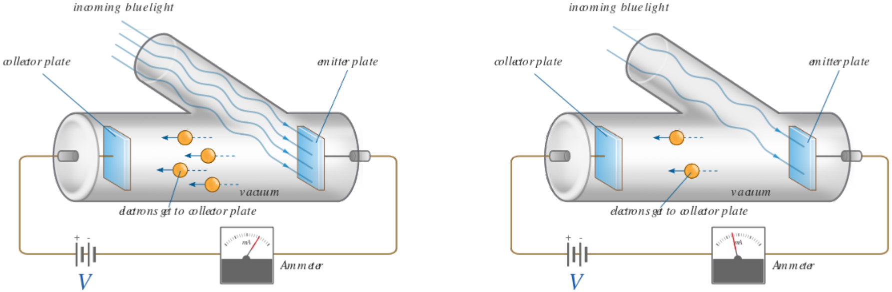

Figure: Lenard's photoelectric experiment.

- Lenard's photoelectric experiment: Kinetic energy doesn't depend on intensity.

Photoelectric Effect

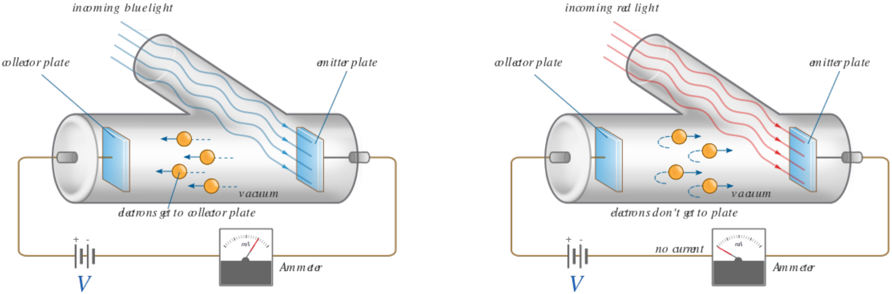

Figure: Millikan's photoelectric experiment.

- Millikan's photoelectric experiment: Kinetic energy depends on intensity.

Photoelectric Effect

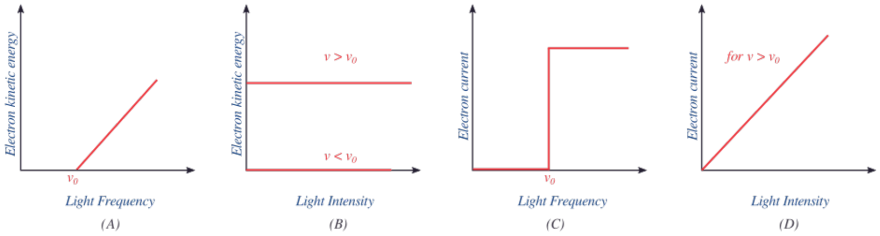

Figure: Photoelectric observations.

- Einstein's photoelectric kinetic energy: $K = (h\nu - W_0)u(\nu - \nu_0)$
- Einstein's photoelectric current: $J = AIu(\nu - \nu_0)$

Bohr's Hydrogen Atom Model

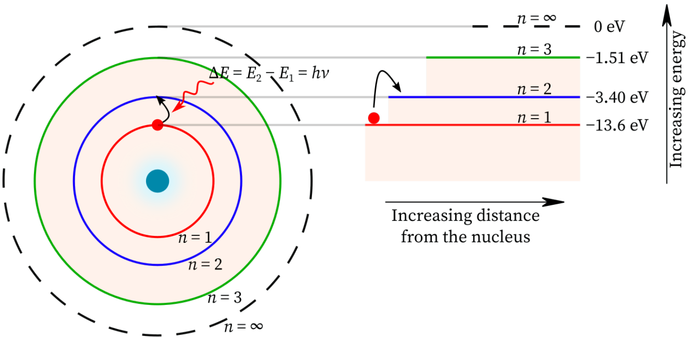

Figure: Bohr's atomic model.

- Energy level; $E_n = -13.6/n^2$ eV
- Orbital radius; $r_n = 52.9n^2$ pm
- Orbital velocity; $v_n = 2.187 \times 10^6/n$ m/s
- Radiation (absorption) frequency; $\nu = \frac{1}{\lambda} = R_H(\frac{1}{n_i^2} - \frac{1}{n_f^2})$ Hz

Bohr's Hydrogen Atom Model

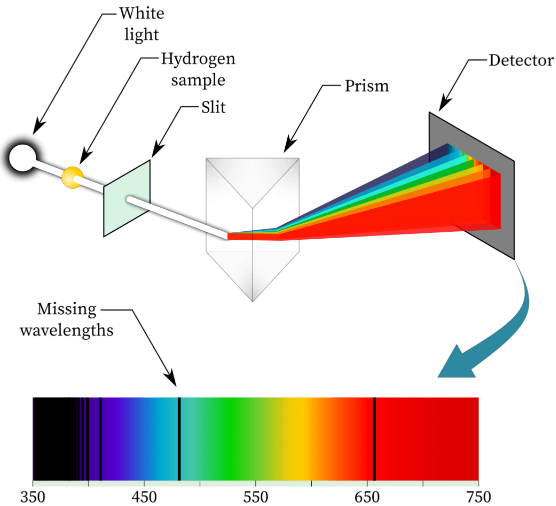

Figure: Hydrogen absorption spectrum.

Bohr's Hydrogen Atom Model

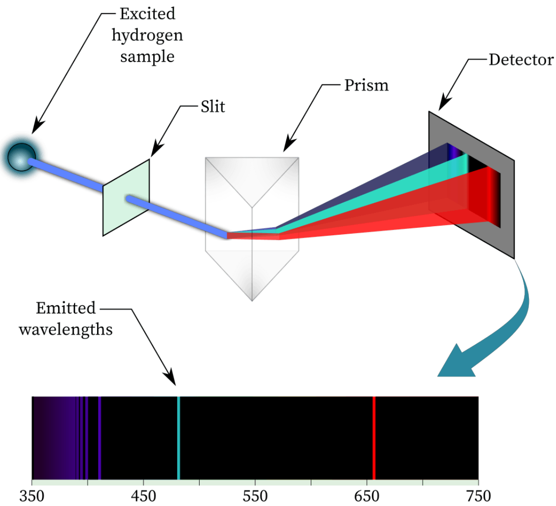

Figure: Hydrogen emission spectrum.

Schrodinger's Hydrogen-like Atom Model

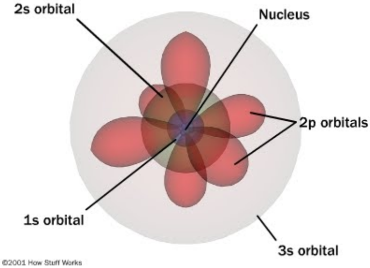

Figure: Schrodinger's atomic model.

- Schrodinger's equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = j\hbar\frac{\partial\Psi}{\partial t}$
- Time-independent Schrodinger's equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$
- Potential function: $V(r) = -Ze^2/r$
- Energy levels: $E_n = -\frac{M_r Z^2 e^4}{(4\pi\epsilon_0)^2 2\hbar^2}\frac{1}{n^2}$
- Wavefunctions: $\psi(r,\theta,\phi) = R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\phi)$

Schrodinger's Hydrogen-like Atom Model

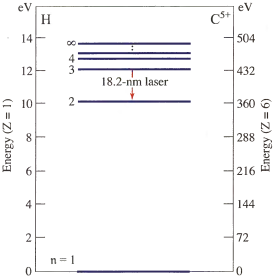

Figure: Energy levels of a hydrogen-like atomic stricture.

- Associated Laguerre function: $R_{nl}(r)$
- Associated Legendre function: $\Theta_{lm}(\theta)$
- Phase function: $\Phi_m(\phi)$
- Principal quantum number (shell): $n = 1, 2, 3 \cdots$
- Azimuthal quantum number (sub-shell): $l = 0, 1, \cdots, n-1 \equiv s, p, d, f, \cdots$
- Magnetic quantum number: $m = 0, \pm 1, \cdots, \pm l$
- Electron configuration: nl^u
- Spin quantum number: $s = \pm \frac{1}{2}$

Multi-electron Atom Model

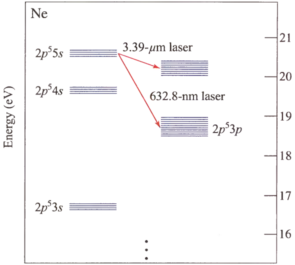

Figure: Energy levels of a multi-electron atom.

- Schrodinger's equation: Hartree approximated method
- Manifold: Collection of closely spaced fine-structure energy-level splittings
- Pauli exclusion principle: No two electrons may have the same quantum numbers.
- Minimum energy principle: Energy minimization while satisfying the Pauli exclusion.
- Valence electron: Electrons in the outermost shell.

Molecules

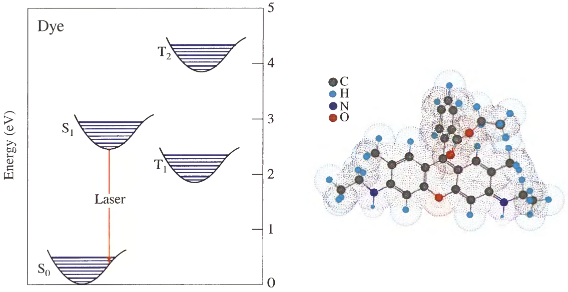

Figure: Sample energy levels of molecule.

- Common molecular binding: Ionic binding and covalent binding.
- Energy levels: Arise from rotational transitions, vibrational transitions, and electronic transitions
- Minimum energy principle: A stable molecule emerges when the sharing of valence electrons by the constituent atoms reduces the overall energy.

Crystal Lattices

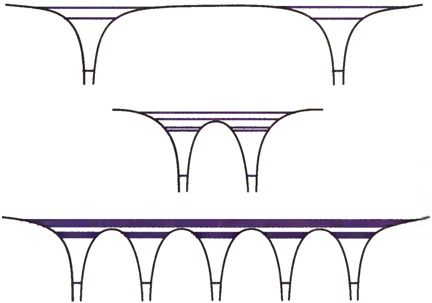

Figure: Change of energy levels by proximity of individual atoms in a solid.

- Common solid binding: Ionic, covalent, and metallic binding.
- Solid structures: Crystal and non-crystal
- Crystals: A solid with periodic arrangement of molecules.
- Energy levels: Arise from individual and neighboring atoms

Band Theory

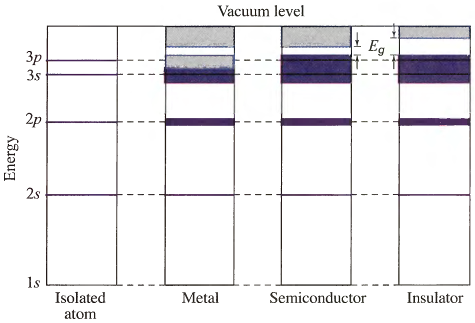

Figure: Broadening of the discrete energy levels of an isolated atom into energy bands in a crystal. The bands can also be described by solving the Schrodinger equation for Kronig-Penny model of crystals.

- Conduction band: Lowest unoccupied, or partially occupied, energy band
- Valence band: Highest fully occupied energy band.
- Forbidden band: Separation distance between conduction and valence bands
- Bandgap energy: Energy extent between conduction and valence bands
- Conductivity: Metal, Semi-conductor, isolator

Occupancy of energy levels

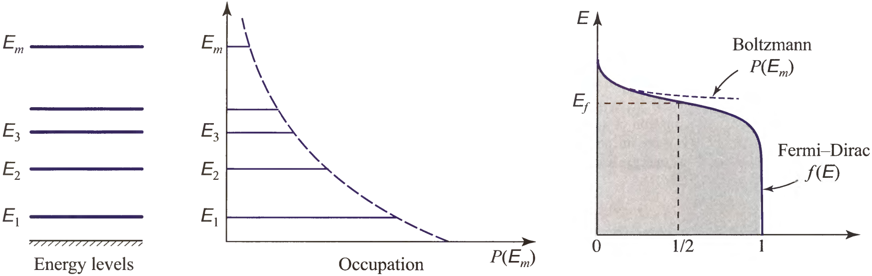

Figure: Occupancy of energy levels are described by Boltzman and Fermi equations.

- Boltzman distribution: $P(E_m) \propto \exp(-E_m/kT), \quad m = 1, 2, \cdots$
- Fermi function: $f(E) = \frac{1}{\exp[(E-E_f)/kT]+1}$

Intrinsic Semi-conductors

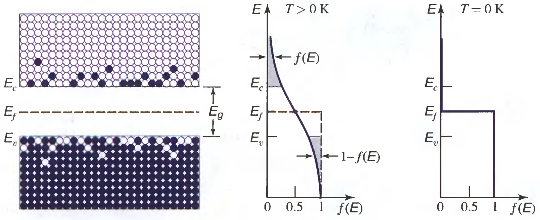

Figure: Creation of electron-hole pairs for thermal excitation in an intrinsic semi-conductor crystal. The electron and holes creates current under external applied field. The conductivity increases as temperature goes up.

- Zero-conductivity at zero temperature: Full valance band
- Thermal excitation: Electron-hole generation
- Thermal equilibrium: Radiative/nonradiative electron-hole recombination
- Probability of electron occupancy in valence band: $f(E)$
- Probability of hole occupancy in valence band: $1 - f(E)$
- Intrinsic carrier concentration: $n = p = n_i = \sqrt{N_c N_v} \exp\left(-\frac{E_g}{2kT}\right)$

Doped Semi-conductors

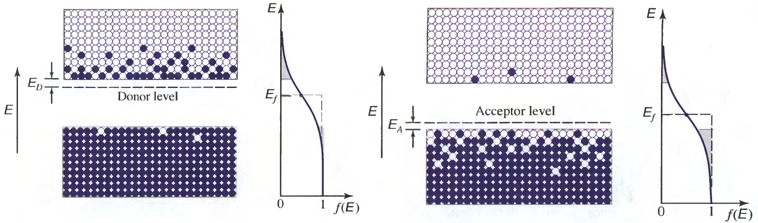

Figure: N-type and P-type doped semi-conductors.

- Donor dopants: Impurities with excess valence electron
- Acceptor dopants: Impurities with deficiency of valence electron
- Doped carrier concentration: $np = n_i^2$

PN Junction

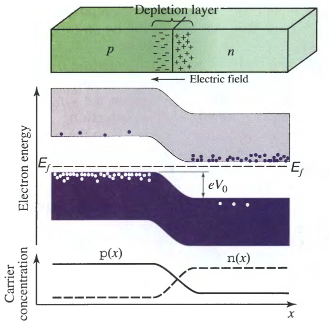

Figure: Unbiased PN junction.

- Depletion layer: A region without mobile carriers
- Majority carriers: Holes in p-type and electrons in n-type
- Minority carriers: Holes in n-type and electrons in p-type
- Inoized atoms: Positive ions in n-type and negative atom in p-type
- Built-in field: Electric field from n-type to p-type
- No net current: Cancellation of diffusion and drift currents

PN Junction

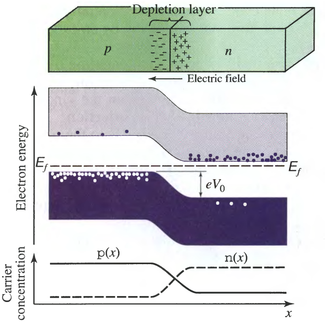

Figure: Biased PN junction.

- Shockly diode equation: $i = i_s[\exp(eV/kT) - 1]$
- Avalanche breakdown: Current multiplication via free-electron creation by collision

Interaction of Photons and Material

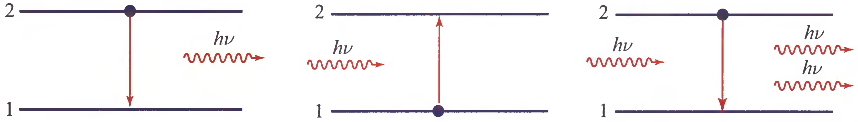

Figure: Three main interactions of a photon with energy $h\nu = E_g = E_2 - E_1$ and atom, spontaneous emission, absorption, and stimulated emission.

- Two-state transitions: Arise from Schrodinger equation
- Fermi's golden rule: Transition rate from one energy state to another due to a weak perturbation
- Absorption/stimulated emission probability density: $W_i = \frac{dP_i}{dt} \propto I(t, \boldsymbol{r})\Delta \boldsymbol{r}\delta(E_g - h\nu)$

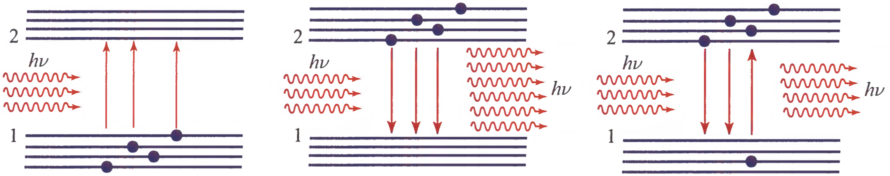

Figure: Absorption and stimulated emission for bands.

- Absorption/stimulated emission probability density:
  $W_i = \frac{dP_i}{dt} \propto I(t, \boldsymbol{r})\Delta \boldsymbol{r}, \quad E_g \geq h\nu$
- Independency of time
- Proportionality to state density
- Proportionality to coupling strength

Physical Description of Photo-Detection

Phototube

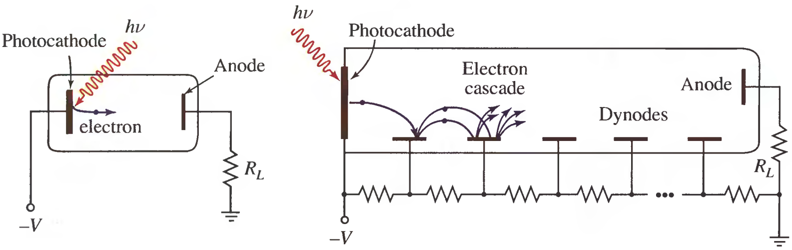

Figure: Vacuum photodiode tube and photomultiplier tube.

- Photoelectric effect: The emission of electrons from a material by light.
- Photon multiplication: Secondary emission of electrons from dynodes using photo-emitted electrons emitted from photo-cathode.

Photodiode

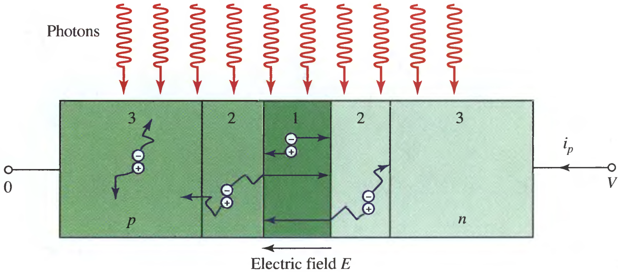

Figure: Photodiode.

- Generation: Absorbed photons generate free carriers.
- Transport: Built-in field causes these carriers to move and create current.

Avalanche Photodiode

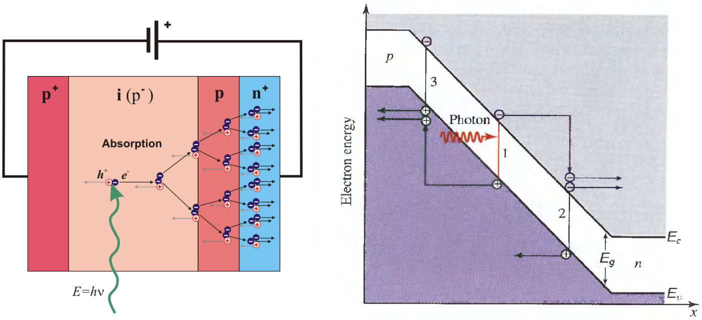

Figure: Multiplication process in avalanche photodiode.

- Generation: Absorbed photons generate free carriers.
- Transport: Built-in field causes these carriers to move and create current.
- Gain: Large electric fields impart sufficient energy to the carriers to free additional carriers.

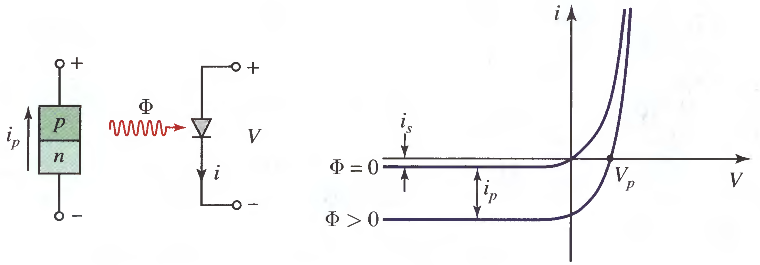

Figure: Generic photodiode and its i-V relation. .

- Photo-diode characteristic curve: $i = i_s[\exp(eV/kT) - 1] - i_p$
- Dark current: i_s arisen from thermally-excited random generation of electrons-hole pairs

Quantum Efficiency

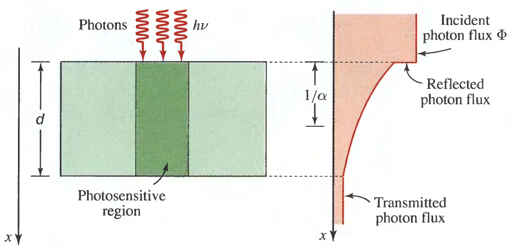

Figure: Effect of surface reflection, incomplete absorption, and penetration depth on the detector quantum efficiency.

- Quantum efficiency: Number of generated electron-hole pairs to the number of incident photons.
- Quantum efficiency: $\eta = (1 - R)\zeta[1 - \exp(-\alpha d)]$.
- Surface reflection: $1 - R$.
- Incomplete absorption: η.
- Penetration depth: $1 - \exp(-\alpha d)$.

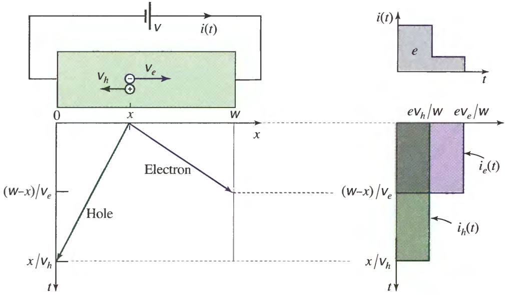

Figure: Electron current and hole current.

- Mean drift velocity: $v = \mu E = \frac{a\tau_{col}}{m}E$
- Carrier mobility: $\mu = \frac{a\tau_{col}}{m}$.
- Ramo's formula: $-QEdx = -Q\frac{V}{w}dx = i(t)Vdt$.
- Ramo's formula: $i(t) = -\frac{Q}{w}v(t)$.
- Transit-time spread: $x/v_h, \quad (W-x)/v_e, \quad v_h < v_e$.

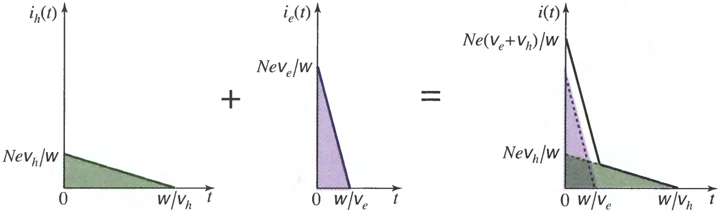

Figure: Impulse response function for a uniformly illuminated detector subject to transit-time spread.

- Total current: $i(t) = i_h(t) + i_e(t)$
- Generated charge: Ne

Responsivity

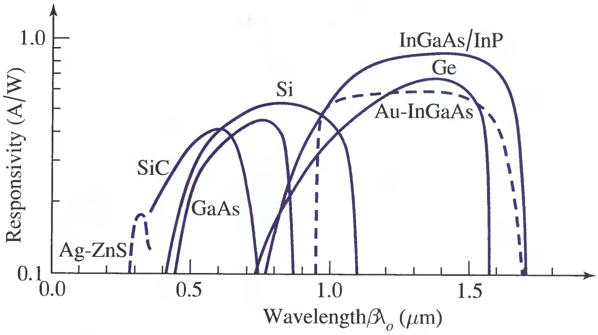

Figure: Responsivity versus wavelength.

- Responsivity: Ratio of the current to the power.
- Responsivity: $\mathcal{R} = \frac{i_p}{P} = \frac{1}{P} d\frac{\eta e \frac{E}{h}}{dt} = \frac{\eta e}{h\nu}$.
- Multiplicative responsivity: $\mathcal{R} = \frac{i_p}{P} = \frac{G\eta e}{h\nu}$.

Statistical Description of Photo-Detection

Poisson Random Variable

Item	Definition	Expression
Probability mass function	$P(k)$	$e^{-m}\frac{m^k}{k!}$
Mean	$\sum_k kP(k)$	m
Mean-square	$\sum_k k^2P(k)$	m^2+m
Variance	$\sum_k k^2P(k)-\left[\sum_k k^2P(k)\right]^2$	m
Characteristic function	$\sum_k e^{j\omega k}P(k)$	$e^{m(e^{j\omega}-1)}$
Moment-generating function	$\sum_k (1-z)^kP(k)$	e^{-zm}
qth moment	$\sum_k k^qP(k)$	$\frac{\partial^q}{\partial z^q}\left[e^{-mz}\right]_{z=1}$

Table: Identities for Poisson random variable.

Definition (Poisson Stochastic Process)

The counting process $\{N(t), t \in [0, \infty)\}$ is called a Poisson process with fixed rate $\lambda > 0$ if all the following conditions hold

- $N(0) = 0$
- $N(t)$ has independent increments.
- The number of arrivals in any interval of length $\tau > 0$ has a Poisson distribution of mean $\lambda\tau$.

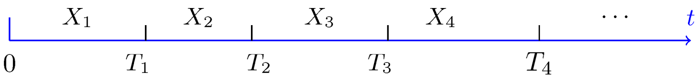

Figure: Poisson stochastic process.

- Independent inter-arrival times: $X_n \sim \text{Exponential}(\lambda) = \lambda e^{-\lambda x}$
- Dependent arrival times: $T_n \sim \text{Gamma}(\lambda) = \frac{\lambda^n t^{n-1} e^{-\lambda x}}{(n-1)!}$

Counting Statistics

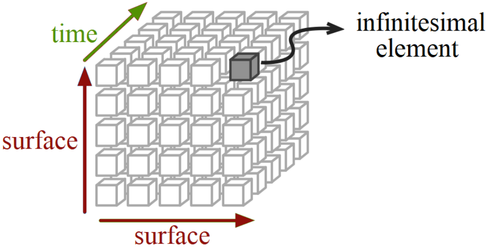

Figure: Random photo-detection process over an infinitesimal area $\boldsymbol{r}$ and infinitesimal interval Δt.

- Fermi's golden rule: $\frac{dP_i}{dt} = \alpha I(t, \boldsymbol{r})\Delta \boldsymbol{r}$.
- No carrier generation probability:
  $P(0) = 1 - P(t_i, \boldsymbol{r}_i) \approx 1 - \alpha I(t_i, \boldsymbol{r}_i)\Delta \boldsymbol{r}\Delta t = 1 - \alpha I(\boldsymbol{v}_i)\Delta \boldsymbol{v}$.
- Single carrier generation probability:
  $P(1) = P(t_i, \boldsymbol{r}_i) \approx \alpha I(t_i, \boldsymbol{r}_i)\Delta \boldsymbol{r}\Delta t = \alpha I(\boldsymbol{v}_i)\Delta \boldsymbol{v}$.
- More carrier generation probability: $P(k) = 0, \quad k \geq 2$.

Counting Statistics

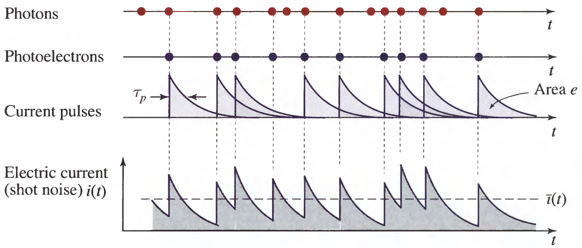

Figure: Random photo-detection process over area A and interval T for a constant intensity I.

- No carrier generation probability: $P(0) = 1 - \alpha I \Delta \boldsymbol{v} = 1 - \alpha I \frac{AT}{N} = 1 - p.$
- Single carrier generation probability: $P(1) = \alpha I \Delta \boldsymbol{v} = \alpha I \frac{AT}{N} = p.$
- k carrier generation probability: $P(k) = \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k}.$
- k carrier generation probability:
  $pN = \alpha IAT, N \to \infty \Rightarrow P(k) = e^{-m_v} \frac{m_v^k}{k!}, \quad m_v = pN = \alpha IAT = \alpha E.$

Counting Statistics

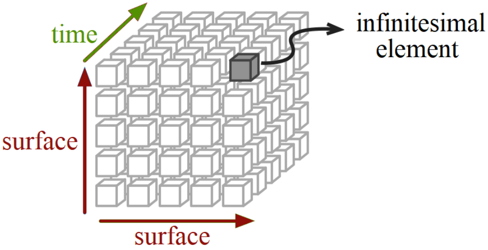

Figure: Random photo-detection process over area A and interval T for a deterministic intensity I.

- No carrier generation probability: $P(\boldsymbol{V}_i = 0) = 1 - \alpha I(\boldsymbol{v}_i)\Delta \boldsymbol{v} = 1 - p_i$.
- Single carrier generation probability: $P(\boldsymbol{V}_i = 1) \approx \alpha I(\boldsymbol{v}_i)\Delta \boldsymbol{v} = p_i$.
- k carrier generation probability: $P(\boldsymbol{V} = k) \approx P(\sum_{i=1}^{N} \boldsymbol{V}_i = k)$.

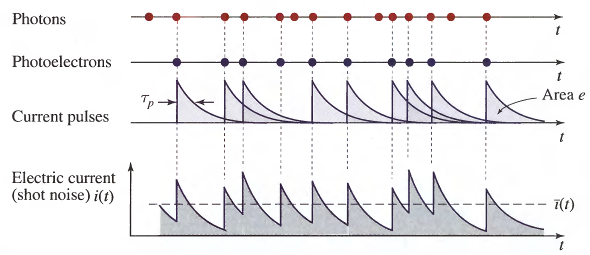

Figure: Random photo-detection process over area A and interval T for a deterministic intensity I.

- Characteristic function: $\Psi_{\boldsymbol{v}_i}(\omega) = \mathcal{E}\{e^{j\omega \boldsymbol{v}_i}\} = 1 - p_i + e^{j\omega} p_i$.
- Characteristic function: $\Psi_{\boldsymbol{v}}(\omega) = \mathcal{E}\{e^{j\omega \sum_i^N \boldsymbol{v}_i}\} = \prod_{i=1}^{N} [1 + (e^{j\omega} - 1)p_i]$.
- Characteristic function: $\Psi_{\boldsymbol{v}}(\omega) = \exp\big(m_v(e^{j\omega} - 1)\big)$.
- k carrier generation probability: $P(k) = e^{-m_v}\frac{m_v^k}{k!}, \quad m_v = \alpha \int_A \int_T I(\boldsymbol{r}, t) d\boldsymbol{r} dt$.
- Calculation lemma:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \ln\Big(1 + (e^{j\omega} - 1)p_i\Big) = \lim_{N\to\infty} \sum_{i=1}^{N} (e^{j\omega} - 1)p_i = (e^{j\omega} - 1) \lim_{N\to\infty} \sum_{i=1}^{N} p_i = m_v(e^{j\omega} - 1)$$

Counting Statistics

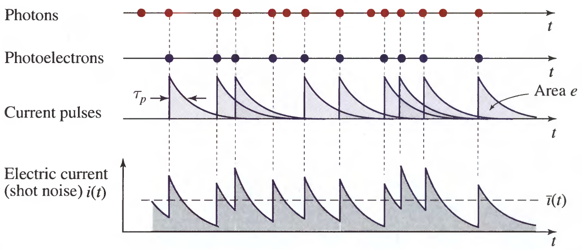

Figure: Random photo-detection process over area A and interval T for a deterministic intensity I.

- Carrier generation variance: $m_v = \alpha \int_A \int_T I(\boldsymbol{r}, t) d\boldsymbol{r} dt$.
- Carrier generation mean: $m_v = \alpha \int_A \int_T I(\boldsymbol{r}, t) d\boldsymbol{r} dt$.
- Carrier generation mean: $\eta \frac{1}{h\nu} \int_A \int_T I(\boldsymbol{r}, t) d\boldsymbol{r} dt$.
- Fermi's proportionality constant: $\alpha = \frac{\eta}{h\nu}$.

Counting Statistics

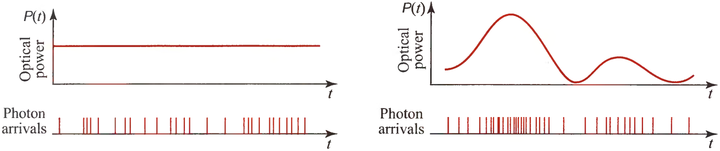

Figure: Random photo-detection process over area A and interval T for a stochastic intensity I.

- Random carrier generation PDF: $p(m)$
- Conditional Poisson distribution: $P(k|m) = e^{-m}\frac{m^k}{k!}$
- Mandel's formula: $P(k) = \int_0^\infty e^{-m}\frac{m^k}{k!}p(m)dm$

Counting Statistics

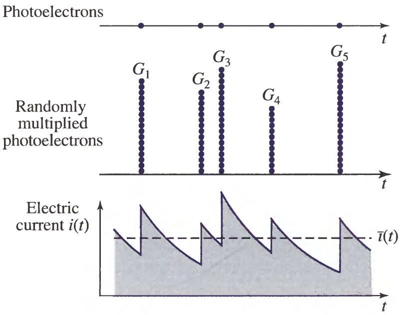

Figure: Random photo-detection process and random multiplication process over area A and interval T for a stochastic intensity I.

- Primary count probability: $P(k_1) = \int_0^\infty e^{-m} \frac{m^{k_1}}{k_1!} p(m) dm$
- Secondary count probability: $P(k_2) = \sum_{k_1=0}^\infty P(k_2|k_1)P(k_1)$

Counting Statistics

Example (Constant monochromatic point source)

The count of generated carriers for a photo-detector illuminated by a constant monochromatic point source at far field has Poisson distribution.

$$f_r(t, \boldsymbol{r}) = ae^{j2\pi\nu t}, \quad \boldsymbol{r} \in A$$

$$m_v = \alpha \int_A \int_T |f_r(t, \boldsymbol{r})|^2 d\boldsymbol{r} dt = \alpha |a|^2 AT = \alpha IAT \Rightarrow p(m) = \delta(m - \alpha IAT)$$

$$P(k) = e^{-m} \frac{m^k}{k!} = e^{-\alpha IAT} \frac{(\alpha IAT)^k}{k!}$$

Short-Term Counting Statistics

Example (Intensity-modulated monochromatic point source)

The count of generated carriers for a photo-detector illuminated by an intensity-modulated monochromatic point source at far field has Poisson distribution.

$$
\begin{aligned}
&f_r(t, \boldsymbol{r}) = a(t)e^{j2\pi\nu t}, \quad \boldsymbol{r} \in A \\
&m_v = \alpha \int_A \int_T |f_r(t, \boldsymbol{r})|^2 d\boldsymbol{r} dt = \alpha A \int_0^T |a(t)|^2 dt \Rightarrow p(m) = \delta(m - \alpha A \int_0^T |a(t)|^2 dt) \\
&P(k) = e^{-m}\frac{m^k}{k!} = e^{-\alpha A \int_0^T |a(t)|^2 dt} \frac{(\alpha A \int_0^T |a(t)|^2 dt)^k}{k!} \\
&T \to 0 \Rightarrow P(k) = e^{-m}\frac{m^k}{k!} = e^{-\alpha ATI(t)} \frac{(\alpha ATI(t))^k}{k!}, I(t) = |a(t)|^2
\end{aligned}
$$

Example (Thermal light)

If a zero-mean stationary Gaussian random envelope $n(t) = \text{Re}\{n(t)\} + j\,\text{Im}\{n(t)\}$ illuminates on a photo-detector, the count of carriers has Bose distribution.

$$y = Kx^2, x \geq 0 \Rightarrow f_Y(y) = \frac{f_X(\sqrt{\frac{y}{K}})}{2\sqrt{Ky}}$$

$$f_r(t, \boldsymbol{r}) = n(t)e^{j2\pi\nu t}, \boldsymbol{r} \in A, \quad \text{Re}\{n(t)\} \sim \mathcal{N}(0, \sigma^2), \text{Im}\{n(t)\} \sim \mathcal{N}(0, \sigma^2), \text{Re}\{n(t)\} \perp\!\!\!\perp \text{Im}\{n(t)\}$$

$$|f_r(t, \boldsymbol{r})| = |n(t)| \sim \text{Rayleigh}(\sigma) = \frac{a}{\sigma^2} e^{-\frac{a^2}{2\sigma^2}}$$

$$m_v = \alpha \int_A \int_T |f_r(t, \boldsymbol{r})|^2 d\boldsymbol{r} dt = \alpha A \int_T |n(t)|^2 dt \approx \alpha AT|n(t)|^2$$

$$p(m) = \frac{1}{2\sigma^2 \alpha AT} \exp\left(-\frac{m}{2\sigma^2 \alpha AT}\right)$$

$$P(k) = \int_0^\infty e^{-m} \frac{m^k}{k!} p(m) dm = \int_0^\infty e^{-m} \frac{m^k}{k!} \frac{1}{2\sigma^2 \alpha TA} e^{-\frac{m}{2\sigma^2 \alpha TA}} dm = \frac{1}{2\sigma^2 \alpha TA + 1} \Big[\frac{2\sigma^2 \alpha TA}{2\sigma^2 \alpha TA + 1}\Big]^k$$

Analytical Description of Photo-Detection

Counting Performance Metrics

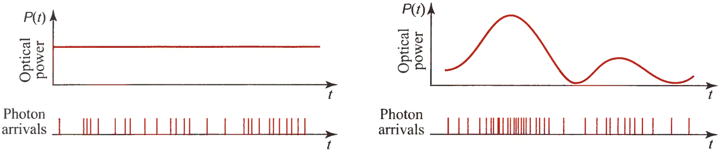

Figure: Average and SNR of the photon number are two main performance metrics of the photo-detector.

- Average carrier count: $\bar{k} = \sum_{k=0}^{\infty} kP(k)$
- Carrier count SNR: $\frac{\bar{k}^2}{\sigma_k^2}$

Example (Intensity-modulated monochromatic point source)

The average carrier count for a photo-detector illuminated by an intensity-modulated monochromatic point source at far field is $\alpha ATI(t)$.

$$P(k) = e^{-m}\frac{m^k}{k!} = e^{-\alpha ATI(t)}\frac{(\alpha ATI(t))^k}{k!}, I(t) = |a(t)|^2$$

$$\bar{k} = \alpha ATI(t)$$

$$\sigma_k^2 = \alpha ATI(t)$$

$$\text{SNR} = \frac{\bar{k}^2}{\sigma_k^2} = \frac{(\alpha ATI(t))^2}{\alpha ATI(t)} = \alpha ATI(t)$$

Short-Term Counting Statistics

Example (Thermal light)

If a zero-mean stationary Gaussian random envelope $n(t) = \text{Re}\{n(t)\} + j\,\text{Im}\{n(t)\}$ illuminates on a photo-detector, the average carrier count is $2\sigma^2\alpha TA$.

$$P(k) = \frac{1}{2\sigma^2\alpha TA + 1}\left[\frac{2\sigma^2\alpha TA}{2\sigma^2\alpha TA + 1}\right]^k$$

$$\bar{k} = 2\sigma^2\alpha TA$$

$$\sigma_k^2 = 2\sigma^2\alpha TA(1 + 2\sigma^2\alpha TA)$$

$$\text{SNR} = \frac{\bar{k}^2}{\sigma_k^2} = \frac{(2\sigma^2\alpha TA)^2}{2\sigma^2\alpha TA(1 + 2\sigma^2\alpha TA)} = \frac{2\sigma^2\alpha TA}{1 + 2\sigma^2\alpha TA}$$

Complex Examples

Gaussian Random Variable

Item	Expression
Probability density function	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in (-\infty, \infty)$
Mean	μ
Variance	σ^2
Characteristic function	$e^{j\mu\omega - \sigma^2\omega^2/2}$

Table: Identities for Gaussian random variable $\mathcal{N}(\mu, \sigma^2)$.

Rayleigh Random Variable

Item	Expression
Probability density function	$\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, x \in [0, \infty)$
Mean	$\sigma\sqrt{\frac{\pi}{2}}$
Variance	$\frac{4-\pi}{2}\sigma^2$
Characteristic function	$1 - \sigma t e^{-\frac{\sigma^2\omega^2}{2}}\sqrt{\frac{\pi}{2}}(\text{erfi}(\frac{\sigma\omega}{\sqrt{2}}))$

Table: Identities for Rayleigh random variable Rayleigh(σ). If $X \sim \mathcal{N}(0, \sigma^2)$, $Y \sim \mathcal{N}(0, \sigma^2)$, and $X \perp\!\!\!\perp Y$, then, $\sqrt{X^2 + Y^2} \sim \text{Rayleigh}(\sigma)$.

- Error function: $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$
- Complementary error function: $\text{erfc}(z) = 1 - \text{erf}(z)$
- Imaginary error function: $\text{erfi}(z) = -j\text{erf}(jz)$

Rice Random Variable

Item	Expression
Probability density function	$\frac{x}{\sigma^2} e^{-\frac{x^2+\nu^2}{2\sigma^2}} I_0(\frac{x\nu}{\sigma^2}), x \in [0, \infty)$
Mean	$\sigma\sqrt{\frac{\pi}{2}} L_{1/2}(-\frac{\nu^2}{2\sigma^2})$
Variance	$2\sigma^2 + \nu^2 - \frac{\pi\sigma^2}{2} L_{1/2}^2(-\frac{\nu^2}{2\sigma^2})$

Table: Identities for Rice random variable Rice(ν, σ). If $X \sim \mathcal{N}(\nu \cos(\theta), \sigma^2)$, $Y \sim \mathcal{N}(\nu \sin(\theta), \sigma^2)$, and $X \perp\!\!\!\perp Y$, then, $\sqrt{X^2 + Y^2} \sim$ Rice(ν, σ).

- Gamma function: $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$
- Modified Bessel function of the first kind: $I_\alpha(x) = \sum_{i=0}^\infty \frac{1}{i!\Gamma(i+\alpha+1)} \left(\frac{x}{2}\right)^{2i+\alpha}$
- Lagurre function of order 1/2: $L_{1/2}(x) = e^{x/2}\left[(1-x)I_0(-\frac{x}{2}) - xI_1(-\frac{x}{2})\right]$

Poisson Random Variable

Item	Expression
Probability mass function	$e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, \cdots$
Mean	λ
Variance	λ
Characteristic function	$e^{\lambda(e^{j\omega}-1)}$

Table: Identities for Poisson random variable Poisson(λ).

Bose Random Variable

Item	**Expression**
Probability mass function	$\frac{1}{p+1}(\frac{p}{p+1})^k, k = 0, 1, \cdots$
Mean	p
Variance	$p(p+1)$
Characteristic function	$\frac{1}{p+1-pe^{j\omega}}$

Table: Identities for Bose random variable Bose(p).

Lagurre Random Variable

Item	Expression
Probability mass function	$\frac{b^k}{(1+b)^{k+c+1}} e^{-\frac{a}{1+b}} L_k^c\left[-\frac{a}{b(1+b)}\right], k=0,1,\cdots$
Mean	$(c+1)b+a$
Variance	$(c+1)(b+1)b+a(2b+1)$
Characteristic function	$\left[\frac{1}{1+b(1-e^{j\omega})}\right]^{c+1} \exp\left(-\frac{a(1-e^{j\omega})}{1+b(1-e^{j\omega})}\right)$

Table: Identities for Lagurre random variable Lagurre(a, b, c).

- Generalized Lagurre polynomial of integer degree c: $L_k^c(x) = \sum_{i=0}^{k} \binom{c+k}{k-i} \frac{(-x)^i}{i!}$
- Limiting froms of Lagurre distribution:
  - Lagurre$(0, b, 0) \equiv$ Bose(b)
  - Lagurre$(a, b \to 0, c \to \infty) \equiv$ Poisson$(a+bc)$
  - Lagurre$(a, 0, 0) \equiv$ Poisson(a)

Counting Statistics

Example (Characteristic function of counting variable)

Characteristic function of counting variable $\Psi_k(j\omega)$ is obtained by replacing ω with $-j(e^{j\omega}-1)$ in the characteristic function of the carrier generation variable $\Psi_m(j\omega)$.

$$\Psi_k(j\omega) = \mathcal{E}\{e^{j\omega k}\} = \mathcal{E}_m\{\mathcal{E}\{e^{j\omega k}\,|\,m\}\} = \mathcal{E}_m\{e^{m(e^{j\omega}-1)}\} = \mathcal{E}_m\{e^{j[-j(e^{j\omega}-1)]}\} = \Psi_m\big(j[-j(e^{j\omega}-1)]\big)$$

Short-Term Counting Statistics

Example (Random field with Gaussian decomposition)

For a random field decomposed to its Karhunen-Loeve modes with Gaussian random coefficients, the characteristic function of the overall carrier count is the product of the characteristic functions of carrier count for each mode.

$$m_v = \alpha \int_A \int_0^T I(\boldsymbol{r}, t) dtdA = \alpha \int_A \int_0^T |f_r(\boldsymbol{r}, t)|^2 dtdA = \alpha \int_A \int_0^T |a_r(\boldsymbol{r}, t)|^2 dtdA$$

$$= \alpha \int_A \int_0^T \sum_{i=1}^{\infty} a_i \Phi_i(\boldsymbol{r}, t) \sum_{j=1}^{\infty} a_j^* \Phi_j^*(\boldsymbol{r}, t) dtdA = \alpha \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i a_j^* \int_A \int_0^T \Phi_i(\boldsymbol{r}, t) \Phi_j^*(\boldsymbol{r}, t) dtdA = \alpha \sum_{i=1}^{\infty} |a_i|^2$$

$$a_i \sim \mathcal{N}(\mu_i, \sigma_i^2), \mathcal{E}\{a_i a_j^*\} = \mu_i \mu_j, i \neq j \Rightarrow a_i \perp\!\!\!\perp a_j, i \neq j$$

$$\Psi_m(j\omega) = \mathcal{E}\{e^{j\omega m}\} = \mathcal{E}\{e^{j\omega\alpha \sum_{i=1}^{\infty} |a_i|^2}\} = \prod_{i=1}^{\infty} \mathcal{E}\{e^{j\omega\alpha |a_i|^2}\} = \prod_{i=1}^{\infty} \mathcal{E}\{e^{j\omega m_i}\} = \prod_{i=1}^{\infty} \Psi_{m_i}(j\omega)$$

$$\Psi_k(j\omega) = \Psi_m\left(j[-j(e^{j\omega} - 1)]\right) = \prod_{i=1}^{\infty} \Psi_{m_i}\left(j[-j(e^{j\omega} - 1)]\right)$$

Example (Deterministic field decomposition)

For a deterministic field decomposed to its modes, the corresponding counting process has Poisson distribution.

$$m_v = \alpha \int_A \int_0^T I(\boldsymbol{r}, t) dt dA = \alpha \int_A \int_0^T |f_r(\boldsymbol{r}, t)|^2 dt dA = \alpha \int_A \int_0^T |a_r(\boldsymbol{r}, t)|^2 dt dA$$

$$= \alpha \int_A \int_0^T \sum_{i=1}^{\infty} a_i \Phi_i(\boldsymbol{r}, t) \sum_{j=1}^{\infty} a_j^* \Phi_j^*(\boldsymbol{r}, t) dt dA = \alpha \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i a_j^* \int_A \int_0^T \Phi_i(\boldsymbol{r}, t) \Phi_j^*(\boldsymbol{r}, t) dt dA = \alpha \sum_{i=1}^{\infty} |a_i|^2$$

$$P(k) = e^{-m_v} \frac{m_v^k}{k!} \Rightarrow \text{SNR} = \frac{\bar{k}^2}{\sigma_k^2} = m_v = \alpha \sum_{i=1}^{\infty} |a_i|^2 = \alpha \sum_{i=1}^{\infty} E_i$$

$$f_{m_i}(m) = \delta(m - \alpha |a_i|^2) \Rightarrow \Psi_{m_i} = e^{j\omega\alpha|a_i|^2}$$

$$\Psi_k(j\omega) = \prod_{i=1}^{\infty} \Psi_{m_i}(j[-j(e^{j\omega} - 1)]) = \prod_{i=1}^{\infty} e^{(e^{j\omega}-1)\alpha|a_i|^2} = e^{(e^{j\omega}-1)\alpha \sum_{i=1}^{\infty} |a_i|^2} = e^{(e^{j\omega}-1)m_v}$$

$$P(k) = e^{-m_v} \frac{m_v^k}{k!}$$

Example (Monochromatic field plus thermal light)

Let a zero-mean stationary Gaussian random envelope $n(t) = \mathrm{Re}\{n(t)\} + j\,\mathrm{Im}\{n(t)\}$ pollute a deterministic slow-varying monochromatic envelope $s(t)$ illuminating on a photo-detector. Then, the number of carriers has Laguerre distribution.

$$f_r(t, \mathbf{r}) = [s(t) + n(t)]e^{j2\pi\nu t}, \mathbf{r} \in A, \quad \mathrm{Re}\{n(t)\} \sim \mathcal{N}(0, \sigma^2), \mathrm{Im}\{n(t)\} \sim \mathcal{N}(0, \sigma^2), \mathrm{Re}\{n(t)\} \perp\!\!\!\perp \mathrm{Im}\{n(t)\})$$

$$|f_r(t, \mathbf{r})| \sim \mathrm{Rice}(\sigma, I) = \frac{a}{\sigma^2} e^{-\frac{a^2+I(t)}{2\sigma^2}} I_0(\frac{a\sqrt{I(t)}}{\sigma^2})$$

$$m_v = \alpha \int_A \int_T |f_r(t, \mathbf{r})|^2 d\mathbf{r} dt \approx \alpha AT |f_r(t, \mathbf{r})|^2$$

$$p(m) = \frac{1}{2\sigma^2\alpha TA} e^{-\frac{m+\alpha TAI(t)}{2\sigma^2\alpha TA}} I_0(\frac{\sqrt{m\alpha TAI(t)}}{\sigma^2\alpha TA})$$

$$P(k) = \int_0^\infty e^{-m} \frac{m^k}{k!} p(m) dm = \int_0^\infty e^{-m} \frac{m^k}{k!} \frac{1}{2\sigma^2\alpha TA} e^{-\frac{m+\alpha TAI(t)}{2\sigma^2\alpha TA}} I_0(\frac{\sqrt{m\alpha TAI(t)}}{\sigma^2\alpha TA}) dm$$

$$= \int_0^\infty e^{-m} \frac{m^k}{k!} \frac{1}{b} e^{-\frac{m+a}{b}} I_0(\frac{2\sqrt{ma}}{b}) dm = e^{-\frac{a}{b}} \frac{1}{bk!} \sum_{j=0}^{\infty} \frac{a^j}{(b)^{2j}(j!)^2} \int_0^\infty m^{k+j} e^{-\frac{m(1+b)}{b}} dm$$

Example (Monochromatic field plus thermal light (cont.))

Let a zero-mean stationary Gaussian random envelope $n(t) = \text{Re}\{n(t)\} + j\,\text{Im}\{n(t)\}$ pollute a deterministic slow-varying monochromatic envelope $s(t)$ illuminating on a photo-detector. Then, the number of carriers has Laguerre distribution.

$$
\begin{aligned}
P(k) &= e^{-\frac{a}{b}} \frac{1}{bk!} \sum_{j=0}^{\infty} \frac{a^j}{(b)^{2j}(j!)^2} \Big(\frac{b}{1+b}\Big)^{k+j+1} \int_0^{\infty} x^{k+j+1-1} e^{-x} dx \\
&= \frac{e^{-\frac{a}{b}}}{1+b} \Big(\frac{b}{1+b}\Big)^k \sum_{j=0}^{\infty} \Big[\frac{a}{b(b+1)}\Big]^j \frac{\Gamma(j+k+1)}{(j!)^2 k!} = \frac{e^{-\frac{a}{b}}}{1+b} \Big(\frac{b}{1+b}\Big)^k \sum_{j=0}^{\infty} \Big[\frac{a}{b(b+1)}\Big]^j \frac{(k+j)!}{(j!)^2 k!} \\
&= \frac{e^{-\frac{a}{b}}}{1+b} \Big(\frac{b}{1+b}\Big)^k {}_1F_1(k+1, 1, \frac{a}{b(b+1)}) = \frac{e^{-\frac{a}{b}}}{1+b} \Big(\frac{b}{1+b}\Big)^k e^{\frac{a}{b(b+1)}} {}_1F_1(-k, 1, -\frac{a}{b(b+1)}) \\
&= \frac{e^{-\frac{a}{b}}}{1+b} \Big(\frac{b}{1+b}\Big)^k e^{\frac{a}{b(b+1)}} L_k^0(-\frac{a}{b(b+1)}) = \frac{b^k}{(1+b)^{k+1}} e^{-\frac{a}{1+b}} L_k^0(-\frac{a}{b(b+1)}) \\
&= \frac{1}{2\sigma^2\alpha TA + 1} \Big(\frac{2\sigma^2\alpha TA}{1+2\sigma^2\alpha TA}\Big)^k \exp\left(-\frac{\alpha ATI(t)}{1+2\sigma^2\alpha AT}\right) L_k^0\Big[\frac{-I(t)/(2\sigma^2)}{1+2\sigma^2\alpha AT}\Big]
\end{aligned}
$$

Short-Term Counting Statistics

Example (Monochromatic field plus thermal light (cont.))

Let a zero-mean stationary Gaussian random envelope $n(t) = \text{Re}\{n(t)\} + j\,\text{Im}\{n(t)\}$ pollute a deterministic slow-varying monochromatic envelope $s(t)$ illuminating on a photo-detector. Then, the number of carriers has Laguerre distribution.

$$k \sim \text{Lagurre}(\alpha ATI(t), 2\sigma^2\alpha AT, 0)$$

$$I(t) = 0 \Rightarrow k \sim \text{Lagurre}(0, 2\sigma^2\alpha AT, 0) \equiv \text{Bose}(2\sigma^2\alpha AT)$$

$$\sigma^2 = 0 \Rightarrow k \sim \text{Lagurre}(\alpha ATI(t), 0, 0) \equiv \text{Poisson}(\alpha ATI(t))$$

$$\text{SNR} = \frac{\bar{k}^2}{\sigma_k^2} = \frac{(2\sigma^2\alpha AT + \alpha ATI(t))^2}{2\sigma^2\alpha AT(2\sigma^2\alpha AT + 1) + \alpha ATI(t)(4\sigma^2\alpha AT + 1)} = \frac{\alpha AT(2\sigma^2 + I(t))^2}{2\sigma^2(2\sigma^2\alpha AT + 1) + I(t)(4\sigma^2\alpha AT + 1)}$$

Short-Term Counting Statistics

Example (Decomposed deterministic signal plus Gaussian noise)

Let a zero-mean stationary Gaussian random envelope with the coherence function $R_n(t_1, \boldsymbol{r}_1; t_2, \boldsymbol{r}_2)$ pollute a deterministic slow-varying monochromatic envelope illuminating on a photo-detector. Then, the characteristic function of the overall carrier count is the product of the characteristic functions of carrier count for each mode.

$$f_r(t, \boldsymbol{r}) = f_s(t, \boldsymbol{r}) + f_n(t, \boldsymbol{r}) = [s(t, \boldsymbol{r}) + n(t, \boldsymbol{r})]e^{j2\pi\nu t}$$

$$f_r(t, \boldsymbol{r}) = \sum_{i=0}^{\infty} a_i \Phi_i(t, \boldsymbol{r}), \quad a_i = s_i + n_i = a_R + ja_I, a_R \sim \mathcal{N}(\mathrm{Re}\{s_i\}, \lambda_i/2), a_I \sim \mathcal{N}(\mathrm{Im}\{s_i\}, \lambda_i/2), a_R \perp\!\!\!\perp a_I$$

$$|a_i| \sim f_{|a_i|}(a) = \mathrm{Rice}(\sqrt{\lambda_i/2}, |s_i|) = \frac{2a}{\lambda_i} e^{-\frac{a^2+|s_i|^2}{\lambda_i}} I_0(\frac{2a|s_i|}{\lambda_i})$$

$$m_i = \alpha |a_i|^2 \sim f(m) = \frac{1}{\alpha\lambda_i} \exp\left[-\frac{|s_i|^2 + \frac{m}{\alpha}}{\lambda}\right] I_0\left[\frac{2|s_i|\sqrt{\frac{m}{\alpha}}}{\lambda_i}\right]$$

$$\Psi_{m_i}(j\omega) = \frac{1}{1-\alpha\lambda_i j\omega} \exp\left[\frac{\alpha|s_i|^2 j\omega}{1-\alpha\lambda_i j\omega}\right] \Rightarrow \Psi_m(j\omega) = \prod_{i=0}^{\infty} \Psi_{m_i}(j\omega) = \prod_{i=0}^{\infty} \frac{1}{1-\alpha\lambda_i j\omega} \exp\left[\frac{\alpha|s_i|^2 j\omega}{1-\alpha\lambda_i j\omega}\right]$$

$$\Psi_k(j\omega) = \prod_{i=1}^{\infty} \Psi_{m_i}(j[-j(e^{j\omega}-1)]) = \prod_{i=0}^{\infty} \frac{1}{1+\alpha\lambda_i(1-e^{j\omega})} \exp\left[\frac{-\alpha|s_i|^2(1-e^{j\omega})}{1+\alpha\lambda_i(1-e^{j\omega})}\right]$$

Short-Term Counting Statistics

Example (Decomposed deterministic signal plus Gaussian noise (cont.))

Let a zero-mean stationary Gaussian random envelope with the coherence function $R_n(t_1, \boldsymbol{r}_1; t_2, \boldsymbol{r}_2)$ pollute a deterministic slow-varying monochromatic envelope illuminating on a photo-detector. Then, the characteristic function of the overall carrier count is the product of the characteristic functions of carrier count for each mode.

$$\Psi_k(j\omega) = \prod_{i=0}^{\infty} \frac{1}{1+\alpha\lambda_i(1-e^{j\omega})} \exp\Big[\frac{-\alpha|s_i|^2(1-e^{j\omega})}{1+\alpha\lambda_i(1-e^{j\omega})}\Big]$$

$$P_{k_i}(k) = \frac{(\alpha\lambda_i)^k}{(1+\alpha\lambda_i)^{k+1}} \exp\Big[-\frac{\alpha|s_i|^2}{1+\alpha\lambda_i}\Big] L_k\Big[-\frac{|s_i|^2}{\lambda_i(1+\alpha\lambda_i)}\Big] \sim \text{Lagurre}(\alpha|s_i|^2, \alpha\lambda_i, 0)$$

$$k = \sum_{i=0}^{\infty} k_i$$

Example (Decomposed Wiener noise process)

Let a zero-mean stationary Gaussian random envelope with the wiener coherence function illuminate on a photo-detector. Then, the mean carrier count is $0.5\alpha\sigma^2 T^2$.

$$\cos(z) = \prod_{i=0}^{\infty} \left(1 - \frac{4z^2}{\pi^2(2i-1)^2}\right), \quad \sec(z) = \sum_{i=0}^{\infty} \frac{(-1)^i \epsilon_{2i}}{(2i)!} z^{2i}$$

$$s_i = 0, \quad \lambda_i = \frac{4\sigma^2 T^2}{(2i-1)^2\pi^2}, i = 1, 2, \cdots$$

$$\Psi_m(j\omega) = \prod_{i=0}^{\infty} \frac{1}{1 - \alpha\lambda_i j\omega} = \prod_{i=0}^{\infty} \frac{1}{1 - \frac{4\alpha\sigma^2 T^2 j\omega}{(2i-1)^2\pi^2}} = \sec\left(\sigma T \sqrt{\alpha j\omega}\right) = \sum_{i=0}^{\infty} \frac{(-1)^i \epsilon_{2i}}{(2i)!} (j\alpha\sigma^2 T^2 \omega)^i$$

$$\Psi_k(j\omega) = \prod_{i=0}^{\infty} \frac{1}{1 + \alpha\lambda_i(1 - e^{j\omega})} = \sum_{i=0}^{\infty} \frac{(-1)^i \epsilon_{2i}}{(2i)!} [\alpha\sigma^2 T^2 (e^{j\omega} - 1)]^i$$

$$\bar{k} = \frac{\Psi'_k(j0)}{j} = \frac{1}{2}\alpha\sigma^2 T^2, \quad \bar{k^2} = \frac{\Psi''_k(j0)}{j^2} = \frac{1}{2}\alpha\sigma^2 T^2 + \frac{5}{12}\alpha^2\sigma^4 T^4$$

$$\sigma_k^2 = \frac{1}{2}\alpha\sigma^2 T^2 + \frac{1}{6}\alpha^2\sigma^4 T^4$$

$$\text{SNR} = \frac{\frac{1}{2}\alpha\sigma^2 T^2}{1 + \frac{1}{3}\alpha\sigma^2 T^2}$$

Short-Term Counting Statistics

Example (Decomposed Wiener noise process (cont.))

Let a zero-mean stationary Gaussian random envelope with the wiener coherence function illuminate on a photo-detector. Then, the mean carrier count is $0.5\alpha\sigma^2 T^2$.

$$\Psi_m(j\omega) = \prod_{i=0}^{\infty} \frac{1}{1-\alpha\lambda_i j\omega} = \prod_{i=0}^{\infty} \frac{1}{1-\frac{4\alpha\sigma^2 T^2 j\omega}{(2i-1)^2\pi^2}} = \sec\left(\sigma T\sqrt{\alpha j\omega}\right) = \sum_{i=0}^{\infty} \frac{(-1)^i \epsilon_{2i}}{(2i)!}(j\alpha\sigma^2 T^2\omega)^i$$

$$P_k(k) = \int_0^{\infty} \frac{m^k}{k!} e^{-m} P_m(m)dm = \int_0^{\infty} \frac{m^k}{k!} P_m(m) \sum_{l=0}^{\infty} \frac{(-m)^l}{l!} dm = \frac{1}{k!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathcal{E}\{m^{k+l}\}$$

$$P_k(k) = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2k+2l)!} \binom{k+l}{l} |\epsilon_{2k+2i}| (\alpha\sigma^2 T^2)^{k+l}$$

Example (Decomposed colored noise process)

Let a zero-mean stationary Gaussian random envelope with the colored coherence function illuminate on a photo-detector. Then, the mean carrier count is approximately $(2BT+1)\alpha N_0$.

$$R_t(t_1,t_2) = N_0\text{sinc}[2B(t_1-t_2)] \Rightarrow \lambda\int_{-0.5T}^{0.5T} N_0\text{sinc}[2B(t_1-t_2)]g(t_1)dt_1 = \lambda g(t_2)$$

$$(1-t^2)f''(t) - 2tf'(t) + (\lambda_n - c^2t^2)f(t) = 0, c = \pi BT, |t| < 1$$

$$N_0T\left[R_{0n}^{(1)}(c,1)\right]^2 S_{0n}^{(1)}(c,t_2) = \int_{-1}^{1}\text{sinc}\left(\frac{c(t_1-t_2)}{\pi}\right)S_{0n}^{(1)}(c,t_1)dt_1, \quad |t| < 1$$

$$\lambda_n = N_0T\left[R_{0n}^{(1)}(\pi BT,1)\right]^2 \approx \begin{cases} N_0, & n = 0,1,\cdots,\lfloor 2BT\rfloor \\ 0, & n > \lfloor 2BT\rfloor + 1 \end{cases}, \quad n = 0,1,\cdots$$

$$\Psi_k(j\omega) = \prod_{i=0}^{\infty}\frac{1}{1+\alpha\lambda_i(1-e^{j\omega})} \approx \left[\frac{1}{1-\alpha N_0(e^{j\omega}-1)}\right]^{2BT+1}$$

$$\bar{k} = \frac{\Psi_k'(j0)}{j} = \alpha N_0(2BT+1), \quad \bar{k^2} = \frac{\Psi_k''(j0)}{j^2} = \alpha N_0(\alpha N_0+1)(2BT+1) + \alpha^2N_0^2(2BT+1)^2$$

$$\sigma_k^2 = \alpha N_0(\alpha N_0+1)(2BT+1)$$

$$\text{SNR} = \frac{\alpha N_0}{1+\alpha N_0}(2BT+1)$$

Example (Decomposed colored noise process (cont.))

Let a zero-mean stationary Gaussian random envelope with the colored coherence function illuminate on a photo-detector. Then, the mean carrier count is approximately $(2BT+1)\alpha N_0$.

$$\Psi_k(j\omega) \approx \Big[\frac{1}{1-\alpha N_0(e^{j\omega}-1)}\Big]^{2BT+1}$$

$$P_k(k) = \binom{2BT+k}{k}\Big(\frac{1}{1+\alpha N_0}\Big)^{2BT+1}\Big(\frac{\alpha N_0}{1+\alpha N_0}\Big)^{k}$$

$$2BT \ll 1 \Rightarrow P_k(k) \approx \text{Bose}(\alpha N_0)$$

$$2BT \gg 1 \Rightarrow P_k(k) \approx \text{Poisson}(2BT\alpha N_0)$$

Example (Decomposed deterministic signal plus colored noise process)

Let a zero-mean stationary Gaussian random envelope with the colored coherence function pollute a deterministic envelope illuminating on a photo-detector. Then, the mean carrier count is approximately $(2BT+1)\alpha N_0 + \alpha E_s$.

$$\Psi_k(j\omega) = \prod_{i=1}^{2BT+1} \frac{1}{1+\alpha N_0(1-e^{j\omega})} \exp\left[\frac{-\alpha(1-e^{j\omega})|s_i|^2}{1+\alpha N_0(1-e^{j\omega})}\right]$$

$$\Psi_k(j\omega) = \left[\frac{1}{1+\alpha N_0(1-e^{j\omega})}\right]^{2BT+1} \exp\left[\frac{-\alpha(1-e^{j\omega})\sum_{i=0}^{2BT}|s_i|^2}{1+\alpha N_0(1-e^{j\omega})}\right], E_s = \sum_{i=0}^{2BT}|s_i|^2$$

$$P_k(k) \sim \text{Lagurre}(\alpha E_s, \alpha N_0, 2BT)$$

$$\bar{k} = \alpha N_0(2BT+1) + \alpha E_s$$

$$\sigma_k^2 = \alpha N_0(\alpha N_0+1)(2BT+1) + \alpha(1+2\alpha N_0)E_s$$

$$\text{SNR} = \frac{[\alpha N_0(2BT+1)+\alpha E_s]^2}{\alpha N_0(\alpha N_0+1)(2BT+1)+\alpha(1+2\alpha N_0)E_s}$$

Example (Decomposed deterministic signal plus colored noise process (cont.))

Let a zero-mean stationary Gaussian random envelope with the colored coherence function pollute a deterministic envelope illuminating on a photo-detector. Then, the mean carrier count is approximately $(2BT+1)\alpha N_0 + \alpha E_s$.

$$
\begin{aligned}
& k \sim \text{Lagurre}(\alpha E_s, \alpha N_0, 2BT) \\
& E_s = 0, 2BT \ll 1 \Rightarrow k \sim \text{Lagurre}(0, \alpha N_0, 0) \equiv \text{Bose}(\alpha N_0) \\
& N_0 \ll 0, 2BT \ll 1 \Rightarrow k \sim \text{Lagurre}(\alpha E_s, 0, 0) \equiv \text{Poisson}(\alpha E_s) \\
& N_0 \ll 0, 2BT \gg 1 \Rightarrow k \sim \text{Lagurre}(\alpha E_s, \alpha N_0 \to 0, 2BT \to \infty) \equiv \text{Poisson}(\alpha E_s + \alpha N_0 2BT)
\end{aligned}
$$

Carrier Multiplication Counting Statistics

Example (Photo-multiplier Tube (PMT))

The secondary carrier count in a PMT with mean gain $\bar{g}$ and spreading factor ζ has approximately Gaussian-shaped discrete distribution if the primary carrier count has Poisson distribution.

$$P(k_2|k_1) = C \exp\left[-\frac{(k_2 - \bar{g}k_1)^2}{2(\zeta\bar{g}k_1)^2}\right]$$

$$p(k_2) = \sum_{k_1=0}^{\infty} P(k_2|k_1)p(k_1)$$

$$\bar{k}_1 \gg 1 \Rightarrow P(k_2) = C \exp\left[-\frac{(k_2 - \bar{g}\bar{k}_1)^2}{2(\zeta\bar{g}\bar{k}_1)^2}\right]$$

Carrier Multiplication Counting Statistics

Example (Avalanche Photo-Diode (APD))

The secondary carrier count in a APD with mean gain $\bar{g}$ and avalanche ionization γ has approximately Gaussian-shaped discrete distribution if the primary carrier count has Poisson distribution.

$$P(k_2|k_1) = \frac{k_1\Gamma(\frac{k_2}{1-\gamma}+1)}{k_2(k_2-k_1)!\Gamma(\frac{\gamma k_2}{1-\gamma}+1+k_1)}\left[\frac{1+\gamma(\bar{g}-1)}{\bar{g}}\right]^{\frac{k_1+\gamma k_2}{1-\gamma}}\left[\frac{(1-\gamma)(\bar{g}-1)}{\bar{g}}\right]^{k_2-k_1}$$

$$p(k_2) = \sum_{k_1=0}^{\infty} P(k_2|k_1)p(k_1)$$

$$\bar{g}\sqrt{\bar{k}_1/F} \gg 1 \Rightarrow P(k_2) = \frac{1}{\sqrt{2\pi(\bar{g}\bar{k}_1)^2(F-1)}}\exp\left[-\frac{(k_2-\bar{g}\bar{k}_1)^2}{2(\bar{g}\bar{k}_1)^2(F-1)}\right], \quad F = \gamma\bar{g} + (2-\frac{1}{\bar{g}})(1-\gamma)$$

The End