Shot Noise

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Overview

Physical Description of Shot Noise

Current Pulse

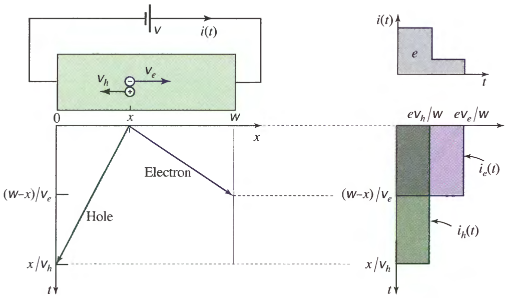


Figure: Electron current and hole current.

- Mean drift velocity: $v = \mu E = \frac{a\tau_{col}}{m} E$
- Carrier mobility: $\mu = \frac{a\tau_{col}}{m}$.
- Ramo's formula: $-QEdx = -Q\frac{V}{w}dx = i(t)Vdt$.
- Ramo's formula: $i(t) = -\frac{Q}{w}v(t)$.
- Transit-time spread: $x/v_h, \quad (W-x)/v_e, \quad v_h < v_e$.

Current Pulse

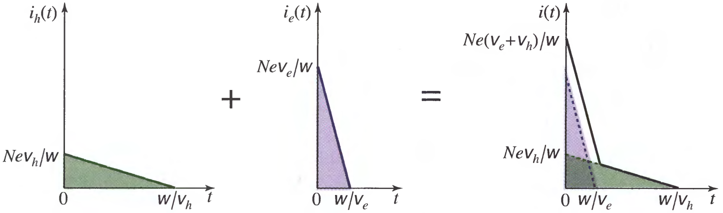


Figure: Impulse response function for a uniformly illuminated detector subject to transit-time spread.

- Total current: $i(t) = i_h(t) + i_e(t)$
- Generated charge: Ne

Characteristic Curve

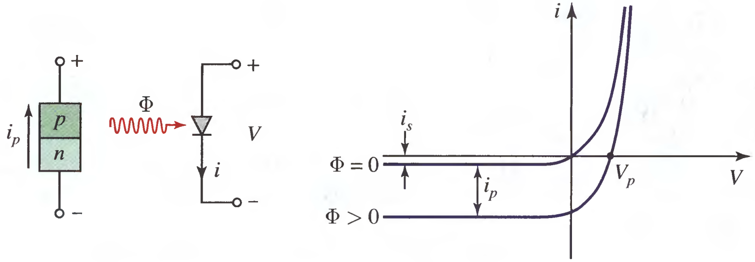


Figure: Generic photodiode and its i-V relation. .

- Photo-diode characteristic curve: $i = i_s[\exp(eV/kT) - 1] - i_p$
- Dark current: i_s arisen from thermally-excited random generation of electrons-hole pairs

Statistical Description of Shot Noise

Shot Noise

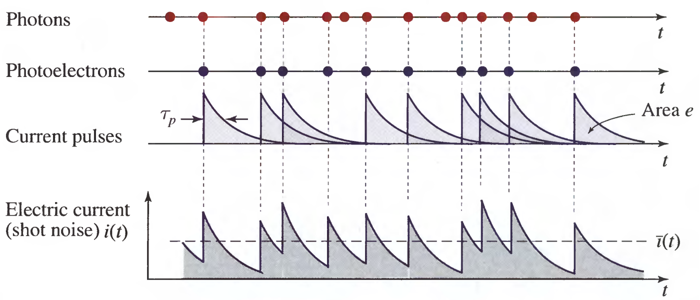


Figure: The photocurrent induced in a photodetector circuit comprises a superposition of current pulses, each associated with a detected photon. The individual pulses illustrated are exponentially decaying step functions but they can assume an arbitrary shape.

- Photocurrent shot noise process: $i(t) = \sum_{j=1}^{k(0,t)} g_j h(t - z_j)$
- Current pulse: $\int_0^\infty h(t)dt = \int_0^\tau h(t)dt = e$
- Mean count: $m_v = \alpha \int_0^T \int_A I(t, \boldsymbol{r}) d\boldsymbol{r} dt = \int_0^T n(t)dt$
- Count intensity: $n(t) = \alpha \int_A I(t, \boldsymbol{r}) d\boldsymbol{r}$

Occurrence Times

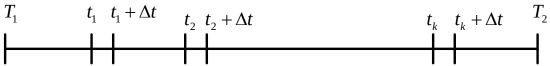


Figure: Time axis model for Poisson shot noise with deterministic incident intensity.

- Disjoint intervals:

$$P(z_1 \in (t_1, t_1 + \Delta t), \cdots, z_k \in (t_k, t_k + \Delta t), k)$$
$$= P(0, t_1 - T_1)P(1, t_1 + \Delta t - t_1) \cdots P(0, t_k - (t_k + \Delta t))P(1, t_k + \Delta t - t_k)P(1, T_2 - (t_k + \Delta t))$$

$$P(0, t_1 - T_1) = \exp\left(-\int_{T_1}^{t_1} n(t)dt\right)$$

$$P(1, t_1 + \Delta t - t_1) = \exp\left(-\int_{t_1}^{t_1+\Delta t} n(t)dt\right)\int_{t_1}^{t_1+\Delta t} n(t)dt \approx \exp\left(-\int_{t_1}^{t_1+\Delta t} n(t)dt\right)n(t_1)\Delta t$$

$$\vdots$$

$$P(z_1 \in (t_1, t_1 + \Delta t), \cdots, z_k \in (t_k, t_k + \Delta t), k) = \exp\left(-\int_{T_1}^{T_2} n(t)dt\right)(\Delta t)^k \prod_{j=1}^{k} n(t_j)$$

Occurrence Times

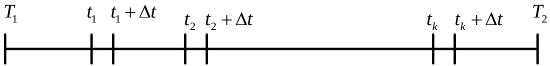


Figure: Time axis model for Poisson shot noise with deterministic incident intensity.

- Disjoint intervals:
  $P(z_1 \in (t_1, t_1 + \Delta t), \cdots, z_k \in (t_k, t_k + \Delta t), k) = \exp\left(-\int_{T_1}^{T_2} n(t)dt\right)(\Delta t)^k \prod_{j=1}^{k} n(t_j)$
- Probability definition:
  $P(z_1 \in (t_1, t_1 + \Delta t), \cdots, z_k \in (t_k, t_k + \Delta t, k) = k!P(z_1 = t_1, \cdots, z_k = t_k)(\Delta t)^k$
- Joint density of occurrence times: $P(z_1, \cdots, z_k, k) = \frac{1}{k!}\exp\left(-\int_{T_1}^{T_2} n(t)dt\right)\prod_{j=1}^{k} n(z_j)$
- Count probability: $P_k(k) = \frac{1}{k!}\exp\left(-\int_{T_1}^{T_2} n(t)dt\right)\left(\int_{T_1}^{T_2} n(t)dt\right)^k$
- Conditional joint density of occurrence times:
  $p(z_1, z_2, \cdots, z_k|k) = \frac{\prod_{j=1}^{k} n(z_j)}{(\int_{T_1}^{T_2} n(t)dt)^k} = \prod_{j=1}^{k} \frac{n(z_j)}{m_v}$

Occurrence Times

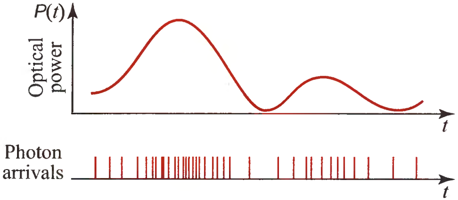


Figure: Occurrence times for Poisson shot noise with deterministic incident intensity.

- Condition joint density of occurrence times:
  $p(z_1, z_2, \cdots, z_k|k) = \frac{\prod_{j=1}^{k} n(z_j)}{(\int_{T_1}^{T_2} n(t)dt)^k} = \prod_{j=1}^{k} \frac{n(z_j)}{m_v}$
- Independent occurrence time: $p_{z_i}(z) = \frac{n(z)}{m_v}$

Occurrence Times

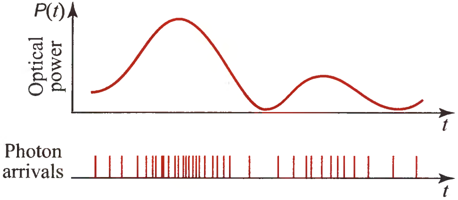


Figure: Occurrence times for conditional Poisson shot noise with stochastic incident intensity.

- Conditional joint density of occurrence times:

$p(z_1, z_2, \cdots, z_k|k, n(t)) = \frac{\prod_{j=1}^{k} n(z_j)}{(\int_{T_1}^{T_2} n(t)dt)^k} = \prod_{j=1}^{k} \frac{n(z_j)}{m_v}$

- Marginal conditional joint density of occurrence times:

$$p(z_1, z_2, \cdots, z_k|k) = \frac{1}{m_v^k} \int \int \cdots \int n(z_1) \cdots n(z_k) P_z(n(z_1), \cdots, n(z_k)) dn(z_1) \cdots dn(z_k)$$

$$p(z_1, z_2, \cdots, z_k|k) = \frac{\mathcal{E}_n\{\prod_{j=1}^{k} n(z_j)\}}{m_v^k}$$

Analytical Description of Shot Noise

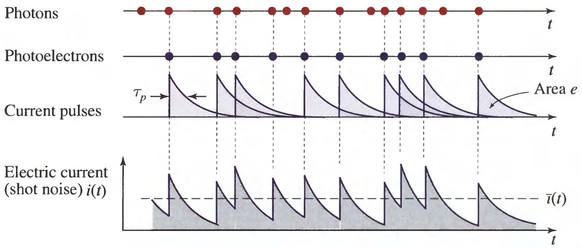


Figure: The photocurrent for a Poisson shot noise with deterministic incident intensity.

- Mean of shot noise:

$$\mathcal{E}\{i(t)\} = \mathcal{E}_k\{\mathcal{E}\{i(t)|k\}\} = \mathcal{E}_k\{\mathcal{E}\{\sum_{j=1}^{k(0,t)} g_j h(t-z_j)|k\}\} = \mathcal{E}_k\{\sum_{j=1}^{k(0,t)} \mathcal{E}\{g_j h(t-z_j)|k\}\}$$

$$= \mathcal{E}_k\{\sum_{j=1}^{k(0,t)} \mathcal{E}\{g_j|k\}\mathcal{E}\{h(t-z_j)|k\}\} = \mathcal{E}_k\{\mathcal{E}\{g_j|k\}\mathcal{E}\{h(t-z_j)|k\}k(0,t)\}$$

$$= \mathcal{E}\{g\}\mathcal{E}\{h(t-z)\}\mathcal{E}_k\{k(0,t)\} = m_v \bar{g}\mathcal{E}\{h(t-z)\} = \bar{g}\int_{-\infty}^{t} h(t-z)n(z)dz$$

Mean of Shot Noise

Example (Mean of shot noise)

If $h(t) = \frac{e}{\tau}[u(t) - u(t-\tau)]$, then $\mathcal{E}\{i(t)\} = \bar{g}\frac{e}{\tau}\bar{k}(t-\tau, t)$, where $\bar{k}(t-\tau, t)$ is the average number of carriers in interval $[t-\tau, t]$.

$$\mathcal{E}\{i(t)\} = \bar{g}\int_{-\infty}^{t} h(t-z)n(z)dz = \bar{g}\int_{t-\tau}^{t} \frac{e}{\tau}n(z)dz = \bar{g}\frac{e}{\tau}\bar{k}(t-\tau, t)$$

Example (Mean of shot noise)

For a photo-detector with infinite bandwidth having $h(t) = e\delta(t)$, $\mathcal{E}\{i(t)\} = \bar{g}en(t)$.

$$\mathcal{E}\{i(t)\} = \bar{g}\int_{-\infty}^{t} h(t-z)n(z)dz = \bar{g}\int_{-\infty}^{t} e\delta(t-z)n(z)dz = \bar{g}en(t)$$

Mean Square of Shot Noise

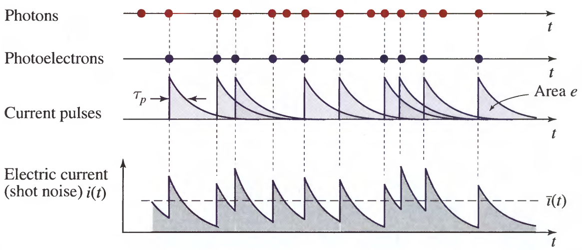


Figure: The photocurrent for a Poisson shot noise with deterministic incident intensity.

- Mean square of shot noise:

$$\mathcal{E}\{i^2(t)\} = \mathcal{E}_k\{\mathcal{E}\{i^2(t)|k\}\} = \mathcal{E}_k\{\mathcal{E}\{[\sum_{j=1}^{k(0,t)} g_j h(t-z_j)]^2|k\}\}$$

$$= \mathcal{E}_k\{\mathcal{E}\{\sum_{j=1}^{k(0,t)} g_j^2 h^2(t-z_j) + \sum_{j,i=1,j\neq i}^{k(0,t)} g_j g_i h(t-z_j)h(t-z_i)|k\}\}$$

$$= \mathcal{E}_k\{k(0,t)\mathcal{E}\{g^2\}\mathcal{E}\{h^2(t-z)\} + [k^2(0,t) - k(0,t)]\mathcal{E}\{g\}\mathcal{E}\{g\}\mathcal{E}\{h(t-z)\}\mathcal{E}\{h(t-z)\}\}$$

$$= \mathcal{E}_k\{k(0,t)\}\mathcal{E}\{g^2\}\mathcal{E}\{h^2(t-z)\} + \mathcal{E}_k\{k^2(0,t) - k(0,t)\}\mathcal{E}\{g\}\mathcal{E}\{g\}\mathcal{E}\{h(t-z)\}\mathcal{E}\{h(t-z)\}$$

$$= \bar{g^2}\int_{-\infty}^{t} h^2(t-z)n(z)dz + \mathcal{E}\{i(t)\}^2$$

Variance of Shot Noise

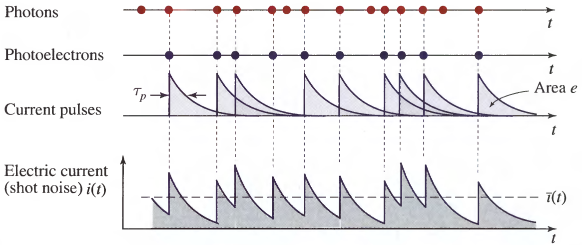


Figure: The photocurrent for a Poisson shot noise with deterministic incident intensity.

- Variance of shot noise: $\text{var}\{i^2(t)\} = \bar{g}^2 \int_{-\infty}^{t} h^2(t-z)n(z)dz$

Variance of Shot Noise

Example (Variance of shot noise)

If $h(t) = \frac{e}{\tau}[u(t) - u(t-\tau)]$, then $\text{Var}\{i(t)\} = \bar{g^2}(\frac{e}{\tau})^2\bar{k}(t-\tau,t)$, where $\bar{k}(t-\tau,t)$ is the average number of carriers in interval $[t-\tau,t]$.

$$\text{Var}\{i(t)\} = \bar{g^2}\int_{-\infty}^{t} h^2(t-z)n(z)dz = \bar{g^2}\int_{t-\tau}^{t}(\frac{e}{\tau})^2 n(z)dz = \bar{g^2}(\frac{e}{\tau})^2\bar{k}(t-\tau,t)$$

Example (Fast photodetector)

If the time variations of $n(t)$ is sufficiently slower than τ, or equivalently, if the bandwidth of the photo-detector $1/\tau$ is sufficiently higher than the bandwidth of $n(t)$, $\mathcal{E}\{i(t)\} = \bar{g}en(t)$ and $\mathcal{E}\{i^2(t)\} = \bar{g^2}n(t)\int_0^\tau h^2(t)dt$.

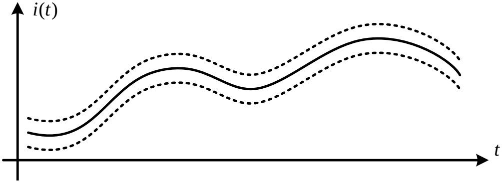
$$\mathcal{E}\{i(t)\} = \bar{g}\int_{-\infty}^{t} h(t-z)n(z)dz\bar{g} = \bar{g}\int_{t-\tau}^{t} h(t-z)n(z)dz = \bar{g}n(t)\int_{t-\tau}^{t} h(t-z)dz = \bar{g}en(t)$$

$$\text{Var}\{i(t)\} = \bar{g^2}\int_{-\infty}^{t} h^2(t-z)n(z)dz = \bar{g^2}\int_{t-\tau}^{t} h^2(t-z)n(z)dz = \bar{g^2}n(t)\int_{0}^{\tau} h^2(t)dt$$

Variance of Shot Noise

Example (Instantaneous power detector)

A fast photo-detector acts like an instantaneous power detector if the incident power is sufficiently high.



$$h(t) = \frac{e}{\tau}[u(t) - u(t-\tau)], g = 1$$

$$\mathcal{E}\{i(t)\} = en(t)$$

$$\text{Var}\{i(t)\} = n(t)\int_0^\tau h^2(t)dt = n(t)\frac{e^2}{\tau}$$

$$\frac{\sqrt{\text{Var}\{i(t)\}}}{\mathcal{E}\{i(t)\}} = \frac{1}{\sqrt{\tau n(t)}} \ll 1 \Rightarrow n(t)\tau \gg 1$$

Correlation of Shot Noise

Example (Correlation of shot noise)

Poisson shot noise process is a non-stationary process.

$$R_i(t,t+\tau)=\mathcal{E}\{i(t)i(t+\tau)\}=\mathcal{E}_k\big\{\mathcal{E}\{\sum_{j=1}^{k(0,t)}g_jh(t-z_j)\sum_{l=1}^{k(0,t+\tau)}g_lh(t+\tau-z_l)|k\}\big\}$$

$$=\mathcal{E}\{\sum_{j=1}^{k(0,t)}g_j^2h(t-z_j)h(t+\tau-z_j)+\sum_{j=1}^{k(0,t)}\sum_{l=1,j\neq l}^{k(0,t+\tau)}g_jg_lh(t-z_j)h(t+\tau-z_l)\}$$

$$=\mathcal{E}_k\big\{\mathcal{E}\{\sum_{j=1}^{k(0,t)}g_j^2h(t-z_j)h(t+\tau-z_j)|k(0,t)\}\big\}+\mathcal{E}_k\big\{\mathcal{E}\{\sum_{j,l=1,j\neq l}^{k(0,t)}g_jg_lh(t-z_j)h(t+\tau-z_l)|k(0,t)\}\big\}$$

$$+\mathcal{E}_k\big\{\mathcal{E}\{\sum_{j=1}^{k(0,t)}\sum_{l=1}^{k(t,t+\tau)}g_jg_lh(t-z_j)h(t+\tau-z_l)|(k(0,t),k(t,t+\tau))\}\big\}$$

$$=\mathcal{E}_k\big\{k(0,t)\mathcal{E}\{g_j^2\}\mathcal{E}\{h(t-z_j)h(t+\tau-z_j)\}\big\}+\mathcal{E}_k\big\{[k(0,t)^2-k(0,t)]\mathcal{E}\{g_j\}\mathcal{E}\{g_l\}\mathcal{E}\{h(t-z_j)\}$$

$$\mathcal{E}\{h(t+\tau-z_l)\}\big\}+\mathcal{E}_k\big\{k(0,t)k(t,t+\tau)\mathcal{E}\{g_j\}\mathcal{E}\{g_l\}\mathcal{E}\{h(t-z_j)\}\mathcal{E}\{h(t+\tau-z_l)\}\big\}$$

$$=m_v(0,t)\overline{g^2}\mathcal{E}\{h(t-z)h(t+\tau-z)\}+m_v^2(0,t)\bar{g}^2\mathcal{E}\{h(t-z)\}\mathcal{E}\{h(t+\tau-z)\}+m_v(0,t)m_v(t,t+\tau)\bar{g}^2$$

$$\mathcal{E}\{h(t-z)\}\mathcal{E}\{h(t+\tau-z)\}=m_v(0,t)\overline{g^2}\mathcal{E}\{h(t-z)h(t+\tau-z)\}+m_v(0,t)\bar{g}\mathcal{E}\{h(t-z)\}$$

$$\bar{g}\mathcal{E}\{h(t+\tau-z)\}\Big[m_v(0,t)+m_v(t,t+\tau)\Big]=\overline{g^2}\int_{-\infty}^{t}h(t-z)h(t+\tau-z)n(z)dz+\mathcal{E}\{i(t)\}\mathcal{E}\{i(t+\tau)\}$$

Power Spectral Density of Shot Noise

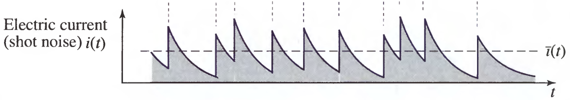


Figure: The photocurrent for a Poisson shot noise with deterministic incident intensity.

- Power spectral density: $S_i(\omega) = \lim_{T\to\infty} \frac{1}{2T}\mathcal{E}\{|X_T(\omega)|^2\}$
- Truncated Fourier transform of a sample shot noise:

$$X_T(\omega) = \int_{-T}^{T} i(t)e^{-j\omega t}dt = \sum_{j=1}^{k} g_j \int_{-T}^{T} h(t-z_j)e^{-j\omega t}dt = \sum_{j=1}^{k} g_j e^{-j\omega z_j} H_T(\omega)$$

$$|X_T(\omega)|^2 = X_T(\omega)X_T^*(\omega) = |H_T(\omega)|^2 \sum_{j,l=1}^{k} g_j g_l e^{-j\omega(z_j-z_l)} = |H_T(\omega)|^2 \Big[\sum_{j=1}^{k} g_j^2 + \sum_{j,l=1, j\neq l}^{k} g_j g_l e^{-j\omega(z_j-z_l)}\Big]$$

- Conditional average:

$$\mathcal{E}\{|X_T(\omega)|^2|k\} = |H_T(\omega)|^2 \Big[\bar{g^2} k(-T,T) + (k^2-k)\bar{g}^2 \int_{-T}^{T} e^{-j\omega z_j} p_{z_j}(z_j)dz_j \int_{-T}^{T} e^{j\omega z_l} p_{z_l}(z_l)dz_l\Big]$$

Power Spectral Density of Shot Noise

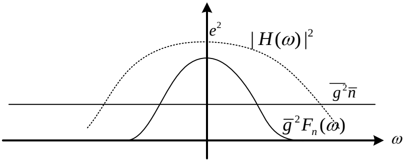


Figure: The power spectral density for a Poisson shot noise with deterministic incident intensity.

- Ensemble average:

$$\mathcal{E}\{|X_T(\omega)|^2\} = |H_T(\omega)|^2\left[\bar{g^2}\bar{k}(-T,T) + \bar{g}^2\int_{-T}^{T} e^{-j\omega z_j} n(z)dz \int_{-T}^{T} e^{j\omega z} n(z)dz\right]$$

$$= |H_T(\omega)|^2\left[\bar{g^2}\bar{k}(-T,T) + \bar{g}^2 N_T(\omega)N_T^*(\omega)\right] = |H_T(\omega)|^2\left[\bar{g^2}\bar{k}(-T,T) + \bar{g}^2|N_T(\omega)|^2\right]$$

- Power spectral density:

$$S_i(\omega) = \lim_{T\to\infty}\frac{1}{2T}\mathcal{E}\{|X_T(\omega)|^2\} = |H_T(\omega)|^2\left[\bar{g^2}\bar{n} + \bar{g}^2 F_n(\omega)\right]$$

$$\bar{n} = \lim_{T\to\infty}\frac{\bar{k}(-T,T)}{2T} = \lim_{T\to\infty}\frac{\int_{-T}^{T} n(t)dt}{2T}$$

$$F_n(\omega) = \lim_{T\to\infty}\frac{|N_T(\omega)|^2}{2T} = \lim_{T\to\infty}\frac{|\int_{-T}^{T} n(u)e^{-j\omega u}du|^2}{2T}$$

Power Spectral Density of Shot Noise

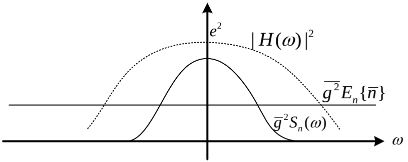


Figure: The power spectral density for a conditional Poisson shot noise with stochastic incident intensity. For a fast photo-detector, $h(t) = e\delta(t)$ and therefore, $|H(\omega)|^2 = e^2, \forall \omega$.

- Power spectral density:

$$S_i(\omega) = \lim_{T\to\infty} \frac{1}{2T} \mathcal{E}\{|X_T(\omega)|^2\} = |H(\omega)|^2 \left[\bar{g^2}\mathcal{E}_n\{\bar{n}\} + \bar{g}^2 S_n(\omega)\right]$$

$$\mathcal{E}_n\{\bar{n}\} = \lim_{T\to\infty} \frac{\mathcal{E}_n\{\bar{k}(-T,T)\}}{2T}$$

$$s_n(\omega) = \lim_{T\to\infty} \frac{\mathcal{E}_n\{|N_T(\omega)|^2\}}{2T}$$

- Shot noise level: $\bar{g^2}\mathcal{E}_n\{\bar{n}\}$
- Shot noise power: $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g^2}\mathcal{E}_n\{\bar{n}\}|H(\omega)|^2 d\omega$

Power Spectral Density of Shot Noise

Example (Shot noise power)

For stationary intensities, the power in the shot noise spectrum is the variance of the output current.

$$\mathrm{Var}\{i(t)\} = \bar{g^2} \int_{-\infty}^{t} h^2(t-z)n(z)dz, \quad n(t) \text{ is deterministic}$$

$$\mathrm{Var}\{i(t)\} = \bar{g^2} \int_{-\infty}^{t} h^2(t-z)\mathcal{E}_n\{n(z)\}dz = \bar{g^2} \int_{-\infty}^{t} h^2(t-z)\bar{n}(z)dz, \quad n(t) \text{ is stochastic}$$

$$\mathcal{E}_n\{\bar{n}\} = \lim_{T\to\infty} \frac{\mathcal{E}_n\{\bar{k}(-T,T)\}}{2T} = \lim_{T\to\infty} \frac{\mathcal{E}_n\{\int_{-T}^{T} n(t)dt\}}{2T} = \lim_{T\to\infty} \frac{\int_{-T}^{T} \mathcal{E}_n\{n(t)\}dt}{2T} = \bar{n}, \quad n(t) \text{ is stationary}$$

$$\mathrm{Var}\{i(t)\} = \bar{g^2} \int_{-\infty}^{t} h^2(t-z)\mathcal{E}_n\{n(z)\}dz = \bar{g^2}\bar{n} \int_{0}^{\infty} h^2(t)dt, \quad n(t) \text{ is stationary stochastic}$$

$$\mathrm{Var}\{i(t)\} = \bar{g^2}\bar{n} \int_{-\infty}^{\infty} h^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{g^2}\bar{n}|H(\omega)|^2 d\omega, \quad n(t) \text{ is stationary stochastic}$$

Power Spectral Density of Dark Current

Example (Shot noise power)

If the fixed count intensity of the dark current is $n_{dc} = \frac{I_{dc}}{e}$, then $S_{dc}(\omega) = |H(\omega)|^2 \left[\bar{g^2} \frac{I_{dc}}{e} + 2\pi \bar{g}^2 (\frac{I_{dc}}{e})^2 \delta(\omega)\right]$. The dark current inserts a DC current into the output currents and increases the shot noise level.

$$S_{dc}(\omega) = |H_T(\omega)|^2 \left[\bar{g^2} n_{dc} + \bar{g}^2 2\pi n_{dc}^2 \delta(\omega)\right] = |H(\omega)|^2 \left[\bar{g^2} \frac{I_{dc}}{e} + 2\pi \bar{g}^2 (\frac{I_{dc}}{e})^2 \delta(\omega)\right]$$

Filtered Shot Noise

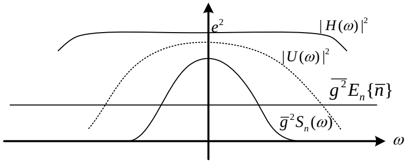


Figure: The power spectral density for a filtered conditional Poisson shot noise with stochastic incident intensity. For a fast photo-detector, $h(t) = e\delta(t)$ and therefore, $|H(\omega)|^2 = e^2, \forall\omega$.

- Filtered power spectral density: $S_y(\omega) = |U(\omega)|^2 S_i(\omega)$
- Filtering condition: Count intensity bandwidth < Filter bandwidth < Photo-detector bandwidth
- Shot noise power: $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}^2 \mathcal{E}_n\{\bar{n}\}|U(\omega)|^2 d\omega$

Complex Examples

Poisson Shot Noise

Example (Characteristic function of Poisson shot noise)

Characteristic function of the Poisson shot noise can be expressed in terms of the characteristic function of $h(t-z)$.

$$\Psi_i(\omega,t) = \mathcal{E}\{e^{j\omega i(t)}\} = \mathcal{E}\Big\{\exp\Big(j\omega\sum_{j=1}^{k} h(t-z_j)\Big)\Big\} = \mathcal{E}_k\Big\{\mathcal{E}\Big\{\exp\Big(j\omega\sum_{j=1}^{k} h(t-z_j)\Big)|k\Big\}\Big\}$$

$$= \mathcal{E}_k\Big\{\prod_{j=1}^{k}\mathcal{E}\{\exp(j\omega h(t-z_j))\}\Big\} = \mathcal{E}_k\Big\{\Big[\mathcal{E}\{\exp(j\omega h(t-z_j))\}\Big]^k\Big\} = \mathcal{E}_k\Big\{\Big[\Psi_n(\omega,t)\Big]^k\Big\}$$

$$\Psi_n(\omega,t) = \int_{-\infty}^{t}\exp(j\omega h(t-\rho))\frac{n(\rho)}{m_v}d\rho$$

$$\Psi_i(\omega,t) = \mathcal{E}_k\Big\{\Big[\Psi_n(\omega,t)\Big]^k\Big\} = \sum_{k=0}^{\infty} e^{-m_v}\frac{m_v^k}{k!}\Big[\Psi_n(\omega,t)\Big]^k = e^{-m_v}\sum_{k=0}^{\infty}\frac{\Big[m_v\Psi_n(\omega,t)\Big]^k}{k!}$$

$$= \exp(m_v(\Psi_n(\omega,t)-1)) = \exp\Big(\int_{-\infty}^{t}\big[\exp(j\omega h(t-\rho))-1\big]n(\rho)d\rho\Big)$$

Poisson Shot Noise

Example (PDF of Poisson shot noise)

For a rectangular current pulse, the PDF of the shot noise is Poisson-like.

$$h(t) = \frac{e}{\tau}[u(t) - u(t-\tau)]$$

$$\Psi_i(\omega, t) = \exp(m_v(\Psi_n(\omega, t) - 1)) = \exp\left(\int_{-\infty}^{t} \left[\exp(j\omega h(t-\rho)) - 1\right] n(\rho) d\rho\right)$$

$$= \exp\left(\left[\exp\left(j\omega\frac{e}{\tau}\right) - 1\right] \int_{t-\tau}^{t} n(\rho) d\rho\right) = \exp\left(m_v\left[e^{j\omega\frac{e}{\tau}} - 1\right]\right), \quad m_v = \int_{t-\tau}^{t} n(\rho) d\rho = \bar{k}(t-\tau, t)$$

$$p(i, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega i}\Psi_i(\omega, t) d\omega = \frac{e^{-m_v}}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega i} e^{m_v e^{j\omega e/\tau}} d\omega$$

$$= \frac{e^{-m_v}}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega i}\sum_{q=0}^{\infty}\frac{(m_v e^{j\omega e/\tau})^q}{q!} d\omega = e^{-m_v}\sum_{q=0}^{\infty}\frac{m_v^q}{q!}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(i-qe/\tau)} d\omega\right]$$

$$= \sum_{q=0}^{\infty} e^{-m_v}\frac{m_v^q}{q!}\delta\left(i - \frac{qe}{\tau}\right)$$

Poisson Shot Noise

Example (Moments of Poisson shot noise)

The moments can be given in terms of the Poisson shot noise semi-invariants χ_q.

$$\Psi_i(\omega, t) = \exp(m_v(\Psi_n(\omega, t) - 1)) = \exp\left(\int_{-\infty}^{t} \left[\exp(j\omega h(t-\rho)) - 1\right] n(\rho) d\rho\right)$$

$$\ln(\Psi_i(\omega, t)) = \int_{-\infty}^{t} \left[\exp(j\omega h(t-\rho)) - 1\right] n(\rho) d\rho = \int_{-\infty}^{t} \left[\sum_{q=0}^{\infty} \frac{(j\omega h(t-\rho))^q}{q!} - 1\right] n(\rho) d\rho$$

$$= \sum_{q=1}^{\infty} \frac{(j\omega)^q}{q!} \int_{-\infty}^{t} h(t-\rho)^q n(\rho) d\rho = \sum_{q=1}^{\infty} \frac{(j\omega)^q}{q!} \chi_q, \quad \chi_q = \int_{-\infty}^{t} h(t-\rho)^q n(\rho) d\rho$$

$$\mathcal{E}\{i(t)\} = \chi_1$$
$$\mathcal{E}\{i^2(t)\} = \chi_2 + \chi_1^2$$
$$\mathcal{E}\{i^3(t)\} = \chi_3 + 3\chi_2\chi_1 + \chi_1^3$$
$$\mathrm{Var}\{i(t)\} = \chi_2$$

The End