Thermal Light

Mohammad Hadi

mohammad.hadi@sharif.edu

@MohammadHadiDastgerdi

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Overview

Preliminaries

Solid Angle

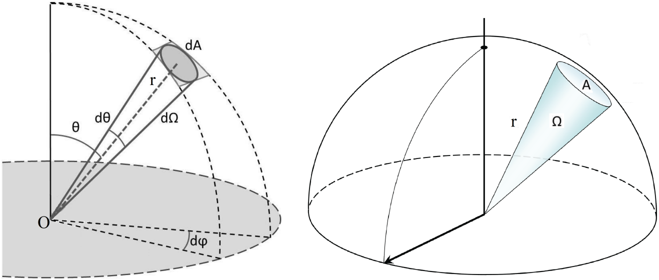


Figure: Solid angle with the unit of steradian sr.

- Infinitesimal area element: $dA = r^2 \sin(\theta) d\theta d\phi$
- Solid angle: $\Omega \triangleq \frac{A}{r^2} = \iint_S \frac{\hat{r}.\hat{n}}{r^2} ds = \iint_S \sin(\theta) d\theta d\phi$
- Infinitesimal solid angle: $d\Omega = \sin(\theta) d\theta d\phi$

Radiance

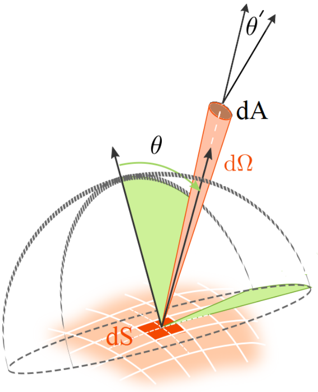


Figure: Geometry of radiance in photometry.

- Time-average Poynting vector: $P = \iint_A \boldsymbol{S}.\hat{\boldsymbol{n}} dA, \quad \boldsymbol{S} = \mathrm{Re}\{\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*\}$
- Transmitted radiance: $I = \frac{\partial^2 P}{\partial\Omega\partial S \cos(\theta)}$
- Transmitted spectral radiance: $I = \frac{\partial^3 P}{\partial\nu\partial\Omega\partial S \cos(\theta)}$

Irrediance

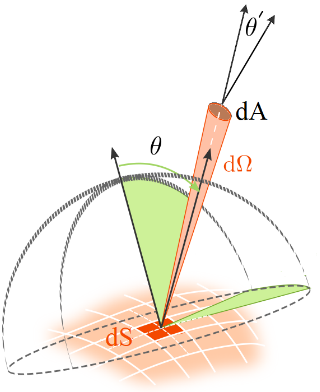


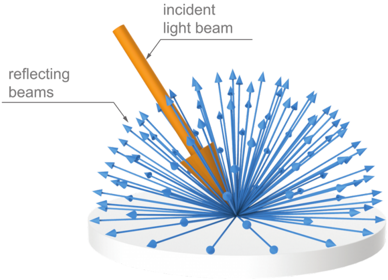
Figure: Geometry of irradiance in photometry.

- Time-average Poynting vector: $P = \iint_A \boldsymbol{S}.\hat{\boldsymbol{n}} dA, \quad \boldsymbol{S} = \text{Re}\{\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*\}$
- Received irradiance: $H = \frac{dP}{dA\cos(\theta')} = |\boldsymbol{S}|$
- Received spectral irradiance: $H = \frac{\partial^2 P}{\partial\nu\partial A\cos(\theta')}$

Lambertian Isotropic Radiators

Example (Lambertian isotropic radiator)

The radiated power of an infinitesimal Lambertian isotropic radiator with the fixed radiance I_0 through the upper hemisphere is πI_0 W/m^2.

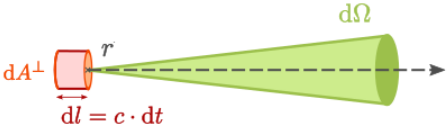


$$\frac{\partial P}{\partial S} = \int_{\Omega} I_0 \cos(\theta) d\Omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} I_0 \cos(\theta) \sin(\theta) d\theta d\phi = -\pi I_0 \frac{1}{2} \cos(2\theta)\Big|_0^{\frac{\pi}{2}} = \pi I_0$$

Energy Density

Example (Energy density)

The energy density of an isotropic infinitesimal radiator is $U = \frac{4\pi}{c} I$.



$$dU = \frac{dE}{dV} = \frac{dPdt}{dA^{\perp}dl} = \frac{IdA^{\perp}d\Omega dl}{cdA^{\perp}dl} = \frac{I}{c}d\Omega$$

$$U = \int_{\Omega} \frac{I}{c}d\Omega = \frac{I}{c}\int_{\Omega} d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{I}{c}\sin(\theta)d\theta d\phi = \frac{4\pi}{c}I$$

Boltzmann Energy Distribution

Figure: Configuration possibilities in an isolated system with N particles and the fixed macroscopic overall energy E. Each particle may have microscopic energy $\epsilon_i, i \in \mathbb{W}$.

- Configuration weight: $W = \frac{N!}{n_0!n_1!n_2!\cdots} \Rightarrow \ln(W) = \ln(N!) - \sum_i \ln(n_i!)$
- Stirling's approximation: $\ln(W) \approx N\ln(N) - \sum_i n_i \ln(n_i)$
- Second thermodynamic law: $d\ln(W) = 0$ s.t. $\sum_i n_i\epsilon_i = E, \sum_i n_i = N$
- Boltzmann energy distribution: $P(\epsilon_i) = \frac{n_i}{N} \propto \exp(-\beta\epsilon_i)$

Boltzmann Energy Distribution

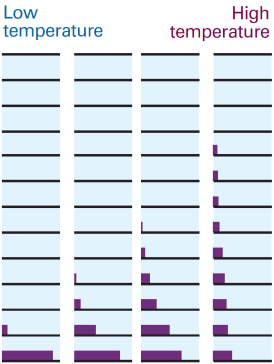


Figure: The populations of microscopic energy levels in Boltzmann distribution.

- Boltzmann energy distribution: $P(\epsilon_i) \propto \exp(-\beta\epsilon_i)$
- Average particle energy: $\bar{E} = \sum_i \epsilon_i P(\epsilon_i)$
- Evaluation for ideal gas: $\bar{E} = \frac{3}{2}KT = \frac{3}{2\beta} \Rightarrow \beta = \frac{1}{KT}$

Boltzmann Energy Distribution

Example (Average particle energy for $\epsilon_i \in \{0, \epsilon_0, 2\epsilon_0, \cdots\}$)

Assuming $\epsilon_i \in \{0, \epsilon_0, 2\epsilon_0, \cdots\}$, the average particle energy is $\bar{E} = \frac{\epsilon_0}{\exp\left(\frac{\epsilon_0}{KT}\right)-1}$.

$$P(n\epsilon_0) = A\exp\left(-\frac{n\epsilon_0}{KT}\right) = Ax^n, \quad x = \exp\left(-\frac{\epsilon_0}{KT}\right) \Rightarrow \sum_{n=0}^{\infty} p(n\epsilon_0) = A\sum_{n=0}^{\infty} x^n = 1, \quad |x| < 1 \Rightarrow$$

$$\Rightarrow A = 1 - x \Rightarrow P(n\epsilon_0) = (1-x)x^n \Rightarrow \bar{E} = \sum_{n=0}^{\infty} n\epsilon_0 p(n\epsilon_0) = \epsilon_0(1-x)\sum_{n=0}^{\infty} nx^n = \frac{\epsilon_0 x}{1-x} = \frac{\epsilon_0}{\exp\left(\frac{\epsilon_0}{KT}\right) - 1}$$

Example (Average particle energy for $\epsilon_i \in [0, \infty)$)

Assuming $\epsilon_i \in [0, \infty)$, the average particle energy is $\bar{E} = KT$.

$$p(\epsilon) = A\exp\left(-\frac{\epsilon}{KT}\right) \Rightarrow \int_0^{\infty} p(\epsilon)d\epsilon = A\int_0^{\infty} \exp\left(-\frac{\epsilon}{KT}\right)d\epsilon = 1 \Rightarrow -AKT\exp\left(-\frac{\epsilon}{KT}\right)\Big|_0^{\infty} = 1$$

$$\Rightarrow A = \frac{1}{KT} \Rightarrow p(\epsilon) = \frac{1}{KT}\exp\left(-\frac{\epsilon}{KT}\right) \Rightarrow \bar{E} = \int_0^{\infty} \epsilon p(\epsilon)d\epsilon = KT$$

Cavity Mode Density

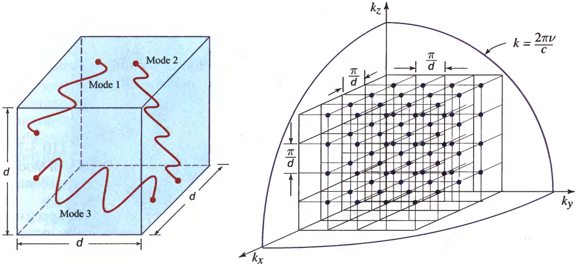


Figure: Mode density in a cavity.

- Monochromatic wave function: $u(\boldsymbol{r}, t) = \text{Re}\{U(\boldsymbol{r})e^{j2\pi\nu t}\}$
- Helmholtz's wave equation: $\nabla^2 U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0, \quad k = 2\pi/\lambda = 2\pi\nu/c$
- Boundary conditions: $U(\boldsymbol{r}) = 0, \boldsymbol{r} \in$ cavity walls

Cavity Mode Density

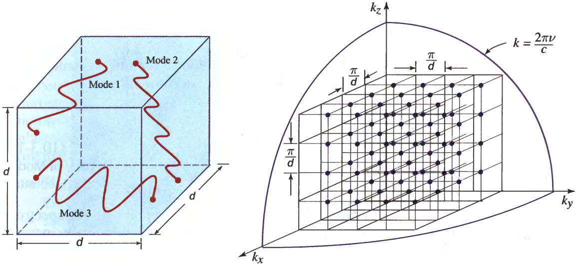


Figure: Mode density in a cavity.

- Complex amplitude: $U(\boldsymbol{r}) \propto \sin(k_x x)\sin(k_y y)\sin(k_z z)$
- Wave numbers: $k_x = \frac{q_x \pi}{d}, k_y = \frac{q_y \pi}{d}, k_z = \frac{q_z \pi}{d}, \quad q_x, q_y, q_z \in \mathbb{N}$
- Mode frequency: $k_x^2 + k_y^2 + k_z^2 = k^2 = (\frac{2\pi\nu}{c})^2 \Rightarrow \nu_q = \frac{c}{2d}\sqrt{q_x^2 + q_y^2 + q_z^2}$
- Mode density: $M(\nu) = 2\frac{\partial^2}{\partial\nu\partial(d^3)}\left[\frac{4\pi}{24}\frac{8\pi^3\nu^3}{c^3}/(\frac{\pi^3}{d^3})\right] = \frac{8\pi\nu^2}{c^3}$

Physical Description of Thermal Light

Black Body

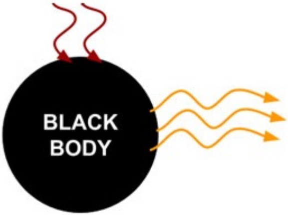


Figure: Perfect black body.

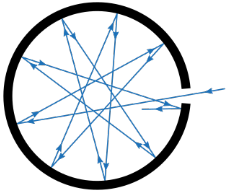


Figure: Practical black body.

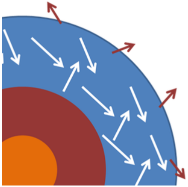


Figure: Natural black body.

- Radiation-matter interactions: Absorption, reflection, and transmission
- Black body: An idealized body absorbing all electromagnetic radiation in all incident angles
- Kirchhoff's thermal radiation law: Equality of absorptivity and emisitivity at thermodynamic equilibrium
- Thermal light: Radiation emitted from a black body

Wien Experimental Law

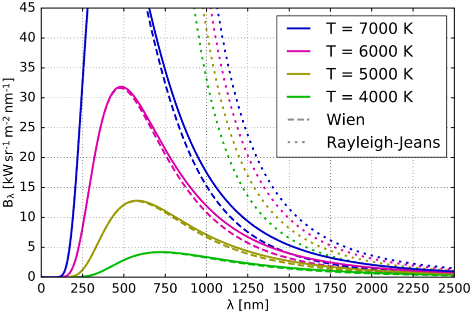


Figure: Black body radiation.

- Stephen-Boltzmann radiation law: $I(T) = \int_0^\infty I(\lambda, T)d\lambda = \sigma T^4$
- Wien's displacement law: $\lambda_{max} T = \text{constant}.$
- Wien's approximation: $I(\lambda, T) = \frac{a_1}{\lambda^5} \exp\left(-\frac{a_2}{\lambda T}\right)$
- $I(\nu, T)d\nu = -I(\lambda, T)d\lambda, \nu = \frac{c}{\lambda} \Rightarrow I(\nu, T) = b_1 \nu^3 \exp\left(-b_2 \frac{\nu}{T}\right)$

Rayleigh–Jeans Classical Law

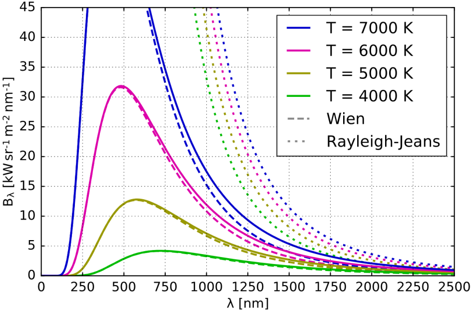


Figure: Black body radiation.

- Rayleigh–Jeans law: $I(\nu, T) = \frac{c}{4\pi} M(\nu)\bar{E} = \frac{2\nu^2}{c^2} KT$
- $I(\nu, T)d\nu = -I(\lambda, T)d\lambda, \nu = \frac{c}{\lambda} \Rightarrow I(\lambda, T) = \frac{2c}{\lambda^4} KT$

Ultraviolet Catastrophe

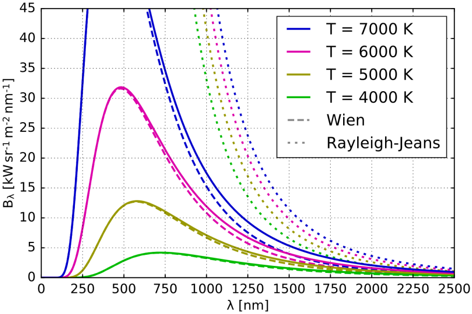


Figure: Black body radiation.

- Wien's approximation
  - Fitting mismatch at long wavelengths
  - No theoretical support
- Rayleigh–Jeans law
  - Fitting mismatch at short wavelengths
  - $I(T) = \int_0^\infty I(\nu, T) d\nu = \int_0^\infty \frac{2\nu^2}{c^2} KT d\nu = \infty$

Planck's Quantum Law

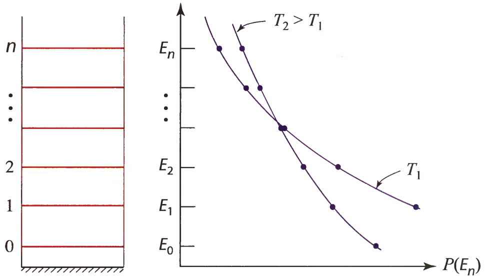


Figure: Quantized energy levels accessible for each mode.

- Planck's quantum assumption: $E_n = nh\nu$
- Boltzmann's distribution: $P(E_n) \propto \exp\left(-\frac{nh\nu}{KT}\right) = x^n, \quad x = \exp\left(-\frac{h\nu}{KT}\right)$
- Probability axioms: $\sum_n P(E_n) = \sum_n Ax^n = 1, |x| < 1 \Rightarrow P(E_n) = (1-x)x^n$
- Average energy: $\bar{E} = \sum_n E_n P(E_n) = h\nu \frac{x}{1-x}$
- Planck's radiation law: $I(\nu, T) = \frac{c}{4\pi} M(\nu)\bar{E} = \frac{2\nu^2}{c^2} \frac{h\nu}{\exp\left(\frac{h\nu}{KT}\right)-1}$

Planck's Quantum Law

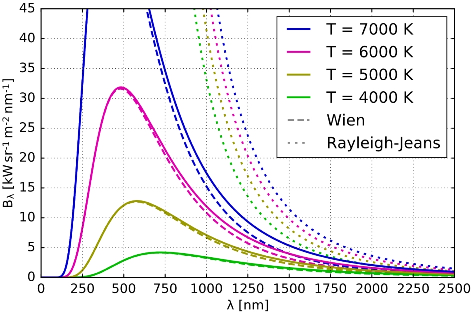


Figure: Black body radiation.

- Planck's law: $I(\nu, T) = \frac{2\nu^2}{c^2} \frac{h\nu}{\exp\left(\frac{h\nu}{KT}\right) - 1}, \quad \forall \nu$
- Jean's law: $I(\nu, T) \approx \frac{2\nu^2}{c^2} \frac{h\nu}{1 + \frac{h\nu}{KT} - 1} = \frac{2\nu^2}{c^2} KT, \quad \forall \nu \to 0$
- Wien's law: $I(\nu, T) \approx \frac{2\nu^2}{c^2} \frac{h\nu}{\exp\left(\frac{h\nu}{KT}\right)} = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{KT}\right), \quad \forall \nu \to \infty$

Planck's Quantum Law

Example (Stephen-Boltzmann radiation law)

Stephen-Boltzmann radiation law is derived from Planck's radiation law.

$$I(T) = \int_0^\infty I(\nu, T) d\nu = \frac{2h}{c^2} \int_0^\infty \frac{\nu^3}{\exp\left(\frac{h\nu}{KT}\right) - 1} d\nu = \frac{2K^4 T^4}{h^3 c^2} \int_0^\infty \frac{y^3}{\exp(y) - 1} dy, \quad y = \frac{h\nu}{KT}$$

$$I(T) = \frac{2K^4 T^4}{h^3 c^2} \int_0^\infty \frac{y^3}{\exp(y) - 1} dy = \frac{2K^4 T^4}{h^3 c^2} \int_0^\infty y^3 e^{-y} \sum_{n=0}^\infty e^{-ny} = \frac{2K^4 T^4}{h^3 c^2} \sum_{n=0}^\infty \int_0^\infty y^3 e^{-(n+1)y} dy$$

$$I(T) = \frac{2K^4 T^4}{h^3 c^2} \sum_{n=0}^\infty \frac{1}{(n+1)^4} \int_0^\infty z^3 e^{-z} dz = \frac{2K^4 T^4}{h^3 c^2} 6 \sum_{n=1}^\infty \frac{1}{n^4} = \frac{2\pi^4 K^4}{15 h^3 c^2} T^4 = \sigma T^4, \quad z = (n+1)y$$

Planck's Quantum Law

Example (Wien's displacement law)

Wien's displacement law is derived from Planck's radiation law.

$$\frac{dI(\nu,T)}{d\nu}=0\Rightarrow [\frac{h\nu}{KT}-3]\exp\left(\frac{h\nu}{KT}\right)+3=(y-3)e^{y}+3=0,\quad y=\frac{h\nu}{KT}$$

$$\Rightarrow y=\frac{h\nu_{max}}{KT}=y_0=2.82\Rightarrow \nu_{max}=\frac{y_0KT}{h}=\text{constant}$$

$$I(\lambda,T)=-I(\nu,T)\frac{d\nu}{d\lambda}\Rightarrow I(\lambda,T)=\frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda KT}\right)-1}$$

$$\frac{dI(\lambda,T)}{d\lambda}=0\Rightarrow \frac{hc}{\lambda KT}\frac{\exp\left(\frac{hc}{\lambda KT}\right)}{\exp\left(\frac{hc}{\lambda KT}\right)-1}-5=\frac{ye^{y}}{e^{y}-1}-5=0,\quad y=\frac{hc}{\lambda KT}$$

$$\Rightarrow y=\frac{hc}{\lambda_{max}KT}=y_0=4.96\Rightarrow \lambda_{max}T=\frac{hc}{y_0K}=\text{constant}$$

Statistical Description of Thermal Light

Thermal Light

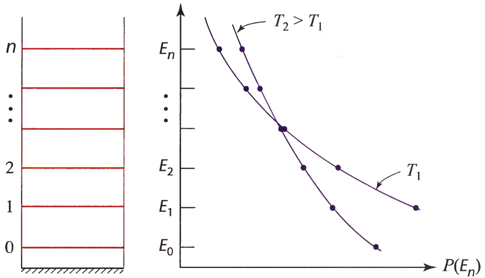


Figure: Each photon can have a quanta of energy $h\nu$. A mode with energy level E_n has n photons.

- Photon distribution: $P(n) \equiv P(E_n) = (1-x)x^n, \quad x = \exp\left(-\frac{h\nu}{KT}\right)$
- Average photon number: $\bar{n} = \sum_n nP(n) = \frac{x}{1-x} = \frac{\bar{E}}{h\nu} \Rightarrow x = \frac{\bar{n}}{\bar{n}+1}$
- Bose-Einstein (thermal, geometric) distribution: $P(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n, n \in \mathbb{W}$

Analytical Description of Thermal Light

Thermal Light

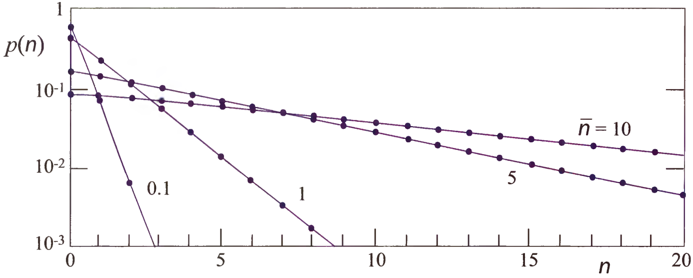


Figure: Bose-Einstein distribution $p(n)$ of the photon number n.

- Bose-Einstein (thermal, geometric) distribution: $P(n) = \frac{1}{\bar{n}+1}(\frac{\bar{n}}{\bar{n}+1})^n, n \in \mathbb{W}$
- Signal to noise ratio: $\text{SNR} = \frac{\text{mean}^2}{\text{variance}} = \frac{\bar{n}^2}{\bar{n}^2+\bar{n}} = \frac{\bar{n}}{\bar{n}+1} < 1$

Thermal Light

Example (Thermal light communication)

In a simple on-off keying digital optical communication system using monochromatic thermal light, the bit error rate 10^{-9} corresponds to to a mean number of photons $\bar{n} = 5 \times 10^8$.

$$P_e = P_1 P_{e|1} + P_0 P_{e|0} = \frac{1}{2} P(0) = \frac{1}{2} \frac{1}{\bar{n}+1} \leq 10^{-9} \Rightarrow \bar{n} \geq 499999999$$

Example (Laser light communication)

In a simple on-off keying digital optical communication system using monochromatic laser light, where $P(n) = \exp(-\bar{n})\frac{\bar{n}^n}{n!}$, the bit error rate 10^{-9} corresponds to to a mean number of photons $\bar{n} = 20$.

$$P_e = P_1 P_{e|1} + P_0 P_{e|0} = \frac{1}{2} P(0) = \frac{1}{2} \exp(-\bar{n}) \leq 10^{-9} \Rightarrow \bar{n} \geq 20$$

Applications

Background Noise

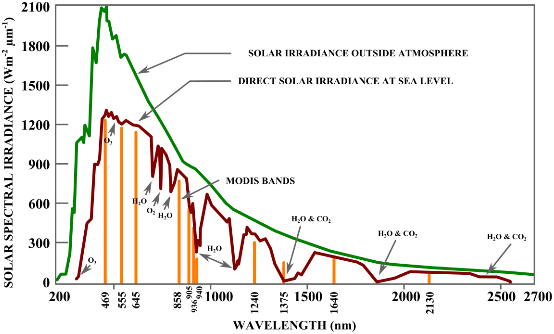


Figure: Sun's spectral irrediance.

- The sun's temperature is 5778 K or 5505 °C.

Background Noise

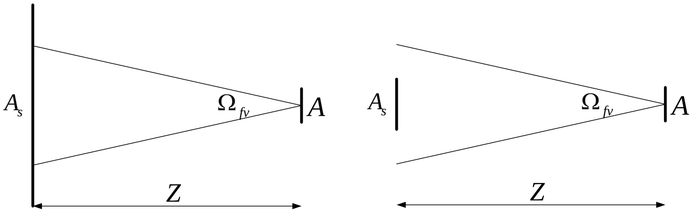


Figure: Background noise source geometry for $Z \to \infty$ for $\Omega_s > \Omega_{fv}$ and $\Omega_s \leq \Omega_{fv}$.

$$P_b = \begin{cases} I(\lambda, T)(\Delta\lambda)(\Omega_{fv} Z^2)(A/Z^2), & A_s > \Omega_{fv} Z^2 \\ I(\lambda, T)(\Delta\lambda) A_s (A/Z^2), & A_s \leq \Omega_{fv} Z^2 \end{cases} = \begin{cases} I(\lambda, T)(\Delta\lambda)\Omega_{fv} A, & \Omega_s > \Omega_{fv} \\ I(\lambda, T)(\Delta\lambda)\Omega_s A, & \Omega_s \leq \Omega_{fv} \end{cases}$$

$$I(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda K T}\right) - 1}$$

Background Noise

Example (Background noise)

6.2×10^{-24} W background noise is collected from the night sky using a receiver operating at $\lambda = 10\ \mu m$ with a 10 cm-diameter lens, 100 $\mathring{A}$ bandwidth, and 100 μrad field of view angle.

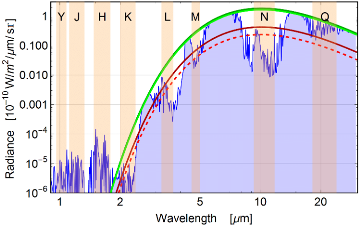


Figure: Night sky spectral radiance.

$$\Omega_{fv} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\theta_{fv}}{2}} \sin(\theta) d\theta d\phi = 2\pi(1 - \cos\left(\frac{\theta_{fv}}{2}\right)) \approx \frac{\pi}{4}\theta_{fv}^2$$

$$P_b = I(\lambda, T)(\Delta\lambda)(\Omega_{fv} Z^2)(A/Z^2) = (0.1 \times 10^{-10})(100 \times 10^{-4})(\frac{\pi}{4} \times 100^2 \times 10^{-12})(\pi \times 0.05^2) = 6.2 \times 10^{-24}$$

Background Noise

Statement (Background Noise Statistical Model)

Background noise is assumed to have the following properties,

1. *Background noise is additive.*
2. *Background noise is a zero-mean process.*
3. *Background noise is a Gaussian process.*
4. *Background noise is a stationary process.*
5. *Background noise may have an approximated colored power spectral density over a narrow working spectrum band.*
6. *Background noise may have an approximated white power spectral density if the spectrum outside a narrow working spectrum band is not important.*

Background Noise

Example (Background noise statistical model)

The background noise collected from the night sky using a receiver operating at $\lambda = 10\ \mu$m with a 10 cm-diameter lens, 100 $\AA$ bandwidth, and 100 μrad field of view angle can be modeled as an additive colored Gaussian noise with the noise power spectral density $N_0 = 2.1 \times 10^{-34}$ W/Hz $\equiv -337$ dB/Hz $\equiv -307$ dBm/Hz.

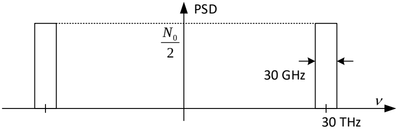


Figure: Power spectral density of the involved background noise.

$$N_0 = I(\nu, T)(\Omega_{fv} Z^2)(A/Z^2) = I(\lambda, T)(\lambda^2/c)(\Omega_{fv} Z^2)(A/Z^2) = \frac{P_b}{\Delta\lambda}\frac{\lambda^2}{c} = \frac{P_b}{\Delta\nu} = 2.1 \times 10^{-34}$$

Thermography

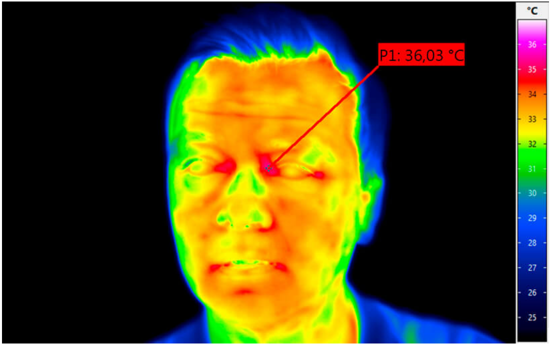


Figure: A thermography image for COVID-19 detection.

- Theremography is used in astronomy, industry, biology, medical science, etc.

The End