Microwave Magnetics

Graduate Course

Electrical Engineering (Communications)

2nd Semester, 1394-1395

Sharif University of Technology

General information

- **Information about the instructor:**
  - **Instructor:** Behzad Rejaei (Salmassi)
  - **Affiliation:** Sharif University of Technology
  - **Email**: rejaei@sharif.ir
  - **Room number**: 620

General information

- **Course focus:** application of magnetic materials in microwave components
- **General outline of the contents:**
  - Magnetism and magnetic phenomena (short review)
  - High frequency properties of magnetic films
  - Electrodynamics of gyrotropic media
  - Waveguides and resonators employing magnetic elements
  - Non-reciprocal magnetic devices (circulators and isolators)
  - Magnetostatic waves and their applications

General information

- **Course structure:** oral lectures
- **Course material:**
  - Lecture notes (Power point slides)
  - A.G. Gurevich, G.A. Melkov, *Magnetization Oscillations and Waves*, CRC Press, Boca Raton, 1996
- **Pre-requisites:**
  - Electromagnetic theory
  - Microwave techniques (at the level of Pozar)

General information

- **Times & dates:** 2nd semester, 1394-1395, every Sunday and Tuesday, 15:00-16:30
- **Remarks:**
  - Definitions, major results and equations presented on slides (transparencies or beamer) which will be sent to students by email
  - Derivations written on white- (or black) board, notes to be taken by students

General information

- **Contents of lecture 1:**
  - Introduction & motivation
    - *Magnetism: brief history*
    - *Magnetic materials in microwave engineering*
  - Magnetism: the elementary magnetic moments
    - *Magnetic field induced by electric currents*
    - *Dipole approximation*
    - *Microscopic magnetization density*
    - *Macroscopic magnetization*
    - *Magnetic field of uniformly magnetized ellipsoids*

General information

- Motion of microscopic moments in a magnetic field
  - *Force and torque exerted on a magnetic dipole*
  - *Classical motion of an atomic dipole in a magnetic field*
  - *Damped motion of magnetic dipoles*
  - *Equation of motion of macroscopic magnetization in a magnetic field*

Introduction

- Magnetism (brief history)
  - **< 600 BC:** Greeks report magnetic properties of lodestone
  - **16th century:** Magnetic field of Earth discovered by Gilbert
  - **18th century:** Studies by Coulomb reveal inverse square law of force between magnetic poles
  - **19th century:** Link between electricity and magnetism discovered by Oersted, Ampere, and Faraday; Foundation of electromagnetic theory by Maxwell
  - **20th century:** Modern understanding of magnetism due to work of Curie and Weiss; Quantum theory of magnetism by Ising and Heisenberg

Introduction

- Magnetism in Microwaves (brief history)
  - **WW II:** Modern magnetic materials at Philips Laboratories (Snoek and coworkers) for transformer cores
  - **1948:** Further theoretical understanding due to Neel
  - **1949:** Polder theory of ferromagnetic resonance
  - **1952:** First microwave device based on Faraday rotation
  - **1970's:** Magnetostatic waves and devices
  - **1980's**: Attention shifted away due to emergence of microwave IC's and technological difficulties associated with materials
  - **2000's**: Revival of interest due to thin film IC applications, and novel magnetic phenomena in magnetic nano-structures

Magnetism: the elementary magnetic moments

- Asymmetry of Maxwell equations:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{B} = \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J} \qquad \nabla \cdot \boldsymbol{B} = 0$$

- Isolated magnetic charges and currents do not exist
- Magnetic fields can only be generated by motion of electric charges (current)
- Magnetic phenomena have their origin in electronic currents inside atoms

(i) *Magnetic field induced by electric currents*

- To compute the magnetic field of localized microscopic currents, we first consider the static case

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

- No net charge density is permitted

$$\nabla \cdot \boldsymbol{J} = 0$$

- Equations best solved by using the vector potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

- Resulting equation for the vector potential (in Coulomb gauge):

$$-\nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

(i) *Magnetic field induced by electric currents*

- Solution in infinite space using Green's function

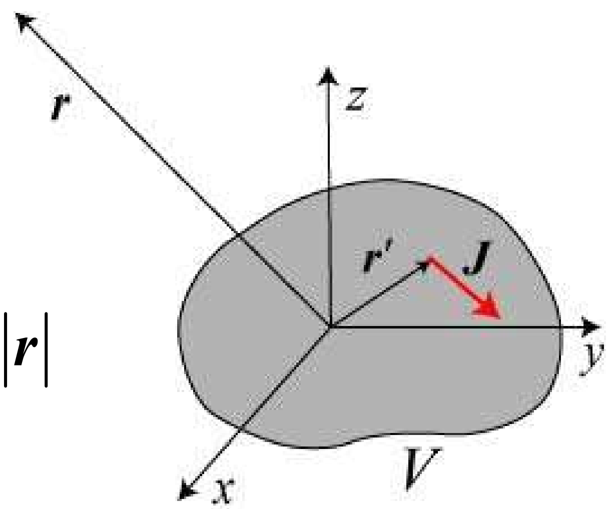
$$\boldsymbol{A}(\boldsymbol{r}) = \mu_0 \int_V G_0(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')dV'$$

- V: volume of the region containing the currents

$$G_0(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

- If the observation point is far away; i.e., if distance is much larger than linear dimensions of V, then dipole approximation can be used

$$|\boldsymbol{r}'| \ll |\boldsymbol{r}|$$



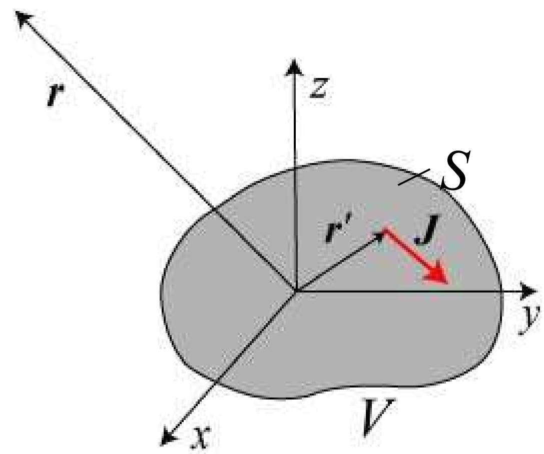
$$G_0(\boldsymbol{r}-\boldsymbol{r}') \approx G_0(\boldsymbol{r}) - \boldsymbol{r}' \cdot \nabla G_0(\boldsymbol{r})$$

(ii) *Dipole approximation*

- Vector potential in dipole approximation

$$\boldsymbol{A}(\boldsymbol{r}) \approx \mu_0 G_0(\boldsymbol{r}) \int_V \boldsymbol{J}(\boldsymbol{r}')dV' - \mu_0 \int_V \nabla G_0(\boldsymbol{r}) \cdot \boldsymbol{r}' \boldsymbol{J}(\boldsymbol{r}')dV'$$

- Current is divergence-less and has no normal component on S →



$$\int_V \boldsymbol{J}(\boldsymbol{r}')dV' = 0$$

$$\int_V \nabla G_0(\boldsymbol{r}) \cdot \boldsymbol{r}' \boldsymbol{J}(\boldsymbol{r}')dV' = -\frac{1}{2}\int_V \nabla G_0(\boldsymbol{r}) \times \left[\boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}')\right]dV'$$

(ii) *Dipole approximation*

- Derivation: for any two arbitrary functions *f(x), g(x):*

$$\int_V fg\nabla'\cdot\boldsymbol{J}dV' + \int_V f\boldsymbol{J}\cdot\nabla' g dV' + \int_V g\boldsymbol{J}\cdot\nabla' f dV' = 0$$

$$\nabla'\cdot\boldsymbol{J} = 0 \rightarrow \int_V f\boldsymbol{J}\cdot\nabla' g dV' + \int_V g\boldsymbol{J}\cdot\nabla' f dV' = 0$$

- Let

$$f = 1, g = x', y', z' \rightarrow \int_V \boldsymbol{J}dV' = 0$$

$$f = r'_\alpha,\ g = r'_\beta,\ \alpha,\beta = 1,2,3,\ r'_1 = x', r'_2 = y', r'_3 = z'$$

$$\rightarrow \int_V r'_\alpha J_\beta dV' + \int_V r'_\beta J_\alpha dV' = 0$$

(ii) *Dipole approximation*

- For any constant vector $\boldsymbol{u}$:

$$\int_V (\boldsymbol{u}\cdot\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')dV' = \sum_{\beta=1}^{3}\int_V u_\beta r'_\beta J_\alpha dV'$$

$$= \frac{1}{2}\sum_{\beta=1}^{3}\left(\int_V u_\beta r'_\beta J_\alpha dV' - \int_V u_\beta J_\beta r'_\alpha dV'\right)$$

$$= \frac{1}{2}\int_V \left[(\boldsymbol{u}\cdot\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}') - (\boldsymbol{u}\cdot\boldsymbol{J}(\boldsymbol{r}'))\boldsymbol{r}'\right]dV' = -\frac{1}{2}\int_V \boldsymbol{u}\times[\boldsymbol{r}'\times\boldsymbol{J}(\boldsymbol{r}')]dV'$$

- This result is valid if (a) current density has no divergence and (b) there is no component of the current density normal to the boundary surface of V

(ii) *Dipole approximation*

- Microscopic magnetization density $\mathfrak{M}(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{r})$
- Magnetic dipole moment: $\mathfrak{m} = \int_V \mathfrak{M}(\boldsymbol{r}) dV = \frac{1}{2}\int_V [\boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{r})] dV$
- Vector potential induced by a dipole at the origin: $\boldsymbol{A}(\boldsymbol{r}) = \mu_0 \nabla G_0(\boldsymbol{r}) \times \mathfrak{m}$
- Magnetic field of the dipole:

$$\boldsymbol{B}(\boldsymbol{r}) = (\mathfrak{m} \cdot \nabla) \nabla G_0(\boldsymbol{r}) + \mu_0 \mathfrak{m} \delta(\boldsymbol{r})$$

For proof use $\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b}$

Discussion

- Although we restricted ourselves to the static case, results obtained are of general nature
- Concept of magnetic dipoles can be derived for time dependent-currents as well (see Jackson), despite the fact that currents may not be divergence-free

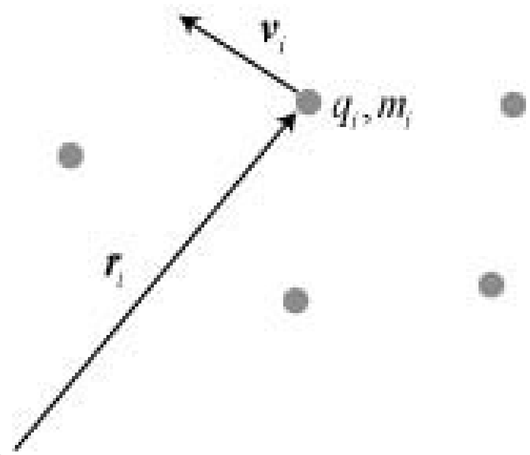
$$\mathfrak{m}(t) = \int_V \mathfrak{M}(\boldsymbol{r},t)\, dV = \frac{1}{2}\int_V \left[\boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{r},t)\right] dV$$

- But time-dependent Green's functions are needed for the calculation of their fields and (vector) potentials

(iii) *Microscopic magnetization density*

- Microscopic definition of current density

$$\boldsymbol{J}(\boldsymbol{r},t)=\sum_i q_i \boldsymbol{v}_i(t)\,\delta\left[\boldsymbol{r}-\boldsymbol{r}_i(t)\right]$$



- Microscopic magnetization density

$$\mathfrak{M}(\boldsymbol{r},t)=\frac{1}{2}\sum_i q_i\left[\boldsymbol{r}_i\times\boldsymbol{v}_i(t)\right]\delta\left[\boldsymbol{r}-\boldsymbol{r}_i(t)\right]$$

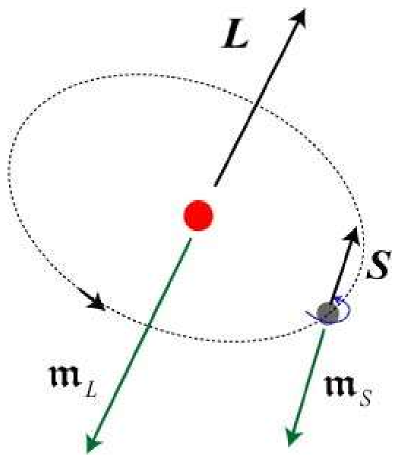
$$=\sum_i \frac{q_i \boldsymbol{L}_i(t)}{2m_i}\delta\left[\boldsymbol{r}-\boldsymbol{r}_i(t)\right]$$

- Angular momentum of each particle: $\boldsymbol{L}_i = m_i\left(\boldsymbol{r}_i\times\boldsymbol{v}_i\right)$

(iii) *Microscopic magnetization density*

- Dipole moment: $$\mathfrak{m} = \int_V \mathfrak{M}(\boldsymbol{r})dV = \sum_i \frac{q_i \boldsymbol{L}_i}{2m_i}$$
- For identical particles: $$\mathfrak{m} = \frac{q\boldsymbol{L}}{2m}, \quad \boldsymbol{L} = \sum_i \boldsymbol{L}_i$$
- For electrons in an atom: $$\mathfrak{m}_{\mathfrak{L}} = -\frac{e\boldsymbol{L}}{2m_e}$$
- But electrons also have an intrinsic spin angular momentum: $$\mathfrak{m}_{\mathfrak{s}} = -\frac{g_e e\boldsymbol{S}}{2m_e}$$

(iii) *Microscopic magnetization density*



- Spin is of (relativistic) quantum mechanical origin. The factor g (Lande g-factor) is nearly 2 due to relativistic effects
- Complete theory must be based on quantum mechanical operators for spin and orbital angular moments
- It turns out that one can still relate the dipole moment to ***J*****,** the total angular momentum

$$\mathfrak{m} = -\gamma_e \boldsymbol{J}$$

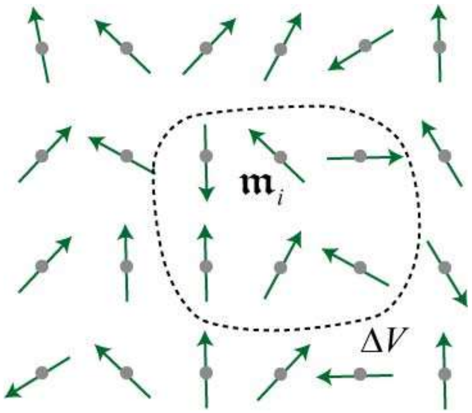
$$\gamma_e = g_J \frac{e}{2m_e}$$

Effective Lande factor

(iv) *Macroscopic magnetization*

- Macroscopic magnetization:

$$\boldsymbol{M}(\boldsymbol{r},t) = \frac{1}{\Delta V}\sum_{i=1}^{N} \mathfrak{m}_i(t)$$



- Vector potential of static macroscopic magnetization:

$$\boldsymbol{A}_M(\boldsymbol{r}) = \mu_0 \int_V \nabla G_0(\boldsymbol{r}-\boldsymbol{r}') \times \boldsymbol{M}(\boldsymbol{r}')dV'$$

- Magnetic field:

$$\boldsymbol{B}_M(\boldsymbol{r}) = \mu_0 \nabla \int_V \nabla G_0(\boldsymbol{r}-\boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}')dV' + \mu_0 \boldsymbol{M}(\boldsymbol{r})$$

$$= -\mu_0 \nabla \int_V \nabla' G_0(\boldsymbol{r}-\boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}')dV' + \mu_0 \boldsymbol{M}(\boldsymbol{r})$$

(iv) *Macroscopic magnetization*

- 1st expression can be written as

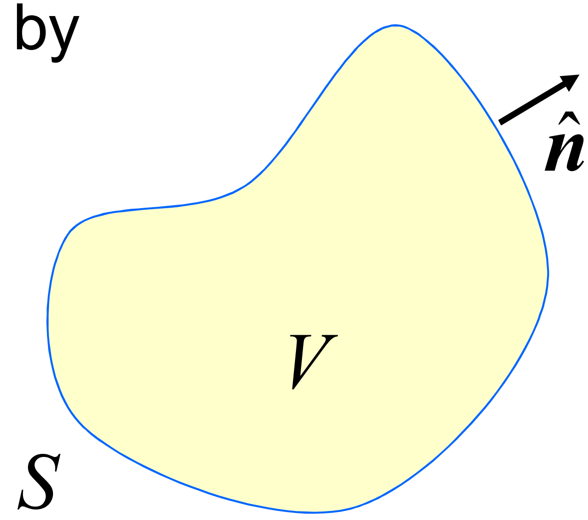
$$\boldsymbol{A}_M(\boldsymbol{r}) = \mu_0 \int_V G_0(\boldsymbol{r}-\boldsymbol{r}')\left[\nabla' \times \boldsymbol{M}(\boldsymbol{r}')\right] dV'$$

$$+\mu_0 \oint_S G_0(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{M}(\boldsymbol{r}') \times d\boldsymbol{s}'$$

- Equal to the vector potential induced by an equivalent bulk current and an equivalent surface current:

$$\boldsymbol{J}_M(\boldsymbol{r}) = \nabla \times \boldsymbol{M}(\boldsymbol{r})$$

$$\boldsymbol{J}_{M,s}(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r}) \times \hat{\boldsymbol{n}}$$



(iv) *Macroscopic magnetization*

- Bulk current directly incorporated into Maxwell equations by separating equivalent magnetization currents from "other" currents

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_c + \mu_0 \boldsymbol{J}_M = \mu_0 \boldsymbol{J}_c + \mu_0 \nabla \times \boldsymbol{M}$$

$$\Rightarrow \begin{cases} \nabla \times \boldsymbol{H} = \boldsymbol{J}_c \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

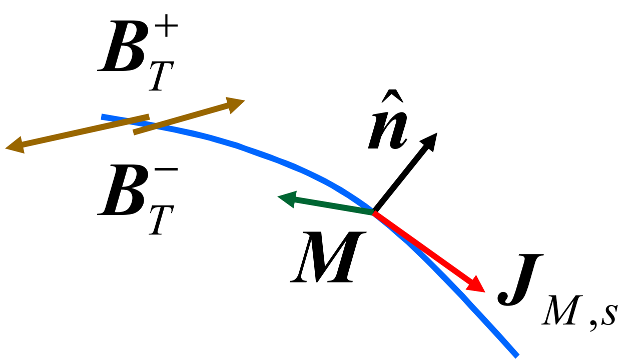
Macroscopic Maxwell equations

$$\boldsymbol{H} = \mu_0^{-1} \boldsymbol{B} - \boldsymbol{M}$$

(iv) *Macroscopic magnetization*

- Equivalent surface current accounted for by boundary conditions: it leads to continuity of tangential $\boldsymbol{H}$ on the surface S

$$\begin{aligned}\boldsymbol{B}_T^+ - \boldsymbol{B}_T^- &= \mu_0 \boldsymbol{J}_{M,s} \times \hat{\boldsymbol{n}} \\ &= \mu_0 \left(\hat{\boldsymbol{n}} \times \boldsymbol{M} \right) \times \hat{\boldsymbol{n}} \\ &= -\mu_0 \boldsymbol{M}_T\end{aligned}$$

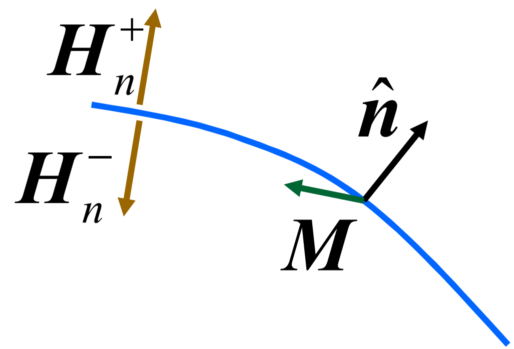


⇒ $$\mu_0^{-1} \boldsymbol{B}_T^+ = \mu_0^{-1} \boldsymbol{B}_T^- - \boldsymbol{M}_T$$

⇒ $$\boldsymbol{H}_T^+ = \boldsymbol{H}_T^-$$

(iv) *Macroscopic magnetization*

- At the surface:



$$B_n^+ = B_n^-$$

$$\Rightarrow \quad H_n^+ - H_n^- = M_n = \hat{\boldsymbol{n}} \cdot \boldsymbol{M}$$

← Equivalent surface magnetic charge density

(iv) *Macroscopic magnetization*

- If there is no free current we can use magnetic potential

$$\boldsymbol{H}_M = -\nabla \varphi_M$$

$$\nabla \cdot \boldsymbol{B} = 0$$

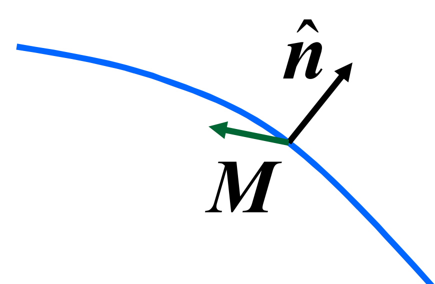
$$\Rightarrow \quad -\nabla^2 \varphi_m = -\nabla \cdot \boldsymbol{M} = \rho_m$$

Equivalent bulk magnetic charge density

- But we should also include the magnetic surface "charge"

$$\rho_{m,s} = \boldsymbol{M} \cdot \hat{\boldsymbol{n}}$$

Equivalent surface magnetic charge density



(iv) *Macroscopic magnetization*

- This will result in

$$\varphi_m(\boldsymbol{r}) = -\int_V \nabla G_0(\boldsymbol{r} - \boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}') dV'$$

$$\boldsymbol{H}(\boldsymbol{r}) = \nabla \int_V \nabla G_0(\boldsymbol{r} - \boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}') dV'$$

- This is identical to what we found before:

$$\boldsymbol{B}(\boldsymbol{r}) = \mu_0 \nabla \int_V \nabla G_0(\boldsymbol{r} - \boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}') dV' + \mu_0 \boldsymbol{M}(\boldsymbol{r})$$

(iv) *Macroscopic magnetization*

- Although the expressions for the Green's functions will be different, the results so far for the vector potential of magnetic dipoles and microscopic magnetization can be generalized to time-dependent Maxwell equations
- Same is true for the macroscopic magnetization
- Time-dependent magnetizations can be treated in Maxwell equations by viewing them as equivalent electric or magnetic currents:

(iv) *Macroscopic magnetization*

- Macroscopic Maxwell equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

- Equivalent electric current as source:

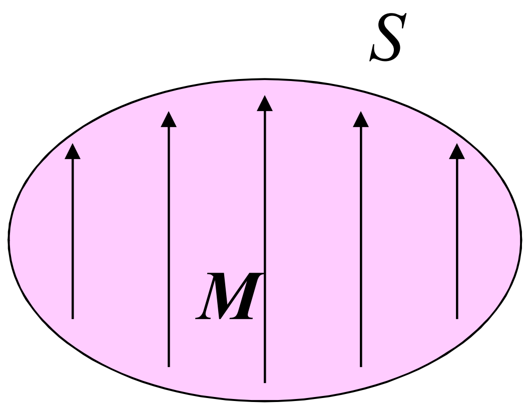
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_0 \frac{\partial \boldsymbol{D}}{\partial t} + \underbrace{\nabla \times \boldsymbol{M}}_{\boldsymbol{J}_e^{eq}}$$

- Equivalent magnetic current as source:

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \underbrace{- \mu_0 \frac{\partial \boldsymbol{M}}{\partial t}}_{\boldsymbol{J}_m^{eq}} \qquad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

(v) *Magnetic field of uniformly magnetized ellipsoids*

- Of particular importance since the field generated inside uniformly magnetized ellipsoids is uniform as well



- Used in model calculations

- Static magnetic field ***H***:

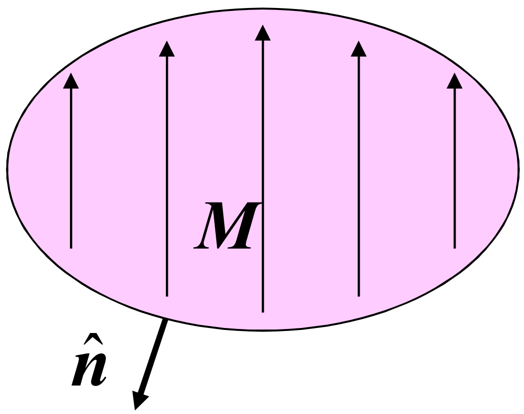
$$\boldsymbol{H}_M = \mu_0^{-1}\boldsymbol{B}_M - \boldsymbol{M} = -\nabla \int_V \nabla' G_0(\boldsymbol{r}-\boldsymbol{r}') \cdot \boldsymbol{M}(\boldsymbol{r}')dV'$$

$$\boldsymbol{H}_M = -\nabla \varphi_M, \quad \varphi_M(\boldsymbol{r}) = \oint_S G_0(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{M}(\boldsymbol{r}') \cdot d\boldsymbol{s}'$$

(v) *Magnetic field of uniformly magnetized ellipsoids*

- Equivalent to the "electrostatic" potential of a surface "electric charge" with the density

$$\rho_M = \boldsymbol{M} \cdot \hat{\boldsymbol{n}}$$



- Instead of computing the integral, we solve the Poisson (Laplace) equation

$$\nabla^2 \varphi_M = 0$$

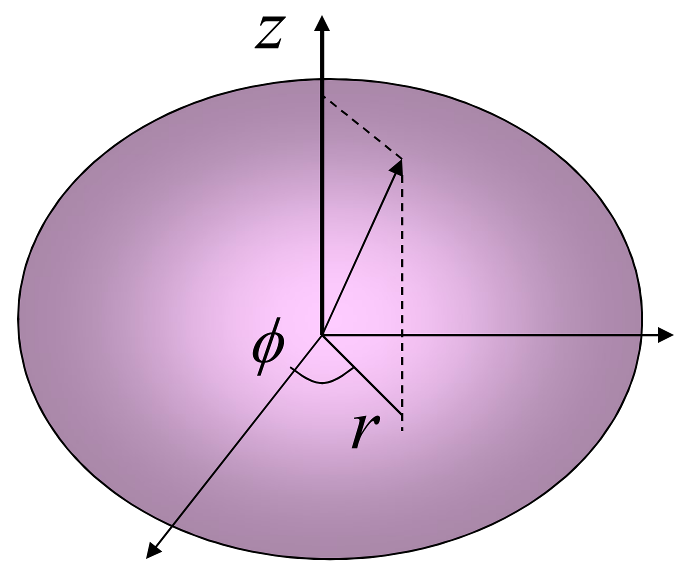
- Together with the surface boundary condition

$$\hat{\boldsymbol{n}} \cdot \left(\boldsymbol{H}_M^+ - \boldsymbol{H}_M^- \right) = \hat{\boldsymbol{n}} \cdot \boldsymbol{M}$$

(v) *Magnetic field of uniformly magnetized ellipsoids*

- Particular case: spheroid (ellipsoid of revolution around z-axis)

$$\boldsymbol{M} = M_z \hat{\boldsymbol{z}}$$



- Use cylindrical-elliptic coordinate system:

$$x = a\cosh(\xi)\sin\theta\cos\phi$$
$$y = a\cosh(\xi)\sin\theta\sin\phi$$
$$z = a\sinh(\xi)\cos\theta$$

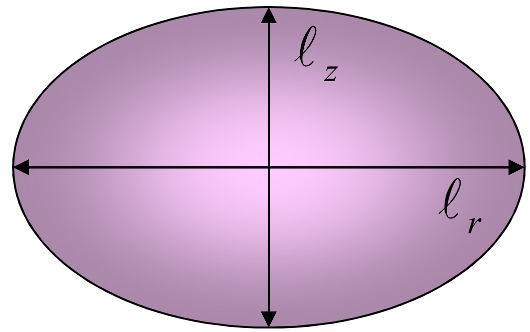
(v) *Magnetic field of uniformly magnetized ellipsoids*

- The demagnetization field:

$$\boldsymbol{H}_M = -N_z M_z \hat{\boldsymbol{z}}$$

$$N_z = \left[1 - s_0 \arctan(1/s_0)\right](1 + s_0^2)$$

$$s_0 = \frac{\ell_r}{\sqrt{\ell_r^2 - \ell_z^2}}$$



- In general for any ellipsoid (with coordinates along the x,y,z axes):

$$\boldsymbol{H}_M = -\begin{bmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & N_z \end{bmatrix} \cdot \boldsymbol{M}$$

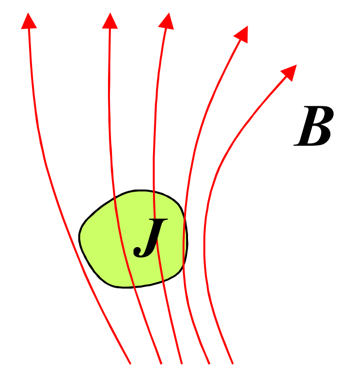
$$N_x + N_y + N_z = 1$$

Motion of microscopic dipoles in a magnetic field

- Dipole experiences force & torque in a magnetic field because of the Lorentz force exerted on moving electric charges forming the dipole current

- Volume density of force exerted on moving electric charges comprising an electric current:

$$\boldsymbol{\mathcal{F}}(\boldsymbol{r}) = \sum_i q_i \boldsymbol{v}_i \times \boldsymbol{B}(\boldsymbol{r}_i)\delta(\boldsymbol{r}-\boldsymbol{r}_i) = \boldsymbol{J}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})$$



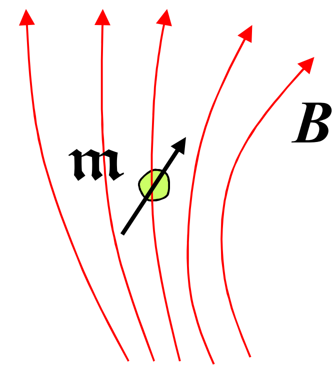
- Total force on a dipole:

$$\boldsymbol{F} = \int_V \boldsymbol{\mathcal{F}}(\boldsymbol{r})dV = \int_V \boldsymbol{J}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})dV$$

(i) *Force and torque exerted on a magnetic dipole*

- Using the fact that current density has no divergence, we find the net force on the dipole to be

$$\boldsymbol{F} = \nabla(\mathfrak{m} \cdot \boldsymbol{B}) = -\nabla U$$



- This is a conservative force. It can be written as the gradient of the dipole potential energy:

$$U(\boldsymbol{r}) = -\mathfrak{m} \cdot \boldsymbol{B}(\boldsymbol{r})$$

- Dipole will tend to align itself parallel to the magnetic field in order to minimize its potential energy

(ii) *Dipole approximation*

- For the derivation we use

$$\int_V r'_\alpha J_\beta dV' = -\int_V r'_\beta J_\alpha dV' = \frac{1}{2}\int_V r'_\alpha J_\beta dV' - \frac{1}{2}\int_V r'_\beta J_\alpha dV' \rightarrow$$

$$\int_V r'_x J_y dV' = -\int_V r'_y J_x dV' = \mathfrak{m}_z, \int_V r'_y J_z dV' = -\int_V r'_z J_y dV' = \mathfrak{m}_x$$

$$\int_V r'_z J_x dV' = -\int_V r'_x J_z dV' = \mathfrak{m}_y$$

$$F_x = \int_V J_y(\boldsymbol{r}) B_z(\boldsymbol{r}) dV - \int_V J_z(\boldsymbol{r}) B_y(\boldsymbol{r}) dV$$

$$\sim \int_V J_y(\boldsymbol{r})\left(\boldsymbol{r}\cdot\nabla\right)B_z(0)dV - \int_V J_z(\boldsymbol{r})\left(\boldsymbol{r}\cdot\nabla\right)B_y(0)dV$$

$$= \mathfrak{m}_z \frac{\partial B_z(0)}{\partial x} - \mathfrak{m}_x \frac{\partial B_z(0)}{\partial z} + \mathfrak{m}_y \frac{\partial B_y(0)}{\partial x} - \mathfrak{m}_x \frac{\partial B_y(0)}{\partial y} =$$

$$= \boldsymbol{\mathfrak{m}} \cdot \frac{\partial \boldsymbol{B}(0)}{\partial x} - \mathfrak{m}_x \nabla\cdot\boldsymbol{B}(0) = \frac{\partial\left[\boldsymbol{\mathfrak{m}}\cdot\boldsymbol{B}(0)\right]}{\partial x}\cdots$$

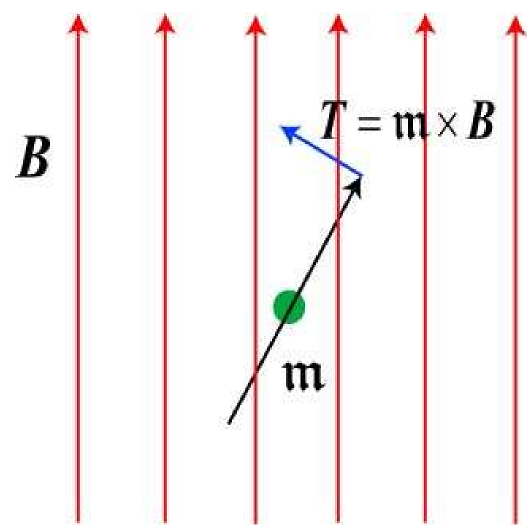
(i) *Force and torque exerted on a magnetic dipole*

- Note that no force is exerted by a uniform magnetic field
- But the dipole always experiences a total torque

$$\boldsymbol{T} = \int_V \boldsymbol{r} \times [\boldsymbol{J}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})] dV$$

- Which can be written as

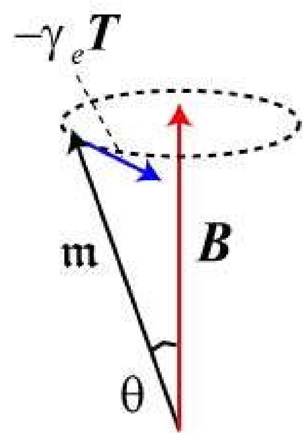
$$\boldsymbol{T} = \mathfrak{m} \times \boldsymbol{B}$$



(ii) *Classical motion of an atomic dipole in a magnetic field*

- Consider an atomic dipole with a total angular momentum $\boldsymbol{J}$ in a magnetic field
- Classical equation of motion of the magnetic moment:

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{T} \quad \Rightarrow \quad \frac{d\mathfrak{m}}{dt} = -\gamma_e \mathfrak{m} \times \boldsymbol{B}$$

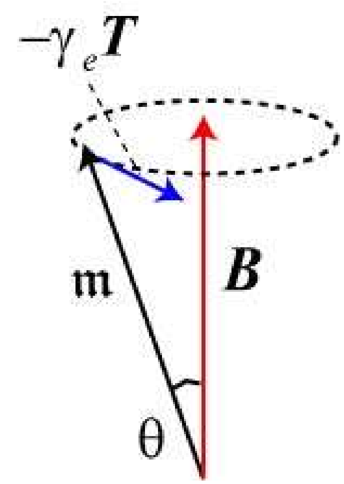


- The dipole rotates around the field vector
- Its angle (and therefore its energy) remains constant during the motion

(ii) *Classical motion of an atomic dipole in a magnetic field*

- The frequency of rotation is the Larmor frequency

$$\omega_0 = \gamma_e \boldsymbol{B}$$



- Note that the magnitude of the dipole moment is preserved in time: the dipole is considered as a rigid body
- In reality the magnetic field also affects the microscopic currents inside an atomic dipole. But this change is negligibly small in practical situations

(iii) *Damped motion of magnetic dipoles*

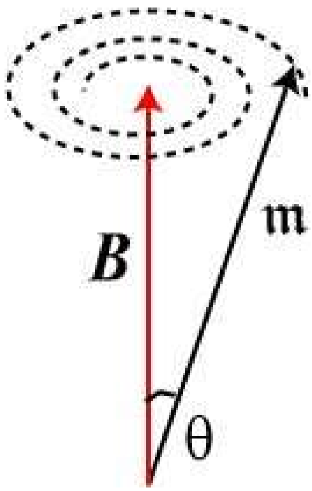
- If the energy remains constant, how will the dipole try to reach the minimum energy direction along the field?
- Tendency of physical systems to reach their minimum energy state is of statistical nature, caused by the interaction with the outside world.
- This interaction is irreversible: it involves the transfer of energy from the system to the environment.
- For the dipole, this effect is taken into account in a phenomenological way by introducing "friction" which slows down the dipole motion

(iii) *Damped motion of magnetic dipoles*

- To the dipole equation of motion, we add a friction term:

$$\frac{d\mathfrak{m}}{dt} = -\gamma_e \mathfrak{m} \times \boldsymbol{B} + \bar{\alpha}\,\mathfrak{m} \times \frac{d\mathfrak{m}}{dt}$$

- As the dipole rotates around the magnetic field, its angle with the field will decrease in time due to damping.
- The dipole looses its (potential) energy, and eventually reaches the minimum energy direction.



(iv) *Equation of motion of macroscopic magnetization*

- The equation of motion of the dipole can de generalized to describe the motion of the macroscopic magnetization *at any point*:

$$\frac{d\boldsymbol{M}}{dt} = -\gamma_e \boldsymbol{M} \times \boldsymbol{B}$$

- This equation can be rewritten as

$$\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{H} \qquad \gamma = \mu_0 \gamma_e$$

- Note that the magnitude of the magnetization vector is again preserved in time

(iv) *Equation of motion of macroscopic magnetization*

- But, there are some issues with this model:
  - Equation of motion describes the evolution of magnetization from an initial state. What is nature of this state?
  - The magnetic field contains both external and internal field. The latter is the collective field generated by all the dipoles in the material.
  - Are the magnetic forces the only forces acting on the dipoles?
  - What about thermal fluctuations?
- Answering these questions requires knowledge of microscopic forces between dipoles, and their relative alignment in magnetic materials