Microwave Magnetics

Graduate Course

Electrical Engineering (Communications)

2^{nd} Semester, 1387-1388

Sharif University of Technology

General information

- **Contents of lecture 5:**
  - Waveguides containing longitudinally magnetized media
    - *Motivation*
    - *Method of solution*
    - *Circular waveguide*
    - *Partially filled circular waveguide*
    - *Faraday ferrite devices*

(i) *Motivation*

- So far we only considered waveguides were waves propagate normal to the direction of magnetization.
- Here we focus on circular waveguides where the magnetic medium is magnetized in the direction along the guide
- We already considered waves along the magnetization in a homogeneous magnetic medium where circular polarization plays an important role and phenomena such as Faraday effect are observed.
- But first we review conventional circular guides

(ii) *Conventional waveguides*

- For TM waves:

$$\tilde{h}_z = 0$$

$$\nabla_\perp^2 \tilde{e}_z + \left(k_0^2 \varepsilon - \beta^2\right) \tilde{e}_z = 0$$

$$\tilde{\mathbf{e}}_\perp = \begin{bmatrix} \tilde{e}_x \\ \tilde{e}_y \end{bmatrix} = \frac{-j\beta}{k_c^2} \nabla_\perp \tilde{e}_z$$

$$\tilde{\mathbf{h}}_\perp = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = -\frac{j\omega\varepsilon_0\varepsilon}{k_c^2} \hat{\boldsymbol{z}} \times \nabla_\perp \tilde{e}_z$$

- For TE waves:

$$\tilde{e}_z = 0$$

$$\nabla_\perp^2 \tilde{h}_z + \left(k_0^2 \varepsilon - \beta^2\right) \tilde{h}_z = 0$$

$$\tilde{\mathbf{e}}_\perp = \begin{bmatrix} \tilde{e}_x \\ \tilde{e}_y \end{bmatrix} = \frac{j\omega\mu_0}{k_c^2} \hat{\boldsymbol{z}} \times \nabla_\perp \tilde{h}_z$$

$$\tilde{\mathbf{h}}_\perp = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = -\frac{-j\beta}{k_c^2} \nabla_\perp \tilde{h}_z$$

$$k_c^2 = k_0^2 \varepsilon - \beta^2$$

(ii) *Conventional Circular waveguide*

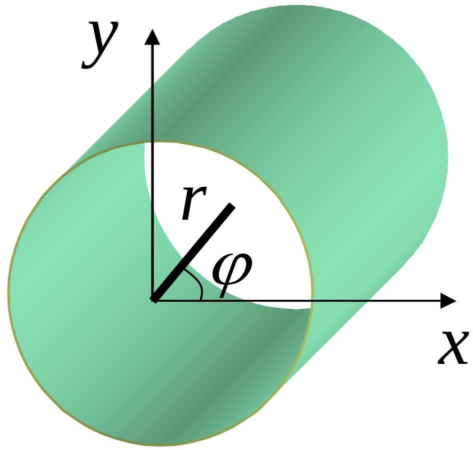
- Circular waveguide
- TM modes:

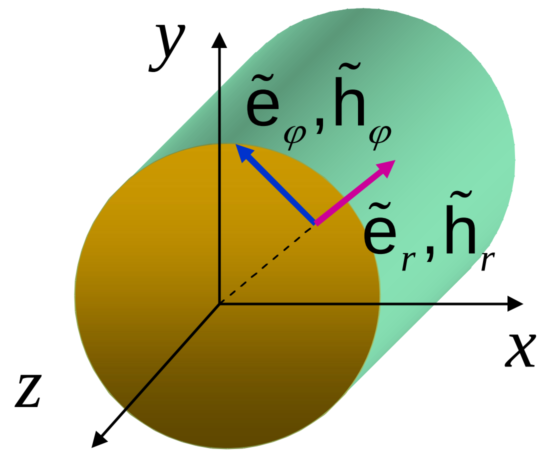
$$\tilde{e}_z = \tilde{e}_z(r,\varphi)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{e}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \tilde{e}_z}{\partial \varphi^2} + k_c^2 \tilde{e}_z = 0$$

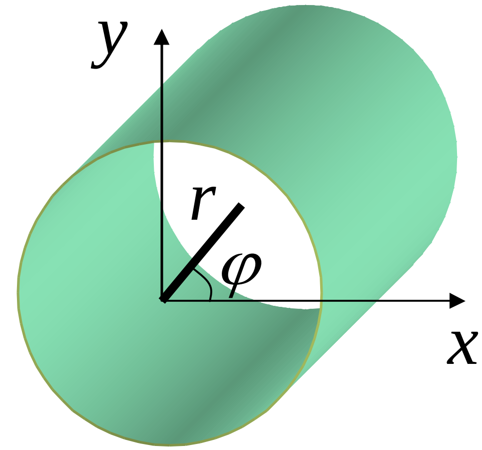
$$\tilde{e}_r = \frac{-j\beta}{k_c^2}\frac{\partial \tilde{e}_z}{\partial r} \qquad \tilde{h}_r = \frac{j\omega\varepsilon_0\varepsilon}{k_c^2}\frac{\partial \tilde{e}_z}{r\partial \varphi}$$

$$\tilde{e}_\varphi = \frac{-j\beta}{k_c^2}\frac{\partial \tilde{e}_z}{r\partial \varphi} \qquad \tilde{h}_\varphi = \frac{-j\omega\varepsilon_0\varepsilon}{k_c^2}\frac{\partial \tilde{e}_z}{\partial r}$$





(ii) *Conventional Circular waveguide*



- Solution by the function

$$\tilde{e}_z(r,\varphi) = \exp(jm\varphi)\, f_m(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_m}{\partial r}\right) + \left(k_c^2 - \frac{m^2}{r^2}\right) f_m = 0$$

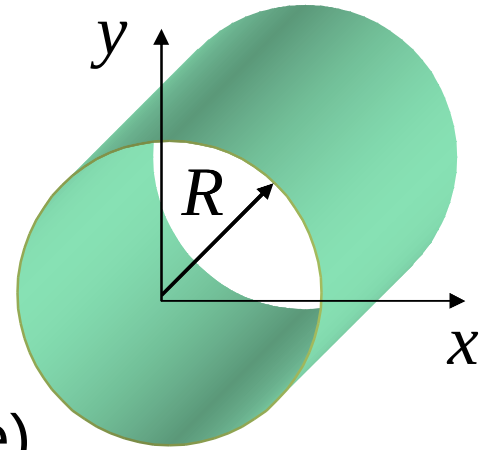
- Solution which remains finite as r → 0 :

$$f_m(r) = J_m(k_c r)$$

Bessel function of the 1st kind

$$\Rightarrow \quad \tilde{e}_z(r,\varphi) = \exp(jm\varphi)\, J_m(k_c r)$$

(ii) *Conventional Circular waveguide*



- TM modes

$$\tilde{e}_z(R,\varphi)=0 \quad \Rightarrow \quad J_m(k_c R)=0$$

- For each *m* (both positive and negative)

$$k_c=\frac{\xi_{m,n}}{R}$$

$\xi_{m,n}$: n-th root of $J_m(z)$ (n>0)

$$\beta=\sqrt{k_0^2\varepsilon-\left(\frac{\xi_{m,n}}{R}\right)^2}$$

- TE modes are similar, but now

$$k_c=\frac{\zeta_{m,n}}{R}$$

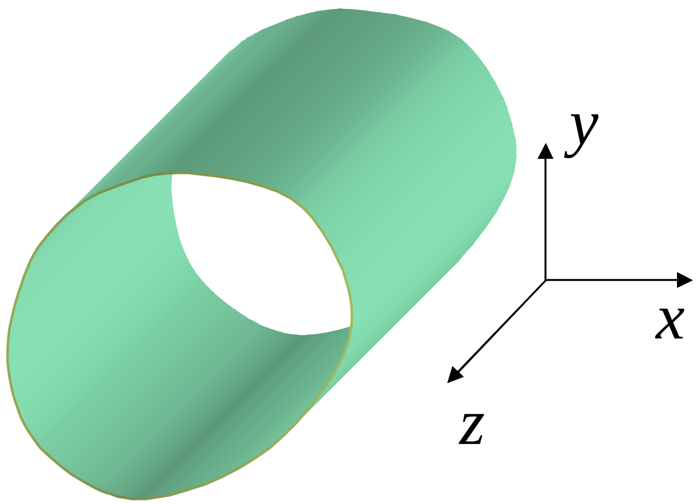
$\zeta_{m,n}$: n-th root of $dJ_m(z)/dz$ (n>0)

(ii) *Method of solution*

- We look for solutions representing propagating waves in the z-direction:

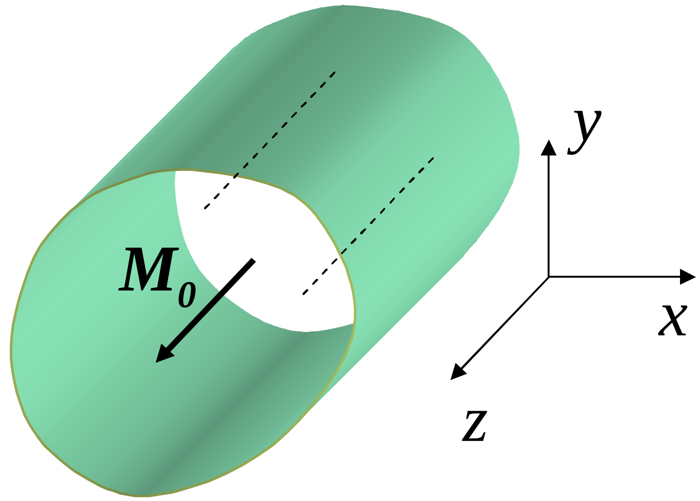
$$\mathbf{e} = \tilde{\mathbf{e}}(r,\varphi)\exp(-j\beta z) \qquad \mathbf{h} = \tilde{\mathbf{h}}(r,\varphi)\exp(-j\beta z)$$

- For a conventional, nonmagnetic waveguide we always have separate TE and TM modes
- They are described in term of longitudinal components of the electric and magnetic fields



(iii) *Magnetic waveguide*

- Consider now a waveguide containing a magnetic medium magnetized along the guide
- We return to the equations describing a magnetic medium



$$\nabla_{\perp}^{2}\mathrm{e}_z+\frac{\partial^2\mathrm{e}_z}{\partial z^2}+\left(k_0^2\varepsilon\mu_{\perp}\right)\mathrm{e}_z+\left(\omega\mu_0\frac{\mu_a}{\mu}\right)\frac{\partial\mathrm{h}_z}{\partial z}=0$$

$$\nabla_{\perp}^{2}\mathrm{h}_z+\frac{1}{\mu}\frac{\partial^2\mathrm{h}_z}{\partial z^2}+\left(k_0^2\varepsilon\right)\mathrm{h}_z-\left(\omega\varepsilon_0\varepsilon\frac{\mu_a}{\mu}\right)\frac{\partial\mathrm{e}_z}{\partial z}=0$$

(iii) *Method of solution*

- Unlike conventional waveguides, resulting equation involves both longitudinal electric and magnetic fields:

$$\mathbf{e} = \tilde{\mathbf{e}}(r,\varphi)\exp(-j\beta z) \qquad \mathbf{h} = \tilde{\mathbf{h}}(r,\varphi)\exp(-j\beta z)$$

$$\nabla_{\perp}^{2}\tilde{\mathrm{e}}_{z} + \left(k_{0}^{2}\varepsilon\mu_{\perp} - \beta^{2}\right)\tilde{\mathrm{e}}_{z} - j\beta\left(\omega\mu_{0}\frac{\mu_{a}}{\mu}\right)\tilde{\mathrm{h}}_{z} = 0$$

$$\nabla_{\perp}^{2}\tilde{\mathrm{h}}_{z} + \left(k_{0}^{2}\varepsilon - \frac{1}{\mu}\beta^{2}\right)\tilde{\mathrm{h}}_{z} + j\beta\left(\omega\varepsilon_{0}\varepsilon\frac{\mu_{a}}{\mu}\right)\tilde{\mathrm{e}}_{z} = 0$$

(ii) *Method of solution*

- Result: $$\left(k_0^2\varepsilon\right)^{-2}\nabla_\perp^4\tilde{h}_z + P\left(k_0^2\varepsilon\right)^{-1}\nabla_\perp^2\tilde{h}_z + Q\,\tilde{h}_z = 0$$

$$P = \mu_\perp + 1 - \left(1 + \frac{1}{\mu}\right)\mu_{ef}$$

$$Q = \mu_\perp - 2\mu_{ef} + \frac{1}{\mu}\mu_{ef}^2$$

$$\mu_{ef} \equiv \frac{\beta^2}{k_0^2\varepsilon}$$

- Equation satisfied by the solutions of

$$\nabla_\perp^2\Psi_1 + \kappa_1^2\Psi_1 = 0 \qquad \nabla_\perp^2\Psi_2 + \kappa_2^2\Psi_2 = 0$$

(ii) *Method of solution*

- Here we have defined $\kappa_{1,2}^2 = k_0^2 \varepsilon \eta_{1,2}$

- η_1 and η_2 are roots of $\eta^2 - P\ \eta + Q\ = 0$

- Solution:

$$\eta_{1,2} = \frac{1}{2}\left(P \pm \sqrt{P^2 - 4Q}\right)$$

$$= \frac{1}{2}\left\{\mu_\perp + 1 - \left(1 + \frac{1}{\mu}\right)\mu_{ef} \pm \sqrt{\left[\mu_\perp - 1 - \left(1 - \frac{1}{\mu}\right)\mu_{ef}\right]^2 + 4\mu_{ef}\frac{\mu_a^2}{\mu^2}}\right\}$$

(ii) *Method of solution*

- We then have to solve

$$\nabla_{\perp}^{2}\Psi_{1,2}+\kappa_{1,2}^{2}\Psi_{1,2}=0$$

- General solution for longitudinal magnetic field:

$$\tilde{h}_z(r,\varphi)=A\ \Psi_1(r,\varphi)+B\ \Psi_2(r,\varphi)$$

(ii) *Method of solution*

- The electric field

$$\tilde{e}_z(r,\varphi) = \frac{j\omega\mu_0\mu}{\beta\mu_a}\left\{A\left[\frac{1}{\mu}\left(\mu-\mu_{ef}\right)-\eta_1\right]\Psi_1(r,\varphi)\right.$$
$$\left.+B\left[\frac{1}{\mu}\left(\mu-\mu_{ef}\right)-\eta_2\right]\Psi_2(r,\varphi)\right\}$$

- β is as yet unknown. Its value must be found by setting up boundary conditions and solving the full problem
- Note also that η_1 and η_2 are independent of the sign of β (propagation direction)

(ii) *Method of solution*

- Remark regarding the cutoff frequency: despite the complicated behavior of the system, the cutoff frequency of a magnetic waveguide is simple to derive
- Cutoff: frequency at which propagation starts to take place; frequency at which $\beta = 0 \rightarrow \mu_{ef} = 0$
- Electric & magnetic z-components decouple at this point

$$\nabla_{\perp}^{2}\tilde{\mathrm{e}}_{z} + \left(k_{0}^{2}\varepsilon\mu_{\perp}\right)\tilde{\mathrm{e}}_{z} = 0$$

$$\nabla_{\perp}^{2}\tilde{\mathrm{h}}_{z} + \left(k_{0}^{2}\varepsilon\right)\tilde{\mathrm{h}}_{z} = 0$$

(ii) *Method of solution*

- These are separate problems for
  - TM mode with permeability $\mu_\perp$
  - TE mode with permeability $\mu_{||}$
- The cutoff in these cases also depend on boundary conditions on waveguide walls
- TM case: e_z=0 on the wall. Problem becomes identical to that of a conventional guide filled with a conventional material with permeability $\mu_\perp$

(ii) *Method of solution*

- TE case: boundary condition imposed on the tangential, transverse electric field on the wall
- These components follows from the original equations

$$\nabla_\perp \tilde{h}_z + j\beta \tilde{\mathbf{h}}_\perp = j\omega\varepsilon_0\varepsilon\, \hat{\boldsymbol{z}} \times \tilde{\mathbf{e}}_\perp$$

$$\nabla_\perp \tilde{e}_z + j\beta \tilde{\mathbf{e}}_\perp = -j\omega\mu_0\mu\, \hat{\boldsymbol{z}} \times \tilde{\mathbf{h}}_\perp + \omega\mu_0\mu_a \tilde{\mathbf{h}}_\perp$$

- But when β → 0, these equations do not care whether we have a magnetic material or not

$$\nabla_\perp \tilde{h}_z = j\omega\varepsilon_0\varepsilon\, \hat{\boldsymbol{z}} \times \tilde{\mathbf{e}}_\perp \rightarrow \tilde{\mathbf{e}}_\perp = -\frac{1}{j\omega\varepsilon_0\varepsilon} \hat{\boldsymbol{z}} \times \nabla_\perp \tilde{h}_z$$

(ii) *Method of solution*

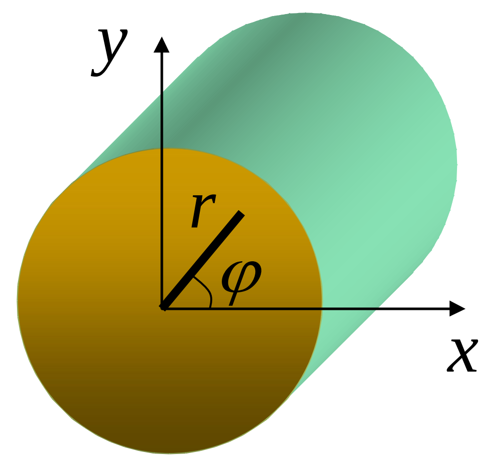
- The resulting boundary condition becomes identical to that of the TE mode of a conventional waveguide filled with a conventional medium with permeability $\mu_{||}$
- Therefore: the cutoff frequency of the modes of the waveguide with magnetic material is identical to those of TM and TE modes in an ordinary waveguide, but now filled with a conventional, isotropic material with permeability $\mu_{\perp}, \mu_{||}$

(iii) *Circular waveguide (longitudinal magnetization)*

- Circular waveguide completely filled with longitudinally magnetized magnetic material can be analytically solved
- Equations solved using cylindrical coordinates

$$\nabla_{\perp}^{2}\Psi_{1,2} + \kappa_{1,2}^{2}\Psi_{1,2} = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \Psi}{\partial r}\right) + \dfrac{1}{r^{2}}\dfrac{\partial^{2}\Psi}{\partial \varphi^{2}} + \kappa^{2}\Psi = 0 \\ \Psi = \Psi_{1,2}(r,\varphi) \qquad \kappa = \kappa_{1,2} \end{cases}$$



(iii) *Circular waveguide (longitudinal magnetization)*

- Let us write the solution as

$$\Psi(r,\varphi) = \exp(jm\varphi)\, f_m(r)$$

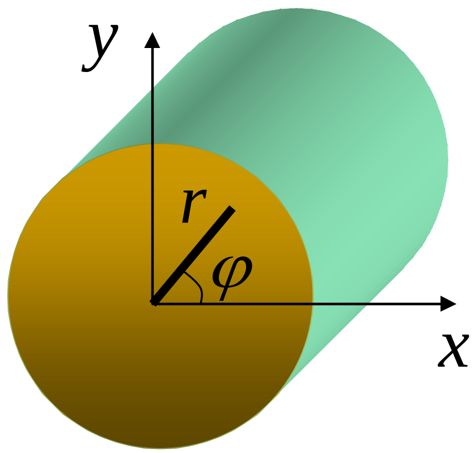
$$r^2 \frac{d^2 f_m(r)}{dr^2} + r\frac{df_m(r)}{dr} + \left(\kappa^2 r^2 - m^2\right) f_m(r) = 0$$

- Solution which remains finite as r → 0 :

$$f_m(r) = J_m(\kappa r)$$

Bessel function of the 1st kind

$$\Psi_{1,2}(r,\varphi) = \exp(jm\varphi) J_m(\kappa_{1,2} r)$$



(iii) *Circular waveguide (longitudinal magnetization)*

- General solution for longitudinal magnetic field for a particular "mode" denoted by m :

$$\tilde{h}_z^m(r,\varphi) = \exp(jm\varphi)\left[A\, J_m(\kappa_1 r) + BJ_m(\kappa_2 r)\ \right]$$

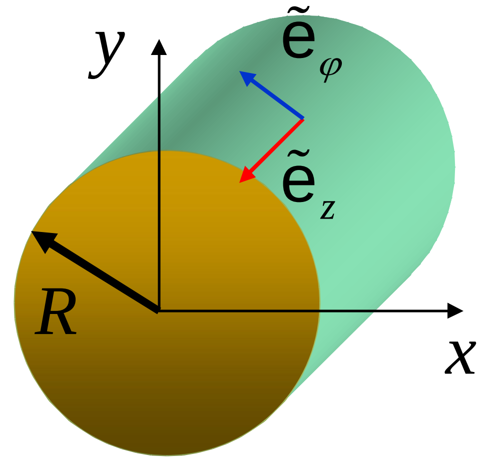
- Longitudinal electric field:

$$\tilde{e}_z^m(r,\varphi) = \frac{j\omega\mu_0\mu}{\beta\mu_a}\exp(jm\varphi)\times$$

$$\left\{\left[\frac{1}{\mu}\left(\mu-\mu_{ef}\right)-\eta_1\right]AJ_m(\kappa_1 r)+\left[\frac{1}{\mu}\left(\mu-\mu_{ef}\right)-\eta_2\right]BJ_m(\kappa_2 r)\right\}$$

(iii) *Circular waveguide (longitudinal magnetization)*

- We assume the waveguide wall to be a perfect conductor



$$\tilde{e}_z^m(R,\varphi) = 0$$

$$\left[\frac{1}{\mu}\left(\mu - \mu_{ef}\right) - \eta_1\right] J_m(\kappa_1 R) A + \left[\frac{1}{\mu}\left(\mu - \mu_{ef}\right) - \eta_2\right] J_m(\kappa_2 R) B = 0$$

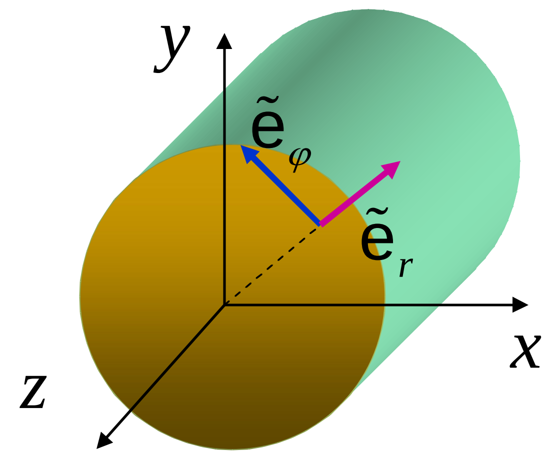
- We should find another equation for A and B. Consider the transverse components of the electric field.

(iii) *Circular waveguide (longitudinal magnetization)*

- We can extract the transverse field by using the 'original' equations:

$$\nabla_\perp \tilde{\mathrm{e}}_z + j\beta \tilde{\mathbf{e}}_\perp = -j\omega\mu_0\mu\, \hat{\mathbf{z}} \times \tilde{\mathbf{h}}_\perp + \omega\mu_0\mu_a \tilde{\mathbf{h}}_\perp$$

$$\nabla_\perp \tilde{\mathrm{h}}_z + j\beta \tilde{\mathbf{h}}_\perp = j\omega\varepsilon_0\varepsilon\, \hat{\mathbf{z}} \times \tilde{\mathbf{e}}_\perp$$



- In cylindrical coordinates:

$$\frac{\partial \tilde{\mathrm{e}}_z}{\partial r} + j\beta \tilde{\mathrm{e}}_r = j\omega\mu_0\mu\, \tilde{\mathrm{h}}_\varphi + \omega\mu_0\mu_a\, \tilde{\mathrm{h}}_r \qquad \frac{\partial \tilde{\mathrm{h}}_z}{\partial r} + j\beta \tilde{\mathrm{h}}_r = -j\omega\varepsilon_0\varepsilon\, \tilde{\mathrm{e}}_\varphi$$

$$\frac{\partial \tilde{\mathrm{e}}_z}{r\partial \varphi} + j\beta \tilde{\mathrm{e}}_\varphi = -j\omega\mu_0\mu\, \tilde{\mathrm{h}}_r + \omega\mu_0\mu_a\, \tilde{\mathrm{h}}_\varphi \qquad \frac{\partial \tilde{\mathrm{h}}_z}{r\partial \varphi} + j\beta \tilde{\mathrm{h}}_\varphi = j\omega\varepsilon_0\varepsilon\, \tilde{\mathrm{e}}_r$$

(iii) *Circular waveguide (longitudinal magnetization)*

- Introduce circularly polarized fields:

$$\tilde{e}_{\pm} = \tilde{e}_r \pm j\tilde{e}_{\varphi} \qquad \tilde{h}_{\pm} = \tilde{h}_r \pm j\tilde{h}_{\varphi}$$

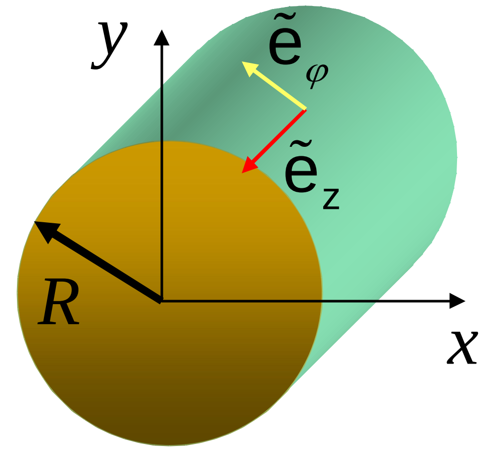
$$\Rightarrow \begin{cases} \dfrac{\partial \tilde{e}_z}{\partial \xi_+} + j\beta \tilde{e}_+ = \omega\mu_0\mu_+ \tilde{h}_+ \\ \dfrac{\partial \tilde{h}_z}{\partial \xi_+} + j\beta \tilde{h}_+ = -\omega\varepsilon_0\varepsilon\, \tilde{e}_+ \end{cases} \qquad \begin{cases} \dfrac{\partial \tilde{e}_z}{\partial \xi_-} + j\beta \tilde{e}_- = -\omega\mu_0\mu_- \tilde{h}_- \\ \dfrac{\partial \tilde{h}_z}{\partial \xi_-} + j\beta \tilde{h}_- = \omega\varepsilon_0\varepsilon\, \tilde{e}_- \end{cases}$$

$$\frac{\partial}{\partial \xi_+} = \frac{\partial}{\partial r} + j\frac{\partial}{r\partial\varphi} \qquad \frac{\partial}{\partial \xi_-} = \frac{\partial}{\partial r} - j\frac{\partial}{r\partial\varphi}$$

(iii) *Circular waveguide (longitudinal magnetization)*

$$\tilde{e}_{\pm} = \frac{1}{\beta^2 - k_0^2 \varepsilon \mu_{\pm}} \frac{\partial}{\partial \xi_{\pm}} \left(\pm \omega \mu_0 \mu_{\pm} \tilde{h}_z + j \beta \tilde{e}_z \right)$$

$$\tilde{h}_{\pm} = \frac{1}{\beta^2 - k_0^2 \varepsilon \mu_{\pm}} \frac{\partial}{\partial \xi_{\pm}} \left(j \beta \tilde{h}_z \mp \omega \varepsilon_0 \varepsilon \tilde{e}_z \right)$$



$$\tilde{e}_{\varphi} = -\frac{j}{2}\left(\tilde{e}_{+} - \tilde{e}_{-}\right) = -\frac{j\omega\mu_0}{\Delta}\left[\mu\left(\beta^2 - k_0^2 \varepsilon \mu_{\perp}\right)\frac{\partial \tilde{h}_z}{\partial r} + j\mu_a \beta^2 \frac{\partial \tilde{h}_z}{r\partial\varphi}\right]$$

$$+\frac{\beta}{\Delta}\left[k_0^2 \varepsilon \mu_a \frac{\partial \tilde{e}_z}{\partial r} + j\left(\beta^2 - k_0^2 \varepsilon \mu\right)\frac{\partial \tilde{e}_z}{r\partial\varphi}\right]$$

$$\Delta = \left(\beta^2 - k_0^2 \varepsilon \mu_{-}\right)\left(\beta^2 - k_0^2 \varepsilon \mu_{+}\right)$$

(iii) *Circular waveguide (longitudinal magnetization)*

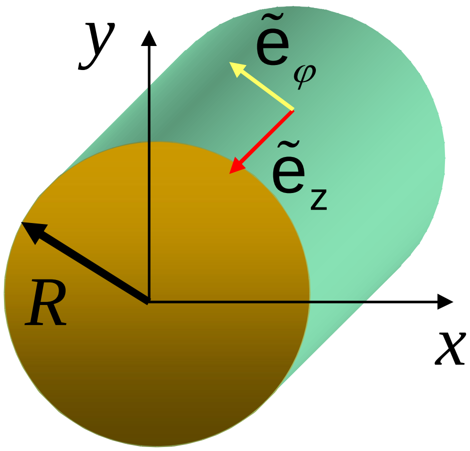
- We rewrite this as

$$\tilde{e}_\varphi = -\frac{j\omega\mu_0 k_0^2 \varepsilon}{\Delta}\left[\mu\left(\mu_{ef} - \mu_\perp\right)\frac{\partial \tilde{h}_z}{\partial r} + j\mu_a \mu_{ef}\frac{\partial \tilde{h}_z}{r\partial\varphi}\right]$$

$$+\frac{\beta k_0^2 \varepsilon}{\Delta}\left[\mu_a \frac{\partial \tilde{e}_z}{\partial r} + j\left(\mu_{ef} - \mu\right)\frac{\partial \tilde{e}_z}{r\partial\varphi}\right]$$

- 2nd equation found by putting

$$\tilde{e}_\varphi(R,\varphi) = 0$$



(iii) *Circular waveguide (longitudinal magnetization)*

- This involves a long and tedious calculation leading to

$$\left(\frac{\mu_\perp - \mu_{ef}}{\eta_1} - 1\right)\frac{\kappa_1 J'_m(\kappa_1 R)}{J_m(\kappa_1 R)} + \frac{\mu_a m}{\mu R}\frac{\mu_{ef}}{\eta_1} = \left(\frac{\mu_\perp - \mu_{ef}}{\eta_2} - 1\right)\frac{\kappa_2 J'_m(\kappa_2 R)}{J_m(\kappa_2 R)} + \frac{\mu_a m}{\mu R}\frac{\mu_{ef}}{\eta_2}$$

- This equation has to be solved for μ_{ef} together with

$$\eta_{1,2} = \frac{1}{2}\left\{\mu_\perp + 1 - \left(1 + \frac{1}{\mu}\right)\mu_{ef} \pm \sqrt{\left[\mu_\perp - 1 - \left(1 - \frac{1}{\mu}\right)\mu_{ef}\right]^2 + 4\mu_{ef}\frac{\mu_a^2}{\mu^2}}\right\}$$

(iii) *Circular waveguide (longitudinal magnetization)*

- This equation must be solved for each mode *m*
- The details of the solution is not important. But note that |β| does not depend on direction of propagation

$$\beta^2 = k_0^2 \varepsilon \mu_{ef}$$

- But, it depends on the sign of the mode number *m.*
- Propagation constant will be different for +m and –m.
- In what follows we try to look for an approximate solution at very high frequencies as an example.

(iii) *Circular waveguide (longitudinal magnetization)*

- Consider the case where $\omega \gg \omega_H, \omega_M$.

$$\mu = 1 + \frac{\omega_H \omega_M}{\omega_H^2 - \omega^2} \approx 1, \quad \mu_a = \frac{\omega_M \omega}{\omega_H^2 - \omega^2} \approx -\frac{\omega_M}{\omega}$$

$$\mu_\perp = \mu - \frac{\mu_a^2}{\mu} \approx 1$$

- In this limit

$$\eta_{1,2} = 1 - \mu_{ef} \pm \frac{\omega_M}{\omega}\sqrt{\mu_{ef}} \qquad \mu_\perp - \mu_{ef} - \eta_{1,2} = \mp \frac{\omega_M}{\omega}\sqrt{\mu_{ef}}$$

$$\kappa_{1,2}^2 = k_0^2 \varepsilon \eta_{1,2} = k_0^2 \varepsilon - \beta^2 \pm \frac{k_0 \omega_M \sqrt{\varepsilon}}{\omega} \beta$$

(iii) *Circular waveguide (longitudinal magnetization)*

- The original equation becomes

$$\left(\kappa_1 R\right) J_m'(\kappa_1 R) J_m(\kappa_2 R) + \left(\kappa_2 R\right) J_m'(\kappa_2 R) J_m(\kappa_1 R) = -\frac{2\mu_{ef}}{1-\mu_{ef}} \frac{m\omega_M}{\omega} J_m(\kappa_1 R) J_m(\kappa_2 R)$$

- To the lowest order→

$$\left(k_c R\right) J_m'(k_c R) J_m(k_c R) = -\frac{\beta^2}{k_c^2} \frac{m\omega_M}{\omega} J_m^2(k_c R)$$

$$k_c^2 = k_0^2 \varepsilon \nu = k_0^2 \varepsilon - \beta^2$$

(iii) *Circular waveguide (longitudinal magnetization)*

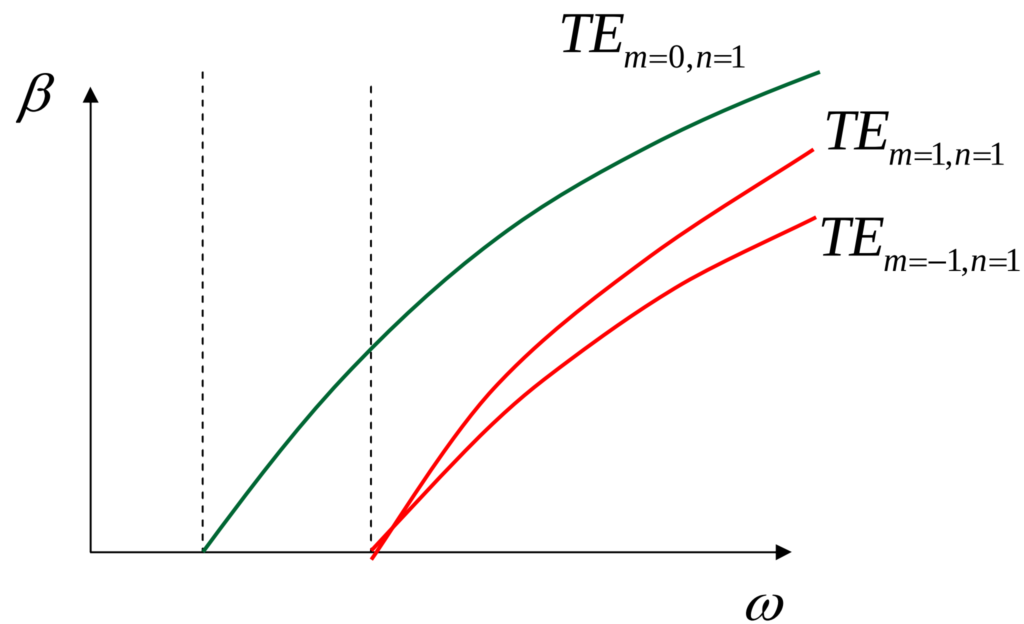
- First solution: $J_m(k_c R) = 0$
- Identical to TM modes in an ordinary waveguide → no influence of the magnetic material on the TM waves in this approximation.
- Second solution found by solving:

$$\Rightarrow \quad (k_c R)\, J'_m(k_c R) = -m \,\frac{\omega_M}{\omega} \frac{\beta^2}{k_c^2} J_m(k_c R)$$

- β depends on the sign of m.
- Remember: in both cases for each m an infinite number of solutions exist denoted by the extra mode number n

(iii) *Circular waveguide (longitudinal magnetization)*

- But, for each n, both $+m$ and $-m$ have the same cutoff since the limit β→0 yields the TE solution in an ordinary waveguide



- Be careful: this drawing is not accurate. It is only valid at very high frequencies. It may not be correct at low frequencies where our approximation is invalid

(iii) *Circular waveguide (longitudinal magnetization)*

- Dependence of the propagation constant on the sign of m is valid in the general case (not only this approximation)
- Now, in each mode any field is represented as

$$F_{m,n}(r,\varphi,z) = \exp(jm\varphi - j\beta_{m,n}z)\,f_{m,n}(r)$$

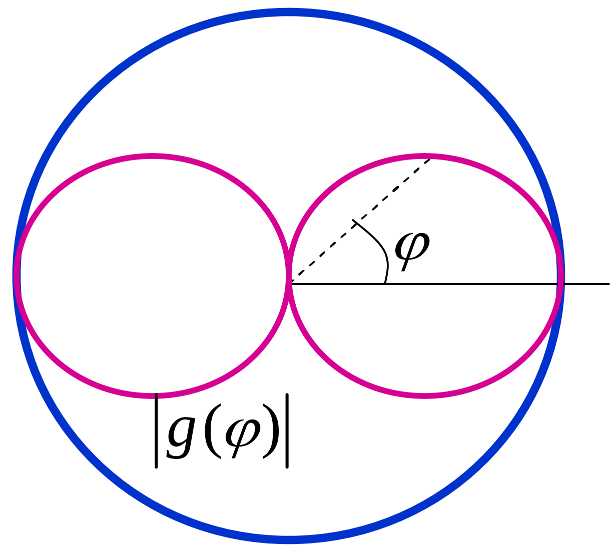
- Consider a linear combination of $+m$ and $-m$ waves traveling in the +z direction inside the waveguide

$$\begin{aligned} g(r,\varphi,z) = C_+\ F_{m,n}(r,\varphi,z) + C_-\ F_{-m,n}(r,\varphi,z) = \\ C_+ \exp(jm\varphi - j\beta_{m,n}z)\,f_{m,n}(r) + \\ C_- \exp(-jm\varphi - j\beta_{-m,n}z)\,f_{-m,n}(r) \end{aligned}$$

(iii) *Circular waveguide (longitudinal magnetization)*

- Consider this solution at z=0. Then

$$g(r_0,\varphi,0) = C_+ \exp(jm\varphi) f_{m,n}(r) + C_- \exp(-jm\varphi) f_{-m,n}(r)$$



- After the waves travel a distance *l*:

$$g(r,\varphi,l) = C_+ \exp(jm\varphi - j\beta_{m,n} l) f_{m,n}(r) + C_- \exp(-jm\varphi - j\beta_{-m,n} l) f_{-m,n}(r)$$

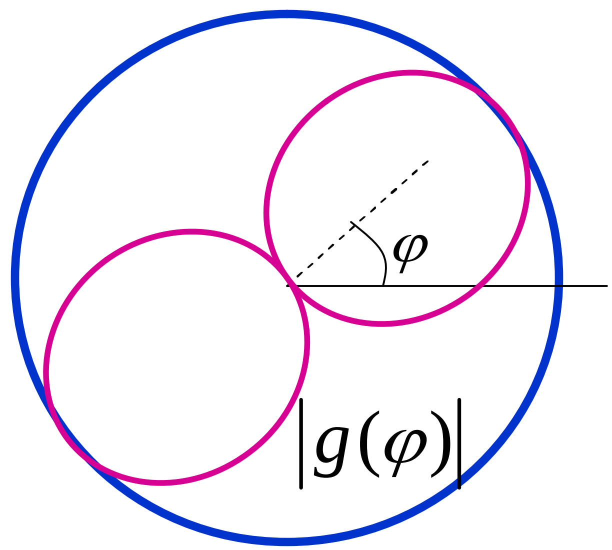
(iii) *Circular waveguide (longitudinal magnetization)*

- After the waves travel a distance *l*:

$$g(r_0,\varphi,l)=\exp\left[-j\left(\beta_{m,n}+\beta_{-m,n}\right)l/2\right]g(r_0,\varphi-\vartheta,0)$$

$$\vartheta=\frac{1}{2|m|}\left(\beta_{m,n}-\beta_{-m,n}\right)l$$

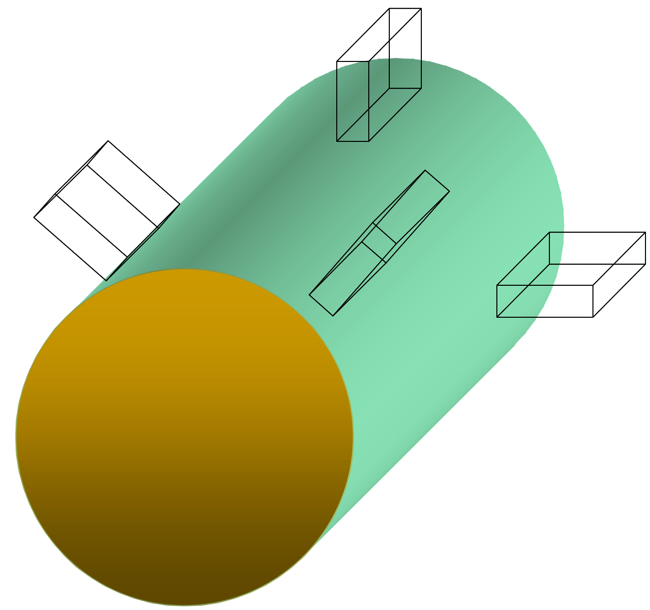
- This angular rotation of field distribution is analogous to Faraday effect. It also remains the same in both propagation directions (+z,-z).

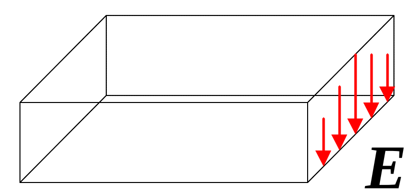


$|m|=1$

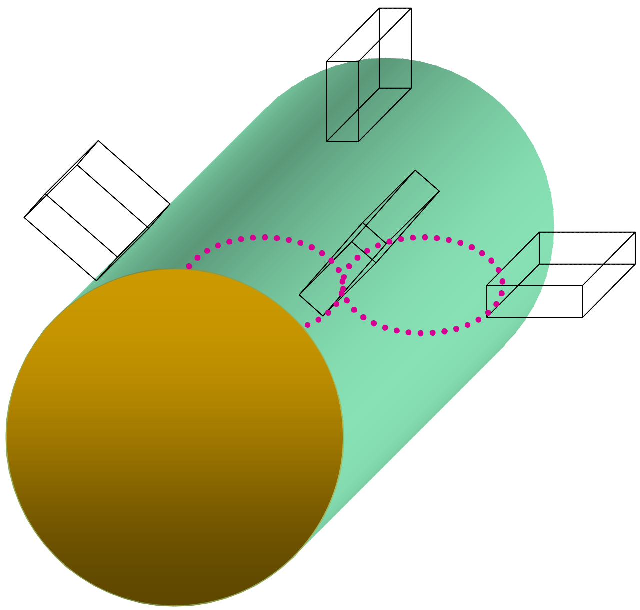
(iv) *Faraday ferrite devices*

- Based on the angular rotation of the field profile in a waveguide containing magnetic materials
- Consider a circular waveguide with a longitudinally magnetized magnetic medium and two pairs of rectangular waveguide ports
- TE_{10} mode of the waveguide ports coupled to modes of the circular guide in this configuration





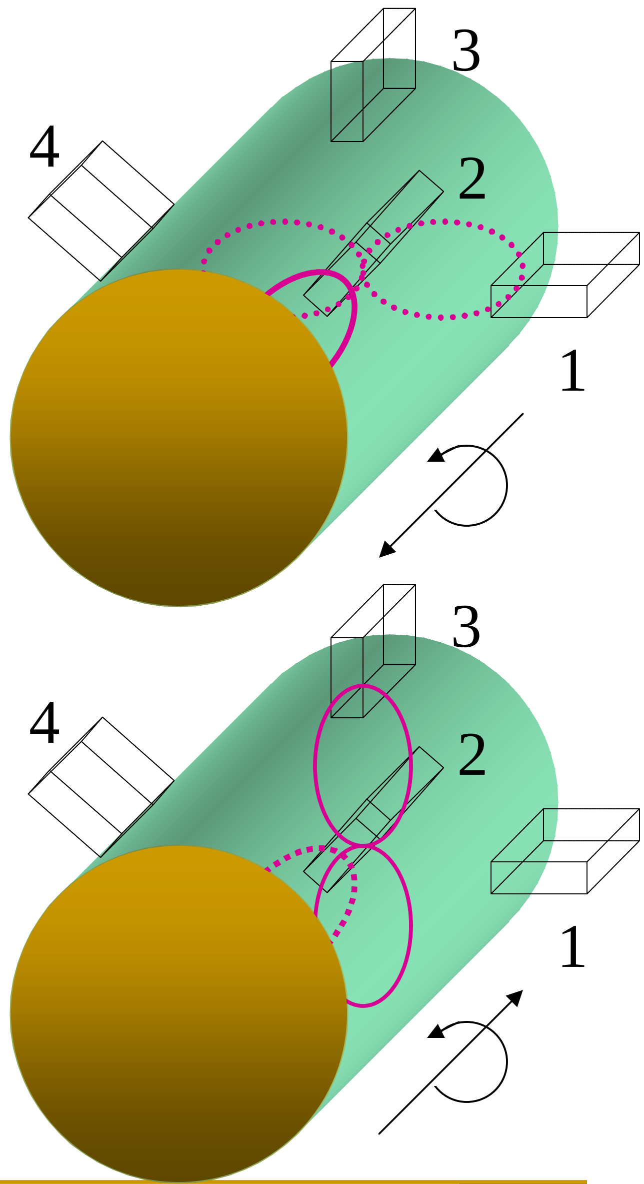
(iv) *Faraday ferrite devices*



- Consider the coupling of the waveguide port modes to the m=1,n=1 and m=-1,n=1 modes of the waveguide
- Of course coupling to other modes exist as well, but we do not consider them now. They may be below cutoff.
- TE_{10} mode of the waveguide ports excites a linear combination of m=1,n=1 and m=-1,n=1 modes. The coupling strength to the two modes is almost equal

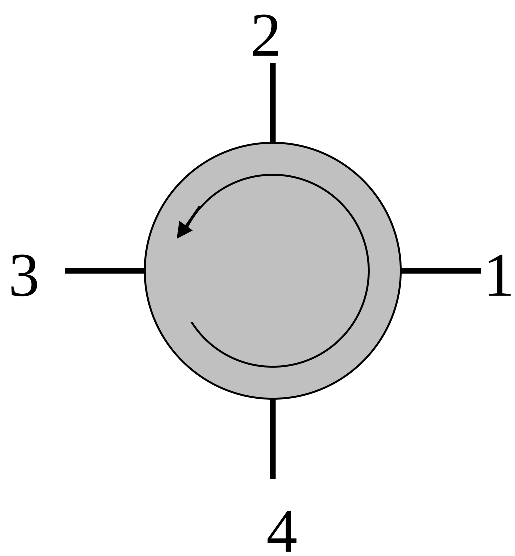
(iv) *Faraday ferrite devices*

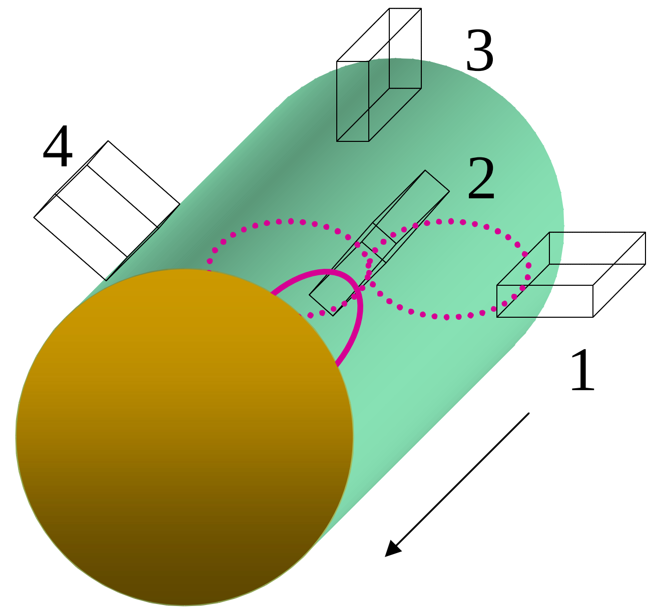
- Imagine that ports 2,4 are at such a distance from ports 1,3 that a 45 degree rotation results
- Wave excited by 1 reaches 2, but not 4. Wave excited by 3 reaches 4 but not 1.
- Waves traveling in opposite direction: sense of rotation does not change → Wave excited by 2 reaches 3, but not 1. Wave excited by 4 reaches 1 but not 3.



(iv) *Faraday ferrite devices*

- Resulting device is a circulator

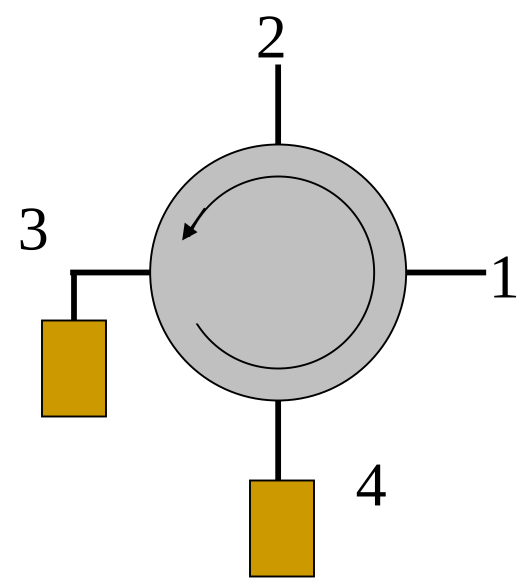




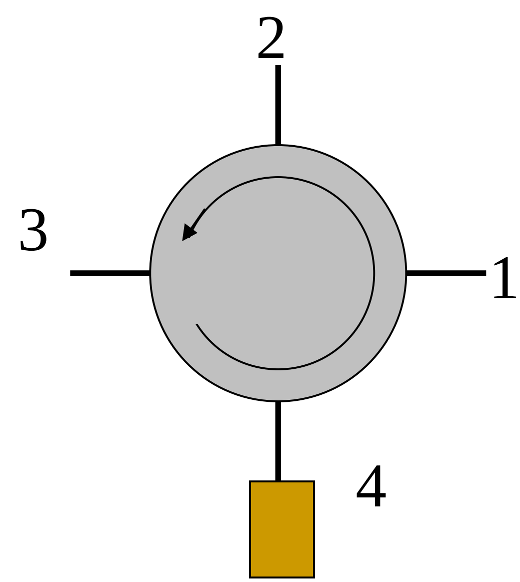
- One can build a number of devices with this component

(iv) *Faraday ferrite devices*

- 3,4 terminated by matched load → isolator: signal goes from 1 to 2 but not from 2 to 1



- 4 terminated by matched load → duplexer: signal goes from 1 to 2 , and from 2 to 3



(iv) *Faraday ferrite devices*

- Note, however, that there may be other propagating modes in the magnetic waveguide. They should be included in the device analysis, or should be avoided by taking appropriate measures (port design, etc)
- Also, in practice, a completely filled magnetic waveguide is not optimal. Losses may be high. In most designs a ferrite rod is used in an otherwise empty waveguide.
- Exact calculation again possible. Dependence of propagation constant on m remains. So does Faraday effect.

(iv) *Faraday ferrite devices*

- The calculation can also be done using perturbation theory if the cross sectional area of the rod is small compared to waveguide cross section