Microwave Magnetics

Graduate Course

Electrical Engineering (Communications)

2nd Semester, 1394-1395

Sharif University of Technology

General information

- **Contents of lecture 3:**
  - The dynamics of magnetization
    - *The Landau-Lifshitz equation*
    - *The susceptibility and permeability tensors*
    - *Inclusion of losses*
    - *Uniform oscillations of a small ellipsoid*

The dynamics of magnetization: *The Landau-Lifshitz equation*

- Now that we know the initial (equilibrium) state of $\boldsymbol{M}$; how does it change under a field?

$$\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{H}_{eff}$$

- This is the Landau-Lifshitz equation, “postulated” in 1935; but there are subtle differences with our previous equation
  - $\boldsymbol{M}$ is averaged over the random thermal motion.
  - The magnitude of $\boldsymbol{M}$ is constant (given by the saturation magnetization M_s well below the Curie temperature)

(i) *The Landau-Lifshitz equation*

- The field $\boldsymbol{H}_{eff}$ contains the total magnetic field
- But also an effective field related to exchange and often magneto-crystalline anisotropy (neglect these terms)

- Solving the exact problem is complicated, even when only magnetic fields are considered

$$\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{H} \qquad \boldsymbol{H} = \boldsymbol{H}_M + \boldsymbol{H}_{ext}$$

- This equation is non-linear because $\boldsymbol{H}_M$ (the demagnetization field) also depends on $\boldsymbol{M}$.

(i) *The Landau-Lifshitz equation*

- In fact, we have to solve the macroscopic Maxwell equations, together with the (nonlinear) LL equation which describes how magnetization and, therefore, the $\boldsymbol{B}$ field, locally depends on the $\boldsymbol{H}$ field.
- Common practice: linearized equations of motion. Justified because in most applications a large DC bias field is applied, besides the “small” microwave field
- This leads to the concept of permeability and susceptibility tensors often applied in microwave devices

(ii) *The susceptibility and permeability tensors*

- Consider a ferromagnetic material in the equilibrium dc (steady) state with the magnetization $\boldsymbol{M_0}(\boldsymbol{r})$.
- Assume that at any point the total field is given by $\boldsymbol{H_0}(\boldsymbol{r})$.
- This field is the sum of the external bias field and the field due to the magnetization itself (demagnetization field).
- From LL equation we should have at any point

$$\boldsymbol{M}_0(\boldsymbol{r}) \times \boldsymbol{H}_0(\boldsymbol{r}) = 0$$

- Therefore, in the steady-state $\boldsymbol{M_0}$ should be parallel to $\boldsymbol{H_0}(\boldsymbol{r})$ everywhere (as we concluded before)

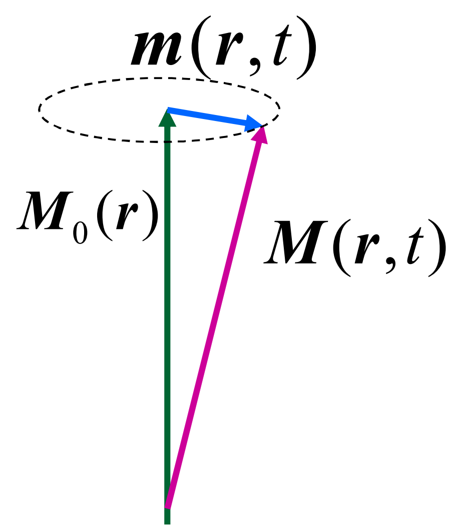
(ii) *The susceptibility and permeability tensors*

- A "small" ac (microwave) field sets the magnetization into motion. This may be the field inside a device. We then write the total field and magnetization as

$$\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{r},t) \qquad \boldsymbol{M}(\boldsymbol{r},t) = \boldsymbol{M}_0(\boldsymbol{r}) + \boldsymbol{m}(\boldsymbol{r},t)$$

- Let us assume that

$$\left|\boldsymbol{h}(\boldsymbol{r},t)\right| \ll \left|\boldsymbol{H}_0(\boldsymbol{r})\right| \qquad \left|\boldsymbol{m}(\boldsymbol{r},t)\right| \ll \left|\boldsymbol{M}_0(\boldsymbol{r})\right|$$



- For $\boldsymbol{M}$ this means that its angle with respect to $\boldsymbol{M}_0$ remains small during its motion

(ii) *The susceptibility and permeability tensors*

- Next we linearize the LL equation:

$$\frac{d\boldsymbol{m}(\boldsymbol{r},t)}{dt} = -\gamma \boldsymbol{M}_0(\boldsymbol{r}) \times \boldsymbol{h}(\boldsymbol{r},t) - \gamma \boldsymbol{m}(\boldsymbol{r},t) \times \boldsymbol{H}_0(\boldsymbol{r})$$

- Consider time-harmonic, sinusoidal signals and use the phasor representation:

$$\boldsymbol{m}(\boldsymbol{r},t) = \tilde{\boldsymbol{m}}(\boldsymbol{r}) \exp(j\omega t) \qquad \boldsymbol{h}(\boldsymbol{r},t) = \tilde{\boldsymbol{h}}(\boldsymbol{r}) \exp(j\omega t)$$

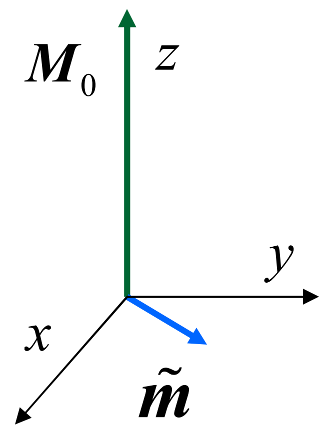
Complex vectors

- Resulting equation:

$$j\omega \tilde{\boldsymbol{m}}(\boldsymbol{r}) = -\gamma \boldsymbol{M}_0(\boldsymbol{r}) \times \tilde{\boldsymbol{h}}(\boldsymbol{r}) - \gamma \tilde{\boldsymbol{m}}(\boldsymbol{r}) \times \boldsymbol{H}_0(\boldsymbol{r})$$

(ii) *The susceptibility and permeability tensors*

- At a point inside ferromagnet, coordinate system chosen such that one axis (here z-axis) coincides with the direction of $\boldsymbol{M_0}$ and $\boldsymbol{H_0}$: $\boldsymbol{M}_0 = M_s\hat{\boldsymbol{z}} \quad \boldsymbol{H}_0 = H_0\hat{\boldsymbol{z}}$



$$
\begin{aligned}
j\omega\tilde{m}_x &= (\gamma M_s)\tilde{h}_y - (\gamma H_0)\tilde{m}_y \\
j\omega\tilde{m}_y &= -(\gamma M_s)\tilde{h}_x + (\gamma H_0)\tilde{m}_x \\
j\omega\tilde{m}_z &= 0
\end{aligned}
$$

$$
\begin{bmatrix} j\omega & \gamma H_0 & 0 \\ -\gamma H_0 & j\omega & 0 \\ 0 & 0 & j\omega \end{bmatrix}
\begin{pmatrix} \tilde{m}_x \\ \tilde{m}_y \\ \tilde{m}_z \end{pmatrix}
= \gamma M_s \begin{pmatrix} \tilde{h}_y \\ -\tilde{h}_x \\ 0 \end{pmatrix}
$$

(ii) *The susceptibility and permeability tensors*

- Inverting the matrix equation results in

$$\Rightarrow \quad \tilde{\boldsymbol{m}} = \bar{\bar{\chi}} \cdot \tilde{\boldsymbol{h}}$$

$$\bar{\bar{\chi}} = \begin{bmatrix} \chi & j\chi_a & 0 \\ -j\chi_a & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Susceptibility tensor

$$\chi = \frac{\omega_H \omega_M}{\omega_H^2 - \omega^2}, \quad \chi_a = \frac{\omega_M \omega}{\omega_H^2 - \omega^2}$$

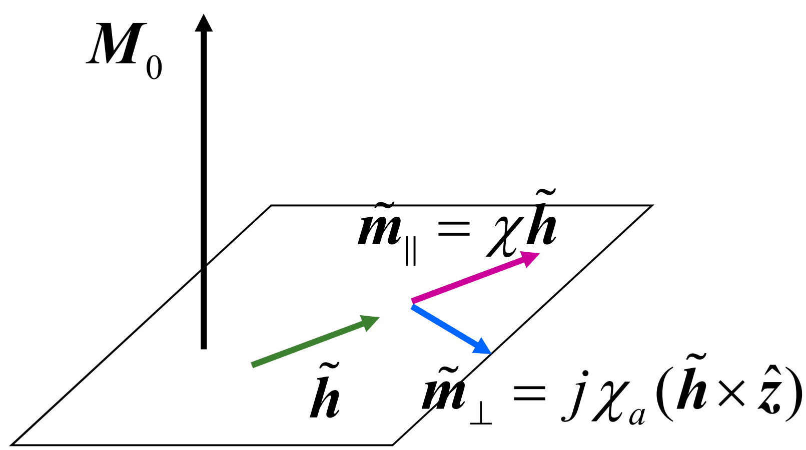
$$\omega_H = \gamma H_0, \quad \omega_M = \gamma M_s$$

- No ac magnetization along the static magnetization $\boldsymbol{M_0}$ (z-direction). Along the dc magnetization, no magnetic ac properties. (Physical reason?)

(ii) *The susceptibility and permeability tensors*

- An ac field applied perpendicular to $\boldsymbol{M_0}$ generates two components of ac magnetization: one parallel to the ac field, and one 90-degrees rotated component.

$$\tilde{\boldsymbol{h}} = \begin{bmatrix} \tilde{h}_x \\ \tilde{h}_y \\ 0 \end{bmatrix}$$

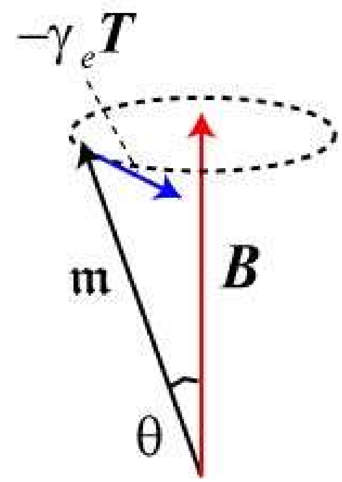


$$\Rightarrow \quad \tilde{\boldsymbol{m}} = \chi \begin{bmatrix} \tilde{h}_x \\ \tilde{h}_y \\ 0 \end{bmatrix} + j\chi_a \begin{bmatrix} \tilde{h}_y \\ -\tilde{h}_x \\ 0 \end{bmatrix}$$

$\tilde{\boldsymbol{h}} \times \hat{\boldsymbol{z}}$

(ii) *The susceptibility and permeability tensors*

- Elements of susceptibility tensor exhibit resonance behavior at ω_H : ***ferromagnetic resonance***
- Physical origin: natural (Larmor) precession of individual dipoles around the magnetic field
- When the frequency of the local field coincides with that of the natural precession of the dipoles, resonant absorption of energy occurs.

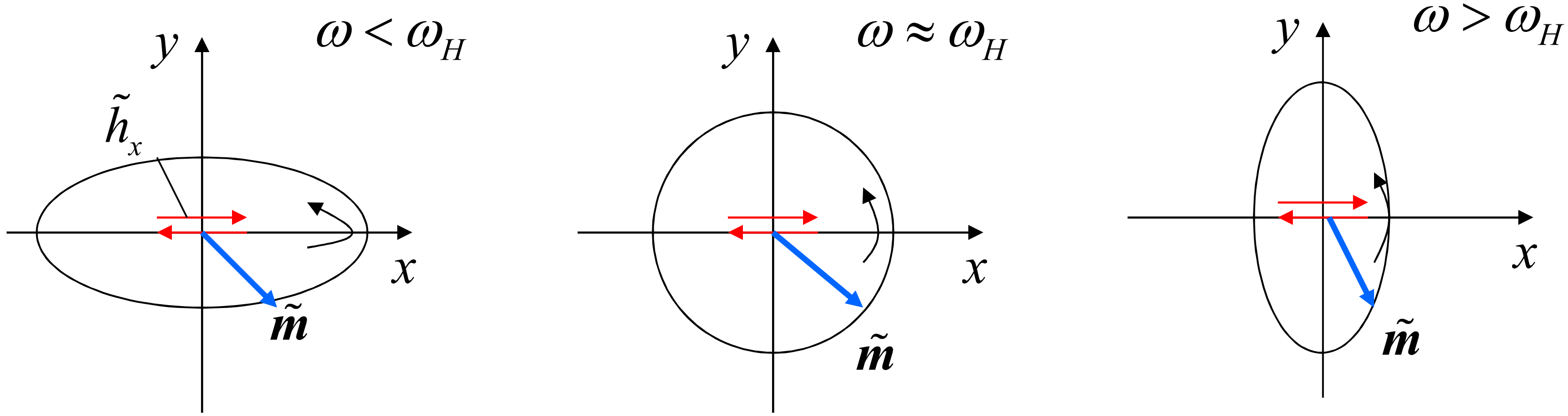


- Of course this is within the linear approximation. Also losses are not included.

(ii) *The susceptibility and permeability tensors*

- Motion of magnetization in a harmonic (sinusoidal field): assume an ac field in, for instance, x-direction

$$\tilde{\boldsymbol{h}} = \begin{bmatrix} \tilde{h}_x \\ 0 \\ 0 \end{bmatrix} \rightarrow \tilde{\boldsymbol{m}} = \begin{bmatrix} \chi \\ -j\chi_a \\ 0 \end{bmatrix} \tilde{h}_x \rightarrow \frac{\tilde{m}_y}{\tilde{m}_x} = -j\frac{\chi_a}{\chi} = -j\frac{\omega}{\omega_H}$$



(ii) *The susceptibility and permeability tensors*

- Permeability tensor:

$$\tilde{\boldsymbol{b}} = \mu_0(\tilde{\boldsymbol{h}} + \tilde{\boldsymbol{m}}) = \mu_0 \bar{\bar{\boldsymbol{\mu}}} \cdot \tilde{\boldsymbol{h}} \qquad \bar{\bar{\boldsymbol{\mu}}} = \begin{bmatrix} \mu & j\mu_a & 0 \\ -j\mu_a & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

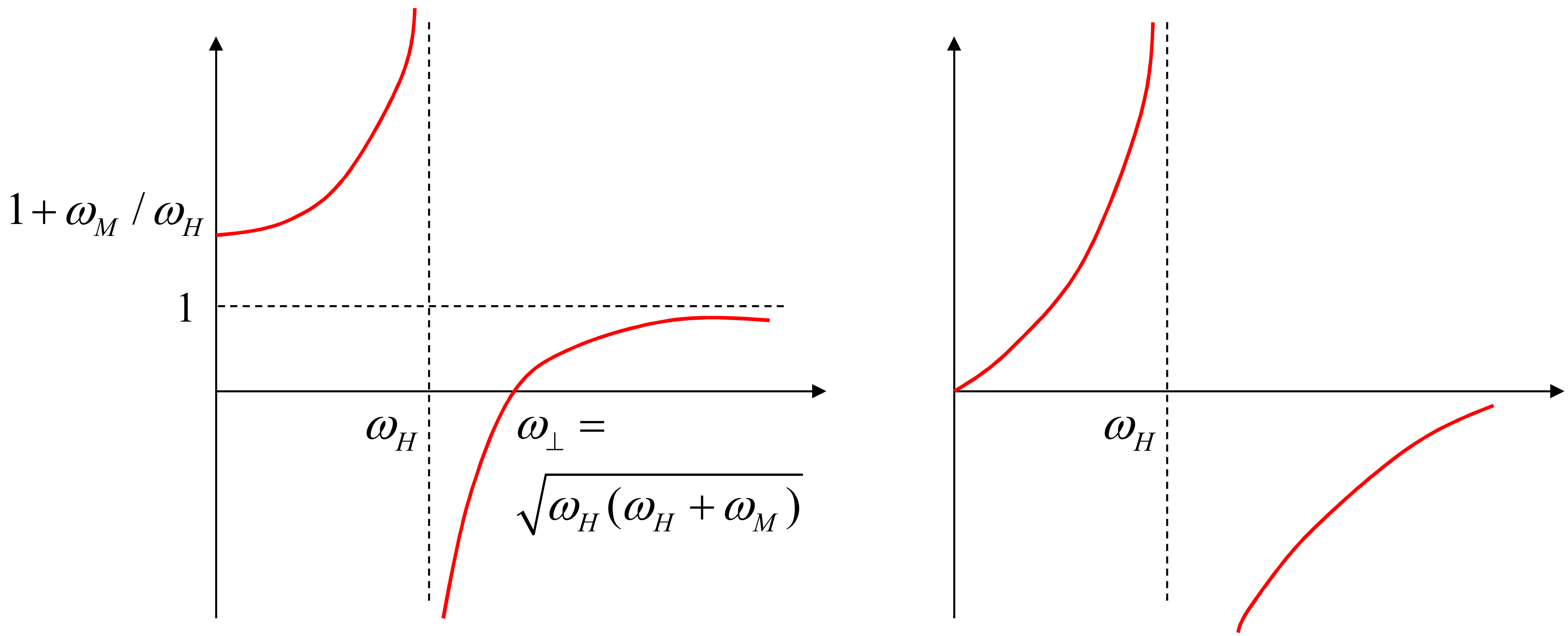
$$\mu = 1 + \chi = \frac{\omega_H(\omega_H + \omega_M) - \omega^2}{\omega_H^2 - \omega^2} \qquad \mu_a = \chi_a = \frac{\omega_M \omega}{\omega_H^2 - \omega^2}$$

- This tensor lies at the basis of the majority of microwave magnetic components
- This is a local relationship between $\boldsymbol{m}$ and $\boldsymbol{h}$. In principle, $\boldsymbol{h}$ should be “solved” using Maxwell equations

(ii) *The susceptibility and permeability tensors*

- Behavior as function of frequency:

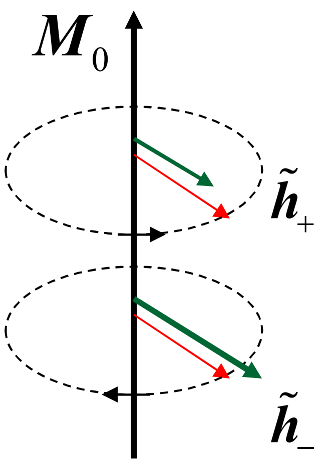
$$\mu = \frac{\omega_H(\omega_H + \omega_M) - \omega^2}{\omega_H^2 - \omega^2} \qquad \mu_a = \frac{\omega_M \omega}{\omega_H^2 - \omega^2}$$



(ii) *The susceptibility and permeability tensors*

- Special case: circularly polarized ac magnetic fields

$$\tilde{\boldsymbol{h}}_+ = \frac{1}{2}\begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \tilde{h}_+ \qquad \tilde{\boldsymbol{m}}_+ = \frac{1}{2}\begin{bmatrix} \chi & j\chi_a & 0 \\ -j\chi_a & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \tilde{h}_+$$

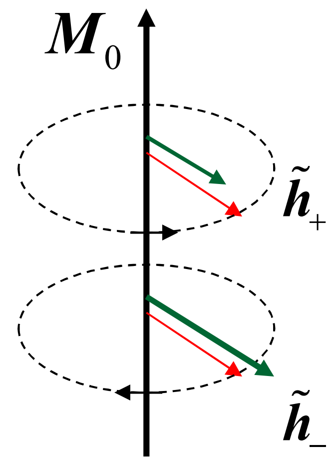


$$\Rightarrow \qquad = \frac{1}{2}\begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \tilde{m}_+ \qquad m_+ = \left(\chi + \chi_a\right) h_+$$

$$\tilde{\boldsymbol{h}}_- = \frac{1}{2}\begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix} \tilde{h}_- \quad \Rightarrow \quad \tilde{\boldsymbol{m}}_- = \frac{1}{2}\begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix} \tilde{m}_- \qquad \tilde{m}_- = \left(\chi - \chi_a\right) \tilde{h}_-$$

(ii) *The susceptibility and permeability tensors*

- Circularly polarized magnetic fields: ac field and magnetization are parallel to each other (they rotate together in time)
- Polarization modes (clockwise and counter clockwise) independent from each other.
- Each mode is characterized by its own susceptibility and permeability



$$\chi_+ = \chi + \chi_a = \frac{\omega_H \omega_M}{\omega_H^2 - \omega^2} + \frac{\omega_M \omega}{\omega_H^2 - \omega^2} = \frac{\omega_M}{\omega_H - \omega}$$

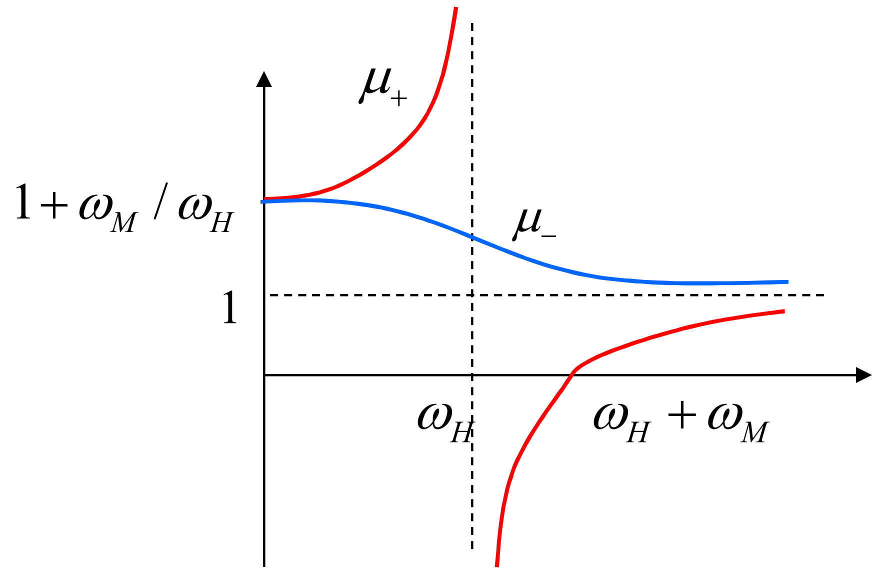
$$\chi_- = \chi - \chi_a = \frac{\omega_M}{\omega_H + \omega} - \frac{\omega_M \omega}{\omega_H^2 - \omega^2} = \frac{\omega_M}{\omega_H + \omega}$$

(ii) *The susceptibility and permeability tensors*

- Effective permeabilities

$$\mu_+ = 1 + \chi_+ = 1 + \frac{\omega_M}{\omega_H - \omega}$$

$$\mu_- = 1 + \chi_- = 1 + \frac{\omega_M}{\omega_H + \omega}$$



- \+ Mode: rotation in the direction of Larmor precession:
  - Permeability shows resonance
  - In can become negative or zero
- \- Mode: rotation opposite to Larmor precession
  - Permeability always positive, no resonance

(ii) *The susceptibility and permeability tensors*

- At each point, any field perpendicular to $\boldsymbol{M_0}$ can be written as the superposition of these modes

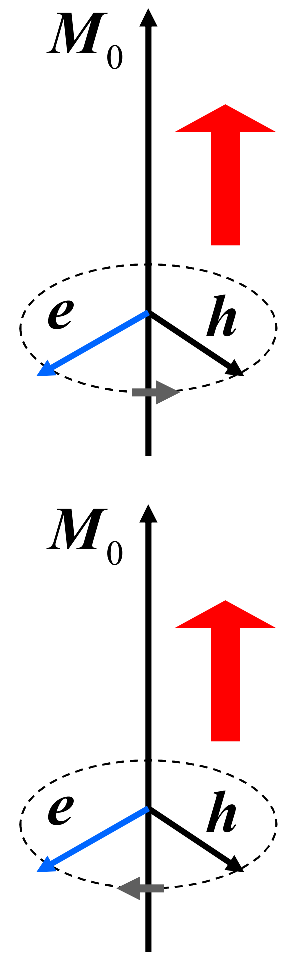
$$\tilde{\boldsymbol{h}} = \begin{bmatrix} \tilde{h}_x \\ \tilde{h}_y \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix}(\tilde{h}_x + j\tilde{h}_y) + \frac{1}{2}\begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix}(\tilde{h}_x - j\tilde{h}_y)$$

$$\tilde{\boldsymbol{m}}_+ = (\tilde{m}_x + j\tilde{m}_y) = \chi_+(\tilde{h}_x + j\tilde{h}_y) \quad \tilde{\boldsymbol{m}}_- = (\tilde{m}_x - j\tilde{m}_y) = \chi_-(\tilde{h}_x - j\tilde{h}_y)$$

- One can state that the ferromagnetic resonance is due to the circular component of the local field rotating in the direction of Larmor precession.

(ii) *The susceptibility and permeability tensors*

- Consider now a circularly-polarized plane wave moving along the equilibrium magnetization (z-direction), in a magnetic medium with a certain dielectric constant
- At the same frequency, each wave will have a different propagation constant and wavelength
- Now assume that we start from a linearly polarized wave (e.g. in the x-direction), what will happen as the wave moves along the z-direction?



(iii) *The effect of loss*

- For a single dipole: dissipation of energy modeled by introducing a phenomenological "friction". Can we extend this to macroscopic magnetization?
- The Landau-Lifshitz-Gilbert equation:

$$\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{H} + \frac{\alpha}{M_s} \boldsymbol{M} \times \frac{d\boldsymbol{M}}{dt}$$

α: dimensionless damping constant

- Original Landau-Lifshitz suggestion:

$$\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{H} - \frac{\gamma\lambda}{M_s^2} \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H})$$

λ: dissipation parameter

(iii) *The effect of loss*

- These two models become exactly equivalent by a redefinition of parameters
- We will use the Landau-Lifshitz-Gilbert (LLG) representation in this course
- For microwave applications it is more important to see the effect of loss on susceptibility/permeability tensors
- Linearization:

$$\frac{d\boldsymbol{m}}{dt} = -\gamma \boldsymbol{m} \times \boldsymbol{H}_0 - \gamma \boldsymbol{M}_0 \times \boldsymbol{h} + \frac{\alpha}{M_s} \boldsymbol{M}_0 \times \frac{d\boldsymbol{m}}{dt}$$

(iii) *The effect of loss*

- Time-harmonic signals:

$$j\omega\tilde{\boldsymbol{m}} = -\gamma\tilde{\boldsymbol{m}}\times\boldsymbol{H}_0 - \gamma\boldsymbol{M}_0\times\tilde{\boldsymbol{h}} + \frac{j\omega\alpha}{M_s}\boldsymbol{M}_0\times\tilde{\boldsymbol{m}}$$

$$j\omega\tilde{\boldsymbol{m}} + \gamma\tilde{\boldsymbol{m}}\times\boldsymbol{H}_0 + \frac{j\omega\alpha}{M_s}\tilde{\boldsymbol{m}}\times\boldsymbol{M}_0 = -\gamma\boldsymbol{M}_0\times\tilde{\boldsymbol{h}}$$

- Since equilibrium magnetization and field are parallel, and $|\boldsymbol{M_0}| = M_s$, we must have $\frac{\boldsymbol{M}_0}{M_s} = \frac{\boldsymbol{H}_0}{H_0}$

$$\Rightarrow \quad j\omega\tilde{\boldsymbol{m}} + \gamma\left(1 + \frac{j\omega\alpha}{H_0}\right)\tilde{\boldsymbol{m}}\times\boldsymbol{H}_0 = -\gamma\boldsymbol{M}_0\times\tilde{\boldsymbol{h}}$$

(iii) *The effect of loss*

- This means that the linearized equations in presence of loss are equivalent to the equations without loss if we make the replacement

$$\omega_H = \gamma H_0 \to \gamma H_0 \left(1 + \frac{j\omega\alpha}{\gamma H_0}\right) = \omega_H + j\omega\alpha$$

- The same replacement applicable to susceptibility and permeability tensors

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu & j\mu_a & 0 \\ -j\mu_a & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mu = 1 + \frac{(\omega_H + j\omega\alpha)\omega_M}{(\omega_H + j\omega\alpha)^2 - \omega^2}, \ \mu_a = \frac{\omega_M \omega}{(\omega_H + j\omega\alpha)^2 - \omega^2}$$

(iii) *The effect of loss*

- Real and imaginary parts

$$\mu = 1 + \frac{\left(\omega_H + j\omega\alpha\right)\omega_M}{\omega_H^2 - (1+\alpha^2)\omega^2 + 2j\omega_H\omega\alpha} = \mu' - j\mu''$$

$$\mu' = 1 + \frac{\omega_M\omega_H\left[\omega_H^2 - (1-\alpha^2)\omega^2\right]}{\left[\omega_H^2 - (1+\alpha^2)\omega^2\right]^2 + 4\left(\omega_H\omega\alpha\right)^2} \qquad \mu'' = \frac{\omega_M\omega\alpha\left[\omega_H^2 + (1+\alpha^2)\omega^2\right]}{\left[\omega_H^2 - (1+\alpha^2)\omega^2\right]^2 + 4\left(\omega_H\omega\alpha\right)^2}$$

$$\mu_a = \frac{\omega_M\omega}{\left(\omega_H + j\omega\alpha\right)^2 - \omega^2} = \frac{\omega_M\omega}{\omega_H^2 - (1+\alpha^2)\omega^2 + 2j\omega_H\omega\alpha} = \mu_a' - j\mu_a''$$

$$\mu_a' = \frac{\omega_M\omega\left[\omega_H^2 - (1+\alpha^2)\omega^2\right]}{\left[\omega_H^2 - (1+\alpha^2)\omega^2\right]^2 + 4\left(\omega_H\omega\alpha\right)^2} \qquad \mu_a'' = \frac{2\omega_H\omega_M\omega^2\alpha}{\left[\omega_H^2 - (1+\alpha^2)\omega^2\right]^2 + 4\left(\omega_H\omega\alpha\right)^2}$$

(iii) *The effect of loss*

- Resonance behavior near ω_H
- Resonance frequency defined at the maximum (peak) of the imaginary part of the diagonal permeability

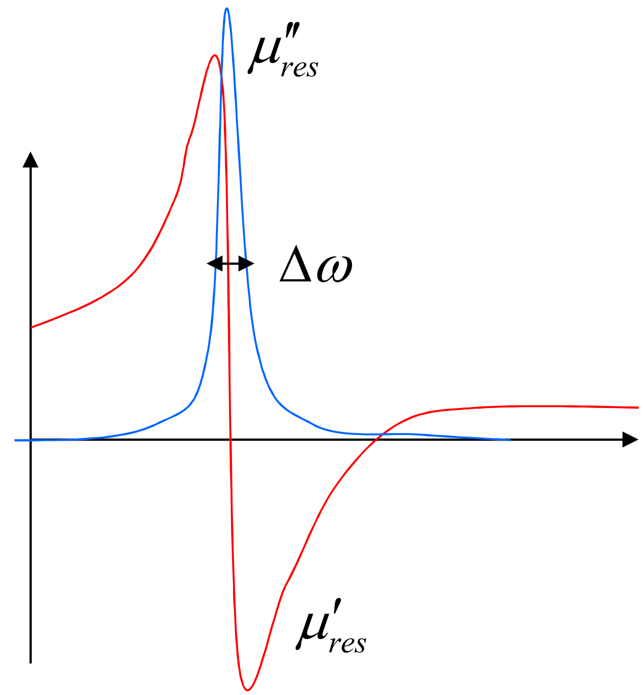
$$\left.\frac{d\mu''}{d\omega}\right|_{\omega_{res}} = 0$$

$$\omega_{res} = \frac{\omega_H}{\sqrt{1+\alpha^2}}$$

$$\mu'_{res} = 1 + \frac{\omega_M}{2\omega_H} \qquad \mu''_{res} = \frac{\omega_M}{2\alpha\omega_{res}}$$

$$\mu'_{a,res} = 0 \qquad \mu''_a = \frac{\omega_M}{2\alpha\omega_H}$$

$$\Delta\omega = 2\alpha\omega_{res}$$



(iii) *The effect of loss*

- Remarks:
  - The LL equation is not a low-frequency (magnetostatic) approximation. The resulting permeability tensor can be used in Maxwell equations to describe high-frequency behavior of the ferromagnetic material
  - For the derivation of permeability tensor we choose the z-direction along steady state magnetization. Any other axis, of course, can be used. If magnetization changes direction, this tensor should be locally adapted, resulting in a position-dependent tensor
  - The damping constant is usually empirically determined and is a function of material properties.

(iv) *Uniform oscillations in small ellipsoids*

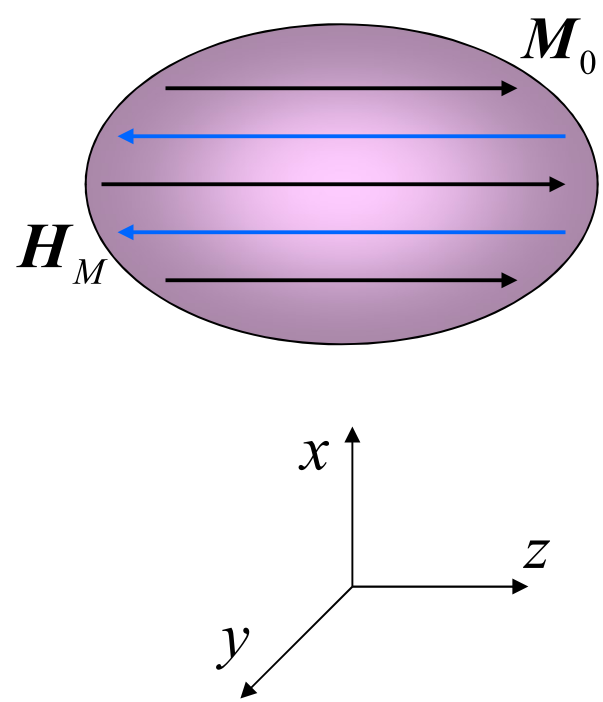
- In general, LL equation and its linear approximation describe the relation between magnetization and local field
- But, local field can only be solved using Maxwell equations and appropriate boundary conditions
- Then, the permeability matrix can be used to define the constitutive relationship between $\boldsymbol{b}$ and $\boldsymbol{h}$
- This is complicated in general. But there is one example where exact solutions can be found within certain approximations

(iv) *Uniform oscillations in small ellipsoids*

- We saw that the demagnetization field generated inside a uniformly magnetized ellipsoid is constant within the magneto-static approximation

$$\boldsymbol{H}_M = -\nabla \int_V \nabla' G_0(\boldsymbol{r}-\boldsymbol{r}') \cdot \boldsymbol{M}_0 dV' = -\bar{\bar{\mathbf{N}}} \cdot \boldsymbol{M}_0$$

$$\bar{\bar{\mathbf{N}}} = \begin{bmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & N_z \end{bmatrix}$$

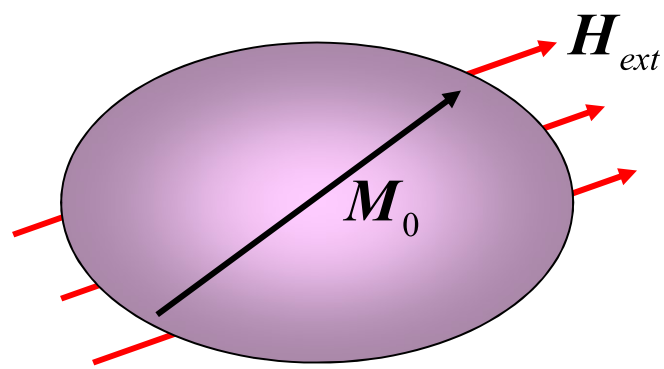


$$N_x + N_y + N_z = 1 \qquad\qquad |\boldsymbol{M}_0| = M_s$$

(iv) *Uniform oscillations in small ellipsoids*

- Assume that a uniform, dc external field is applied to bias the magnetic ellipsoid.
- First, we find the static distribution which is also uniform:

$$\boldsymbol{H}_0 = \boldsymbol{H}_{ext} + \boldsymbol{H}_M = \boldsymbol{H}_{ext} - \bar{\bar{N}} \cdot \boldsymbol{M}_0$$



$$\boldsymbol{H}_0 \times \boldsymbol{M}_0 =$$

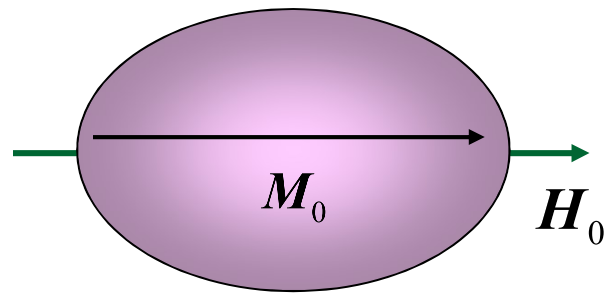
$$\left(\boldsymbol{H}_{ext} - \bar{\bar{N}} \cdot \boldsymbol{M}_0\right) \times \boldsymbol{M}_0 = 0$$

- We consider the simple case of external field applied along one of the axes of the ellipsoid (z-axis)

(iv) *Uniform oscillations in small ellipsoids*

- Now, we have the simple solution:

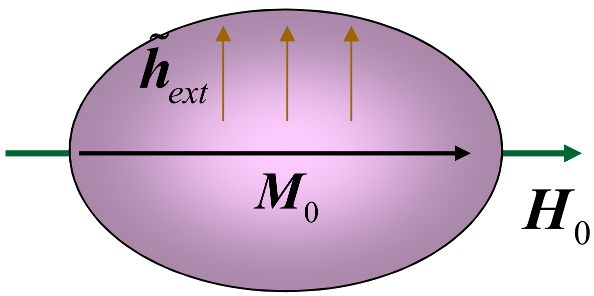
$$\boldsymbol{H}_0 = H_0 \hat{\boldsymbol{z}}$$
$$H_0 = H_{ext} - N_z M_s$$

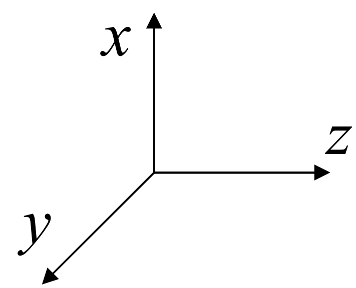


- From here, we can calculate the permeability tensor
- Then, we can solve the Maxwell equations to find the ac field in response to external ac sources
- But, we proceed differently, and use the demagnetization factors to find the response to an external ac field

(iv) *Uniform oscillations in small ellipsoids*

- The necessary conditions for this approximation:
  - The externally applied ac field should also be uniform over the volume of the ellipsoid
  - At the frequencies of interest, the dimensions of the ellipsoid should be small compared to the “electromagnetic” wavelength inside the ellipsoid. This allows us to use the magnetostatic approximation as for the demagnetization factors

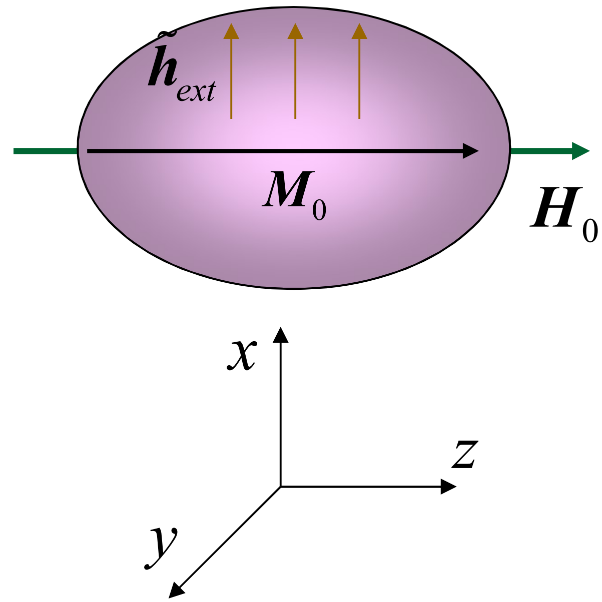




(iv) *Uniform oscillations in small ellipsoids*

- The ac magnetization: $\tilde{\boldsymbol{m}} = \overline{\overline{\chi}} \cdot \tilde{\boldsymbol{h}}$

- But $\boldsymbol{h}$-field is the sum of the external and the internal ac field generated by the ac magnetization



$$\tilde{\boldsymbol{h}} = \tilde{\boldsymbol{h}}_{ext} + \tilde{\boldsymbol{h}}_M$$

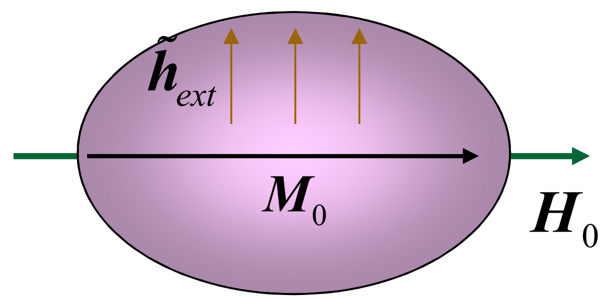
- If the previous conditions apply, we can use the magnetostatic approximation for ac field:

$$\boldsymbol{H}_M + \boldsymbol{h}_M = -\overline{\overline{N}} \cdot \left(\boldsymbol{M}_0 + \boldsymbol{m} \right) \quad \Rightarrow \quad \tilde{\boldsymbol{h}}_M = -\overline{\overline{N}} \cdot \tilde{\boldsymbol{m}}$$

(iv) *Uniform oscillations in small ellipsoids*

- Therefore:

$$\tilde{\boldsymbol{m}} = \underbrace{\left[\bar{\bar{I}} + \bar{\bar{\chi}} \cdot \bar{\bar{N}} \right]^{-1} \cdot \bar{\bar{\chi}}}_{\bar{\bar{\chi}}^{e}} \cdot \tilde{\boldsymbol{h}}_{ext}$$



- This quantity describes the response to an external ac field, not the total ac field. This is the external susceptibility tensor.
- Note: the actual susceptibility (or permeability) tensor does not know whether the fields are internal or external

- Therefore:

$$\bar{\bar{\chi}}^{e} = \left[\bar{\bar{I}} + \bar{\bar{\chi}} \cdot \bar{\bar{N}}\right]^{-1} \cdot \bar{\bar{\chi}} =$$

$$\frac{1}{\Delta}\begin{bmatrix} \chi + N_y(\chi^2 - \chi_a^2) & j\chi_a & 0 \\ -j\chi_a & \chi + N_x(\chi^2 - \chi_a^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta = 1 + (N_x + N_y)\chi + N_x N_y (\chi^2 - \chi_a^2)$$

(iv) *Uniform oscillations in small ellipsoids*

- For the moment neglect the losses, can be included later by letting $\omega_H \to \omega_H + j\omega\alpha$

$$\chi = \frac{\omega_H \omega_M}{\omega_H^2 - \omega^2}, \ \chi_a = \frac{\omega_M \omega}{\omega_H^2 - \omega^2}$$

$$\bar{\bar{\chi}}^e = \frac{1}{\Delta}\begin{bmatrix} \left(\omega_H + N_y \omega_M\right)\omega_M & j\omega_M\omega & 0 \\ -j\omega_M\omega & \left(\omega_H + N_x\omega_M\right)\omega_M & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta = \left(\omega_H + N_x\omega_M\right)\left(\omega_H + N_y\omega_M\right) - \omega^2$$

(iv) *Uniform oscillations in small ellipsoids*

- Note that $H_0 = H_{ext} - N_z M_s$

$$\omega_H = \gamma H_0 = \gamma H_{ext} - N_z \omega_M = \omega_{ext} - N_z \omega_M$$

$$\bar{\bar{\chi}}^e = \frac{1}{\Delta}\begin{bmatrix} \left[\omega_{ext} + \left(N_y - N_z\right)\omega_M\right]\omega_M & j\omega_M\omega & 0 \\ -j\omega_M\omega & \left[\omega_{ext} + \left(N_x - N_z\right)\omega_M\right]\omega_M & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta = \left[\omega_{ext} + \left(N_y - N_z\right)\omega_M\right]\left[\omega_{ext} + \left(N_x - N_z\right)\omega_M\right] - \omega^2$$

(iv) *Uniform oscillations in small ellipsoids*

- Note that the diagonal elements are not equal
- Ferromagnetic resonance now occurs at

$$\omega_0 = \sqrt{(\omega_H + N_x \omega_M)(\omega_H + N_y \omega_M)}$$
$$= \sqrt{\left[\omega_{ext} + (N_y - N_z)\omega_M\right]\left[\omega_{ext} + (N_x - N_z)\omega_M\right]}$$

- It does not occur at ω_H because of the effect of the ac demagnetization field (Physical picture?)

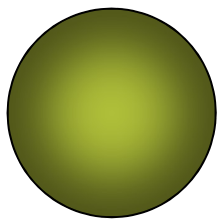
(iv) *Uniform oscillations in small ellipsoids*

- Remarks:
  - We assumed the steady-state magnetization to be along one of the axis of the ellipsoid. If this is not the case, then the expression for either the demagnetization matrix or the susceptibility tensor becomes much more complicated. This case was treated in Gurevich and Melkov
  - This derivation does not mean that the ac magnetization inside the ellipsoid is always uniform. If the external ac field in not uniform, the magnetization will also become non-uniform. Then, we cannot use the simple demagnetization factors as in the uniform case.

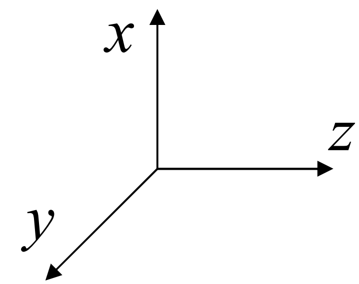
(iv) *Uniform oscillations in small ellipsoids*

- Examples: some limiting cases of ellipsoids

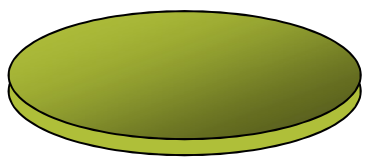
sphere



$$N_x = N_y = N_z = \frac{1}{3}$$



Infinitely thin disk



$$N_x = 1 \quad N_y = N_z = 0$$

Infinitely long rod

$$N_x = N_y = \frac{1}{2} \quad N_z = 0$$

(iv) *Uniform oscillations in small ellipsoids*

- The ferromagnetic resonance frequency of a uniformly magnetized ellipsoid depends on the magnitude direction of static magnetization
- Only for a sphere there is no such dependence:

$$\omega_0 = \omega_{ext} = \gamma H_{ext}$$