Microwave Magnetics

Graduate Course

Electrical Engineering (Communications)

2^{nd} Semester, 1394-1395

Sharif University of Technology

General information

- **Contents of lecture 3:**
  - Electrodynamics of magnetic media
    - *Maxwell equations*
    - *Constitutive relations in ferromagnetic media*
    - *General equations for longitudinal field components*
    - *Uniform plane waves*
      - *Longitudinal waves*
      - *Dispersion relation*
      - *Faraday effect*
      - *Transversal waves*
      - *General propagation direction*

General information

- *Reciprocity*
- *Energy relations*

General equations and boundary conditions

- So far we discussed the dynamics of magnetization in ferromagnetic (and ferrimagnetic) materials, based on the Landau-Lifshitz-Gilbert (LLG) equation

$$\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{H} + \frac{\alpha}{M_s} \boldsymbol{M} \times \frac{d\boldsymbol{M}}{dt}$$

- We linearized this equation by separating the fields into a dc (static) and an ac component, and derived the permeability tensor of the material
- To solve the full problem of devices containing these materials we have to resort to Maxwell equations

(i) *Maxwell equations*

- Maxwell equations (time domain, SI units):

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \cdot \boldsymbol{D} = \varrho \qquad \boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M} \right)$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \qquad \nabla \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

- Continuity equation (charge conservation):

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \varrho}{\partial t} = 0$$

- Note: sources correspond to “free” currents and charges. “Bounded” dipole currents already included in polarization and magnetization

(i) *Maxwell equations*

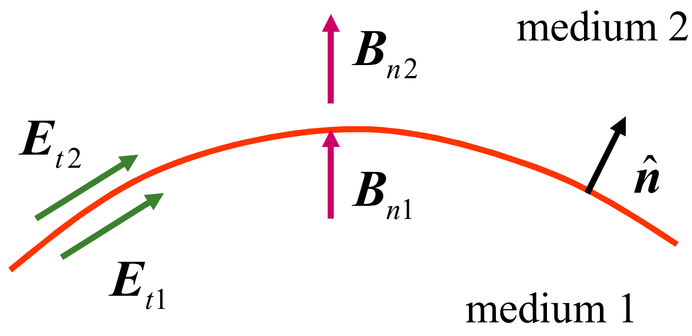
- Boundary conditions at material interfaces:

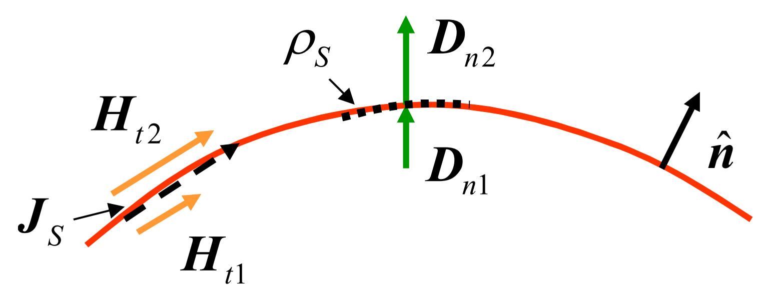
$$\hat{\boldsymbol{n}} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \qquad \text{or} \quad \boldsymbol{E}_{t2} = \boldsymbol{E}_{t1}$$

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \qquad \text{or} \quad B_{n2} = B_{n1}$$

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \rho_s \qquad \text{or} \quad D_{n2} - D_{n1} = \rho_s$$

$$\hat{\boldsymbol{n}} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s \quad \text{or} \quad \boldsymbol{H}_{t2} - \boldsymbol{H}_{t1} = \boldsymbol{J}_s \times \hat{\boldsymbol{n}}$$





(i) *Maxwell equations*

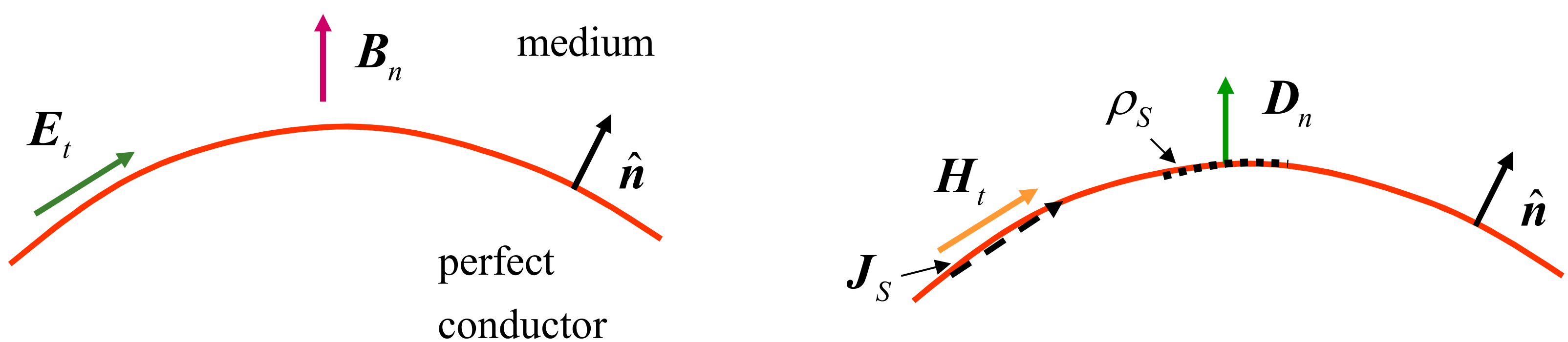
- Conditions on a perfectly conducting surface:

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0 \qquad \text{or} \quad \boldsymbol{E}_t = 0$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0 \qquad \text{or} \quad B_n = 0$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{D} = \rho_s \qquad \text{or} \quad D_n = \rho_s$$

$$\hat{\boldsymbol{n}} \times \boldsymbol{H} = \boldsymbol{J}_s \qquad \text{or} \quad \boldsymbol{H}_t = \boldsymbol{J}_s \times \hat{\boldsymbol{n}}$$



(i) *Maxwell equations*

- When magnetic materials are involved, we divide the problem into a dc and an ac part
- The dc limit of Maxwell equations decouples into separate magnetostatic and electrostatic equations.
- Magnetostatic equations relevant for our problem:

$$\nabla \times \boldsymbol{H}_0 = \boldsymbol{J}_0 \qquad \nabla \cdot \boldsymbol{B}_0 = 0 \qquad \boldsymbol{B}_0 = \mu_0 \left(\boldsymbol{H}_0 + \boldsymbol{M}_0 \right)$$

- Have to be solved together with LLG in static limit:

$$\boldsymbol{M}_0 \times \boldsymbol{H}_0 = 0 \qquad \left| \boldsymbol{M}_0 \right| = M_s$$

(i) *Maxwell equations*

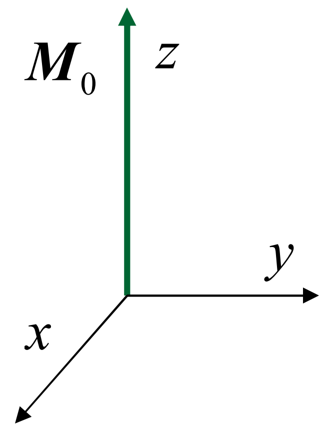
- Once these equations are solved, we can extract the ac permeability tensor (for time-harmonic signals in frequency domain)
- The fields obtained by solving the steady state (dc) problem are not necessarily uniform.
- Direction of $\boldsymbol{M_0}$ and value of $\boldsymbol{H_0}$ may change inside the material → position-dependent permeability tensor
- In practical situations, a strong uniform external magnetic field is applied to wash out these non-uniformities (saturation) → constant permeability tensor

(ii) *Maxwell equations in magnetic media*

- Now consider the frequency domain Maxwell equations for the (small) ac fields in a saturated magnetic medium

$$\nabla \times \mathbf{e} = -j\omega \mathbf{b} \qquad \nabla \cdot \mathbf{d} = \rho$$

$$\nabla \times \mathbf{h} = j\omega \mathbf{d} + \mathbf{j} \qquad \nabla \cdot \mathbf{b} = 0$$



$$\mathbf{b} = \mu_0 \bar{\bar{\boldsymbol{\mu}}} \cdot \mathbf{h} \qquad \bar{\bar{\boldsymbol{\mu}}} = \begin{bmatrix} \mu & j\mu_a & 0 \\ -j\mu_a & \mu & 0 \\ 0 & 0 & \mu_{||} \end{bmatrix}$$

$$\mu = 1 + \frac{(\omega_H + j\omega\alpha)\omega_M}{(\omega_H + j\omega\alpha)^2 - \omega^2}, \quad \mu_a = \frac{\omega_M \omega}{(\omega_H + j\omega\alpha)^2 - \omega^2}, \quad \mu_{||} = 1$$

(ii) *Maxwell equations in magnetic media*

- Relationship between **d** and **e**:
  - For poly-crystalline ferrimagnetic materials, and mono-crystalline materials at microwave frequencies, it is just given by a scalar dielectric constant

$$\mathbf{d} = \varepsilon_0 \varepsilon_d \mathbf{e}$$

  - For mono-crystalline ferrites at infrared, the dielectric constant becomes a tensor similar to permeability

$$\mathbf{d} = \varepsilon_0 \overline{\overline{\varepsilon}} \cdot \mathbf{e} \qquad \overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon & j\varepsilon_a & 0 \\ -j\varepsilon_a & \varepsilon & 0 \\ 0 & 0 & \varepsilon_{\|} \end{bmatrix}$$

(ii) *Maxwell equations in magnetic media*

- Gurevich & Melkov consider the full problem with tensor dielectric constant. But this is not relevant for out case: we just use the scalar dielectric constant
- Also, if the material is conductive, and follows the Ohm law then we write the total current as an Ohmic current and an external, forced current: $\mathbf{j} = \sigma \mathbf{e} + \mathbf{j}_{\mathbf{ext}}$

$$\nabla \times \mathbf{h} = j\omega \mathbf{d} + \mathbf{j} = j\omega \varepsilon_0 \varepsilon_d \mathbf{e} + \sigma \mathbf{e} + \mathbf{j}_{\mathbf{ext}} = j\omega \varepsilon_0 \varepsilon \mathbf{e} + \mathbf{j}_{\mathbf{ext}}$$

$$\varepsilon = \varepsilon_d - j \frac{\sigma}{\omega \varepsilon_0}$$

Complex relative permittivity

(ii) *Maxwell equations in magnetic media*

- Also, we can write

$$\nabla \cdot \mathbf{e} = \frac{\rho_{\mathbf{ext}}}{\varepsilon_0 \varepsilon}$$

$$\nabla \cdot \mathbf{j}_{\mathbf{ext}} + j\omega\rho_{\mathbf{ext}} = 0$$

- Collecting the results:

$$\nabla \times \mathbf{e} = -j\omega\mu_0 \bar{\bar{\mu}} \cdot \mathbf{h} \qquad \nabla \cdot \mathbf{e} = \frac{\rho_{\mathbf{ext}}}{\varepsilon_0 \varepsilon}$$

$$\nabla \times \mathbf{h} = j\omega\varepsilon_0\varepsilon\mathbf{e} + \mathbf{j}_{\mathbf{ext}} \qquad \nabla \cdot \left(\bar{\bar{\mu}} \cdot \mathbf{h}\right) = 0$$

(iii) *Equations for longitudinal field components*

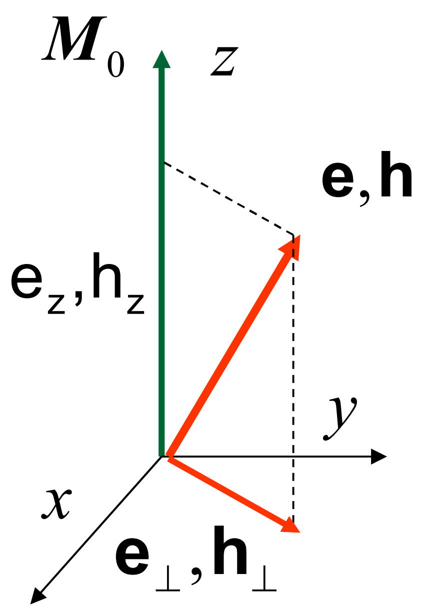
- To find compact equations, we decompose each field in a longitudinal (along $\boldsymbol{M_0}$, in z-direction), and a transversal (perpendicular to $\boldsymbol{M_0}$, parallel to x-y plain) components

$$\mathbf{e} = \mathrm{e}_z \hat{\boldsymbol{z}} + \mathbf{e}_\perp \qquad \mathbf{h} = \mathrm{h}_z \hat{\boldsymbol{z}} + \mathbf{h}_\perp$$

$$\mathbf{j}_{\mathbf{ext}} = \mathrm{j}_{\mathrm{ext},z} \hat{\boldsymbol{z}} + \mathbf{j}_{\mathbf{ext},\perp}$$

$$\mathbf{e}_\perp = \mathrm{e}_x \hat{\boldsymbol{x}} + \mathrm{e}_y \hat{\boldsymbol{y}}$$

$$\mathbf{h}_\perp = \mathrm{h}_x \hat{\boldsymbol{x}} + \mathrm{h}_y \hat{\boldsymbol{y}}$$



(iii) *Equations for longitudinal field components*

- Derivation steps:

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$$\begin{cases} \nabla_\perp \mathrm{e}_z - \dfrac{\partial}{\partial z}\mathbf{e}_\perp = -j\omega\mu_0\mu\, \hat{\boldsymbol{z}} \times \mathbf{h}_\perp + \omega\mu_0\mu_a \mathbf{h}_\perp \\ \left(\nabla_\perp \times \mathbf{e}_\perp\right)_z = -j\omega\mu_0\mu_{\|}\, \mathrm{h}_z \end{cases}$$

$$\nabla_\perp \mathrm{e}_z \equiv \left(\frac{\partial \mathrm{e}_z}{\partial x}, \frac{\partial \mathrm{e}_z}{\partial y}\right) = \frac{\partial \mathrm{e}_z}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial \mathrm{e}_z}{\partial y}\hat{\boldsymbol{y}}$$

$$\left(\nabla_\perp \times \mathbf{e}_\perp\right)_z \equiv \frac{\partial \mathrm{e}_y}{\partial x} - \frac{\partial \mathrm{e}_x}{\partial y} \qquad \hat{\boldsymbol{z}} \times \mathbf{h}_\perp = -\mathrm{h}_y \hat{\boldsymbol{x}} + \mathrm{h}_x \hat{\boldsymbol{y}}$$

(iii) *Equations for longitudinal field components*

2

$$\begin{cases} \nabla_\perp \mathrm{h_z} - \dfrac{\partial}{\partial z}\mathbf{h}_\perp = j\omega\varepsilon_0\varepsilon\ \hat{\boldsymbol{z}} \times \mathbf{e}_\perp + \hat{\boldsymbol{z}} \times \mathbf{j}_{\mathbf{ext},\perp} \\ \left(\nabla_\perp \times \mathbf{h}_\perp\right)_z = j\omega\varepsilon_0\varepsilon\ \mathrm{e}_z + \mathrm{j_{ext,z}} \end{cases}$$

$$\nabla_\perp \mathrm{h_z} \equiv \left(\frac{\partial \mathrm{h_z}}{\partial x}, \frac{\partial \mathrm{h_z}}{\partial y}\right) = \frac{\partial \mathrm{h_z}}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial \mathrm{h_z}}{\partial y}\hat{\boldsymbol{y}}$$

$$\left(\nabla_\perp \times \mathbf{h}_\perp\right)_z \equiv \frac{\partial \mathrm{h_y}}{\partial x} - \frac{\partial \mathrm{h_x}}{\partial y}$$

(iii) *Equations for longitudinal field components*

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$$\nabla_\perp \cdot \mathbf{e}_\perp + \frac{\partial \mathsf{e}_z}{\partial z} = \frac{\rho_{\mathbf{ext}}}{\varepsilon_0 \varepsilon}$$

$$\nabla_\perp \cdot \mathbf{e}_\perp \equiv \frac{\partial \mathsf{e}_x}{\partial x} + \frac{\partial \mathsf{e}_y}{\partial y}$$

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$$\mu \, \nabla_\perp \cdot \mathbf{h}_\perp + j \mu_a \left(\nabla_\perp \times \mathbf{h}_\perp \right)_z + \mu_{\|} \frac{\partial \mathsf{h}_z}{\partial z} = 0$$

$$\nabla_\perp \cdot \mathbf{h}_\perp \equiv \frac{\partial \mathsf{h}_x}{\partial x} + \frac{\partial \mathsf{h}_y}{\partial y}$$

(iii) *Equations for longitudinal field components*

- As a result:

$$\nabla_\perp \cdot \nabla_\perp \mathbf{e}_z - \frac{\partial}{\partial z}\nabla_\perp \cdot \mathbf{e}_\perp = -j\omega\mu_0\mu\nabla_\perp \cdot \hat{z}\times\mathbf{h}_\perp + \omega\mu_0\mu_a\nabla_\perp \cdot \mathbf{h}_\perp$$

$$\rightarrow \left(\nabla_\perp \cdot \nabla_\perp + \frac{\partial^2}{\partial z^2}\right)\mathbf{e}_z = j\omega\mu_0\mu\left(\nabla_\perp \times \mathbf{h}_\perp\right)_z + \omega\mu_0\mu_a\nabla_\perp \cdot \mathbf{h}_\perp$$

$$\rightarrow \left(\nabla_\perp \cdot \nabla_\perp + \frac{\partial^2}{\partial z^2}\right)\mathbf{e}_z = j\omega\mu_0\mu\, j\omega\varepsilon_0\varepsilon\ \mathbf{e}_z + \omega\mu_0\mu_a\nabla_\perp \cdot \mathbf{h}_\perp$$

(iii) *Equations for longitudinal field components*

- Next use

$$\nabla\cdot\mathbf{b}=0\rightarrow\nabla_\perp\cdot\begin{pmatrix}\mu & j\mu_a\\ -j\mu_a & \mu\end{pmatrix}\cdot\mathbf{h}_\perp+\frac{\partial}{\partial z}\mathrm{h}_z=0\rightarrow$$

$$\mu\nabla_\perp\cdot\mathbf{h}_\perp+j\mu_a\left(\nabla_\perp\times\mathbf{h}_\perp\right)_z+\frac{\partial}{\partial z}\mathrm{h}_z=0\rightarrow$$

$$\mu\nabla_\perp\cdot\mathbf{h}_\perp+j\mu_a j\omega\varepsilon_0\varepsilon\;\mathbf{e}_z+\frac{\partial}{\partial z}\mathrm{h}_z=0\rightarrow$$

$$\nabla_\perp\cdot\mathbf{h}_\perp=\frac{1}{\mu}\left(\mu_a\omega\varepsilon_0\varepsilon\;\mathbf{e}_z-\frac{\partial}{\partial z}\mathrm{h}_z\right)$$

(iii) *Equations for longitudinal field components*

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$$\nabla_\perp^2 \mathrm{e_z} + \frac{\partial^2 \mathrm{e_z}}{\partial z^2} + \left(\omega^2 \varepsilon_0 \mu_0 \varepsilon \mu_\perp\right)\mathrm{e_z} + \left(\omega\mu_0 \frac{\mu_a \mu_{\|}}{\mu}\right)\frac{\partial \mathrm{h_z}}{\partial z} = \frac{1}{\varepsilon_0 \varepsilon}\frac{\partial \rho_{\mathbf{ext}}}{\partial z} + \left(j\omega\mu_0\mu_\perp\right)\mathrm{j_{ext,z}}$$

$$\nabla_\perp^2 \mathrm{e_z} \equiv \frac{\partial^2 \mathrm{e_z}}{\partial x^2} + \frac{\partial^2 \mathrm{e_z}}{\partial y^2}$$

$$\mu_\perp = \mu - \frac{\mu_a^2}{\mu}$$

(iii) *Equations for longitudinal field components*

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$$\nabla_\perp^2 \mathrm{h_z} + \frac{\mu_{||}}{\mu}\frac{\partial^2 \mathrm{h_z}}{\partial z^2} + \left(\omega^2 \varepsilon_0 \mu_0 \varepsilon \mu_{||}\right)\mathrm{h_z} - \left(\omega \varepsilon_0 \varepsilon \frac{\mu_a}{\mu}\right)\frac{\partial \mathrm{e_z}}{\partial z}$$
$$= -\frac{j\mu_a}{\mu}\mathrm{j_{ext,z}} - \left(\nabla_\perp \times \mathbf{j}_{\mathbf{ext},\perp}\right)_z$$

$$\nabla_\perp^2 \mathrm{h_z} \equiv \frac{\partial^2 \mathrm{h_z}}{\partial x^2} + \frac{\partial^2 \mathrm{h_z}}{\partial y^2}$$

- These equations involve the longitudinal components (z-components) of the fields only

(iii) *Equations for longitudinal field components*

- Assuming no external sources and using $k_0^2 = \omega^2 \varepsilon_0 \mu_0$

$$\nabla_\perp^2 \mathrm{e_z} + \frac{\partial^2 \mathrm{e_z}}{\partial z^2} + \left(k_0^2 \varepsilon \mu_\perp\right)\mathrm{e_z} + \left(\omega \mu_0 \frac{\mu_a \mu_\parallel}{\mu}\right)\frac{\partial \mathrm{h_z}}{\partial z} = 0$$

$$\nabla_\perp^2 \mathrm{h_z} + \frac{\mu_\parallel}{\mu}\frac{\partial^2 \mathrm{h_z}}{\partial z^2} + \left(k_0^2 \varepsilon \mu_\parallel\right)\mathrm{h_z} - \left(\omega \varepsilon_0 \varepsilon \frac{\mu_a}{\mu}\right)\frac{\partial \mathrm{e_z}}{\partial z} = 0$$

- The longitudinal field components are coupled. Separate TE (e_z=0) or TM (h_z=0) modes do not exist (unless no variation in z-direction)

(iv) *Uniform plane waves in unbounded magnetic media*

- Uniform plane waves: fields do not change in the direction perpendicular to propagation direction
- General plane wave solution:

$$\mathbf{e}(\boldsymbol{r}) = \overline{\mathbf{e}}\ \exp(-j\boldsymbol{k}\cdot\boldsymbol{r}) \qquad \mathbf{h}(\boldsymbol{r}) = \overline{\mathbf{h}}\ \exp(-j\boldsymbol{k}\cdot\boldsymbol{r})$$

Constant vectors

- Assume that there are no external (forced) sources in the unbounded medium

(iv) *Uniform plane waves in unbounded magnetic media*

- Resulting equations for the constant vectors

$$\boldsymbol{k}_\perp \bar{\mathsf{e}}_z - k_z \bar{\mathbf{e}}_\perp = \omega\mu_0 \left(\mu\, \hat{\boldsymbol{z}} \times \bar{\mathbf{h}}_\perp + j\mu_a \bar{\mathbf{h}}_\perp \right)$$

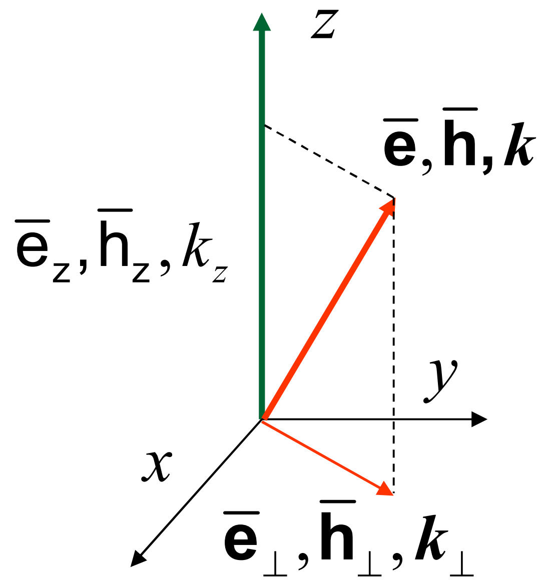
$$\boldsymbol{k}_\perp \bar{\mathsf{h}}_z - k_z \bar{\mathbf{h}}_\perp = -\omega\varepsilon_0\varepsilon\, \hat{\boldsymbol{z}} \times \bar{\mathbf{e}}_\perp$$

$$\left(\boldsymbol{k}_\perp \times \bar{\mathbf{e}}_\perp \right)_z = \omega\mu_0\mu_{\|}\, \bar{\mathsf{h}}_z$$

$$\left(\boldsymbol{k}_\perp \times \bar{\mathbf{h}}_\perp \right)_z = -\omega\varepsilon_0\varepsilon\, \bar{\mathsf{e}}_z$$

$$\boldsymbol{k}_\perp \cdot \bar{\mathbf{e}}_\perp + k_z \bar{\mathsf{e}}_z = 0$$

$$\mu\, \boldsymbol{k}_\perp \cdot \bar{\mathbf{h}}_\perp + j\mu_a \left(\boldsymbol{k}_\perp \times \bar{\mathbf{h}}_\perp \right)_z + \mu_{\|} k_z \bar{\mathsf{h}}_z = 0$$

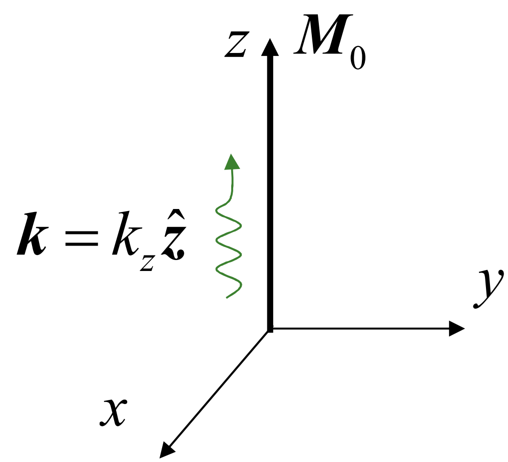


$$\boldsymbol{k}_\perp = (k_x, k_y) = k_x \hat{\boldsymbol{x}} + k_y \hat{\boldsymbol{y}}$$

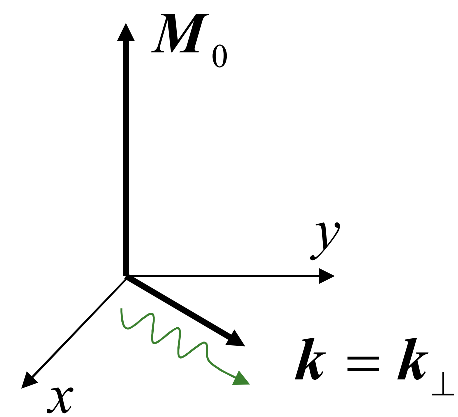
(iv) *Uniform plane waves in unbounded magnetic media*

- Instead of presenting a general solution, we consider first two special cases

- Waves propagating along static magnetization (longitudinal magnetization)



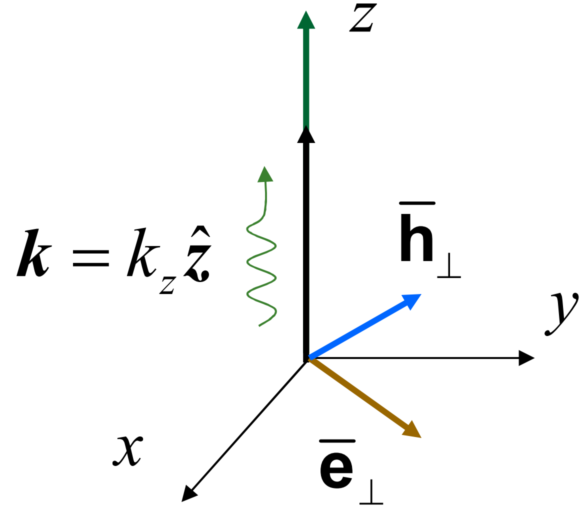
- Waves propagating perpendicular to the static magnetization (transverse magnetization)



(iv) *Uniform plane waves: Longitudinal magnetization*

- Waves propagating along static magnetization $\boldsymbol{k}_{\perp} = 0$

$$\begin{cases} \bar{\mathbf{e}}_z = 0 \qquad \bar{\mathbf{h}}_z = 0 \\ \bar{\mathbf{h}}_{\perp} = \dfrac{\omega \varepsilon_0 \varepsilon}{k_z}\, \hat{\boldsymbol{z}} \times \bar{\mathbf{e}}_{\perp} \\ \bar{\mathbf{e}}_{\perp} = -\dfrac{\omega \mu_0}{k_z}\left(\mu\, \hat{\boldsymbol{z}} \times \bar{\mathbf{h}}_{\perp} + j\mu_a \bar{\mathbf{h}}_{\perp}\right) \end{cases}$$



- These are TEM waves! Electric field perpendicular to magnetic field; both fields perpendicular to direction of propagation.

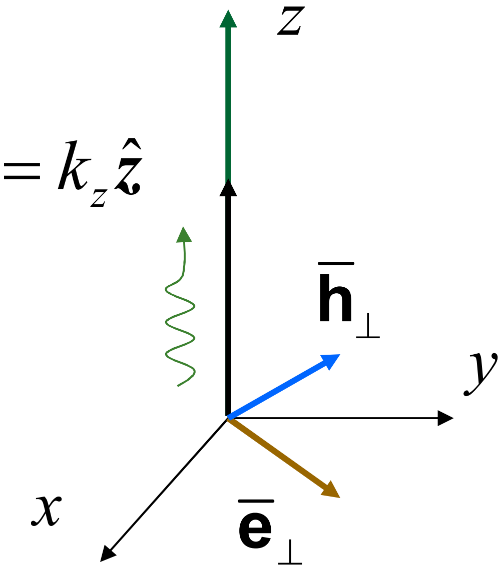
(iv) *Uniform plane waves: Longitudinal magnetization*

- Resulting equation:

$$\overline{\mathbf{h}}_\perp = \frac{k_0^2 \varepsilon}{k_z^2} \left(\mu \, \overline{\mathbf{h}}_\perp - j\mu_a \hat{\boldsymbol{z}} \times \overline{\mathbf{h}}_\perp \right)$$

$$k_z^2 \begin{pmatrix} \overline{\mathrm{h}}_\mathrm{x} \\ \overline{\mathrm{h}}_\mathrm{y} \end{pmatrix} = k_0^2 \varepsilon \begin{bmatrix} \mu & j\mu_a \\ -j\mu_a & \mu \end{bmatrix} \begin{pmatrix} \overline{\mathrm{h}}_\mathrm{x} \\ \overline{\mathrm{h}}_\mathrm{y} \end{pmatrix}$$

$$\boldsymbol{k} = k_z \hat{\boldsymbol{z}}$$



$$\overline{\mathbf{h}}_\perp = \overline{\mathrm{h}}_\mathrm{x} \hat{\boldsymbol{x}} + \overline{\mathrm{h}}_\mathrm{y} \hat{\boldsymbol{y}}$$

$$k_0^2 = \omega^2 \varepsilon_0 \mu_0$$

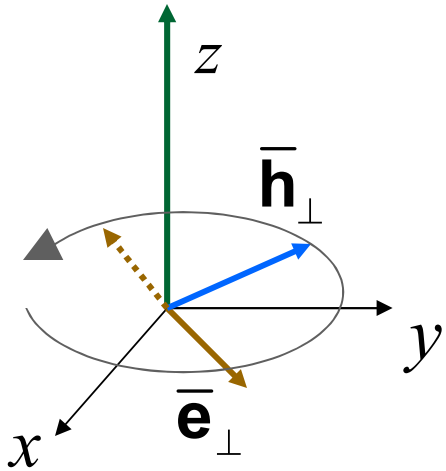
(iv) *Uniform plane waves: Longitudinal magnetization*

- Solutions:

$$\bar{\mathbf{h}}_{\perp}^{+} = \begin{pmatrix} 1 \\ -j \end{pmatrix} h_{+}$$

$$k_z = \pm k_{+}$$
$$k_{+} = k_0\sqrt{\varepsilon(\mu+\mu_a)}$$

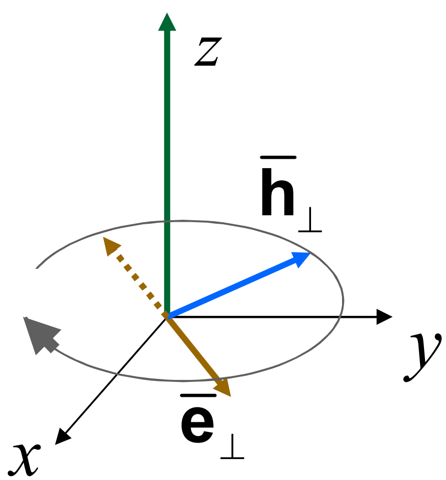
$$\bar{\mathbf{e}}_{\perp}^{+} = -\frac{k_z}{\omega\varepsilon_0\varepsilon}\hat{z}\times\bar{\mathbf{h}}_{\perp}^{+} = -\frac{jk_z h_{+}}{\omega\varepsilon_0\varepsilon}\begin{pmatrix} 1 \\ -j \end{pmatrix}$$



$$\bar{\mathbf{h}}_{\perp}^{-} = \begin{pmatrix} 1 \\ j \end{pmatrix} h_{-}$$

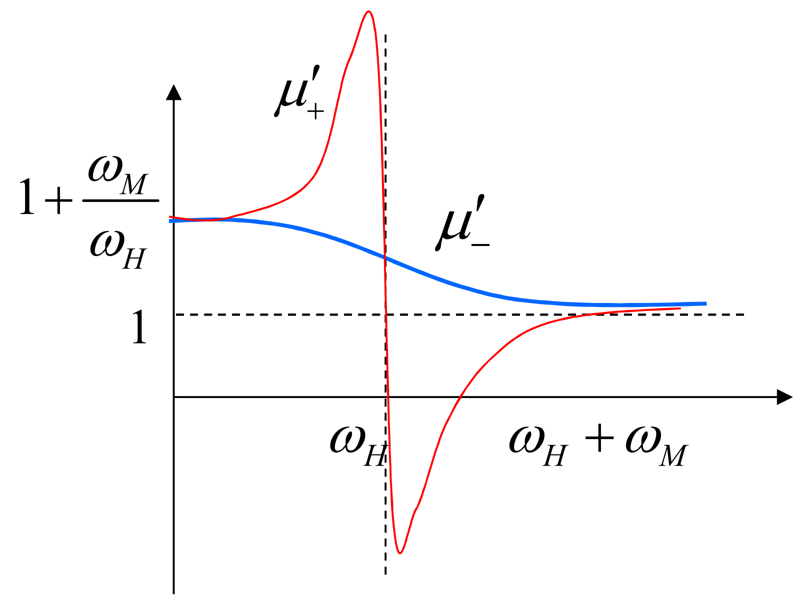
$$k_z = \pm k_{-}$$
$$k_{-} = k_0\sqrt{\varepsilon(\mu-\mu_a)}$$

$$\bar{\mathbf{e}}_{\perp}^{-} = -\frac{k_z}{\omega\varepsilon_0\varepsilon}\hat{z}\times\bar{\mathbf{h}}_{\perp}^{-} = \frac{jk_z h_{-}}{\omega\varepsilon_0\varepsilon}\begin{pmatrix} 1 \\ j \end{pmatrix}$$



(iv) *Uniform plane waves: Longitudinal magnetization*

- These are circularly polarized waves (right-hand and left hand).
- In time, both magnetic and electric fields rotate around z-axis, but remain perpendicular to each other.
- The wave number k_z depends on polarization direction.
- Remember the dependence of the two permeabilities on frequency:



$$\mu_+ = \mu + \mu_a = 1 + \frac{\omega_M}{\omega_H + j\omega\alpha - \omega}$$

$$\mu_- = \mu - \mu_a = 1 + \frac{\omega_M}{\omega_H + j\omega\alpha + \omega}$$

(iv) *Uniform plane waves: Longitudinal magnetization*

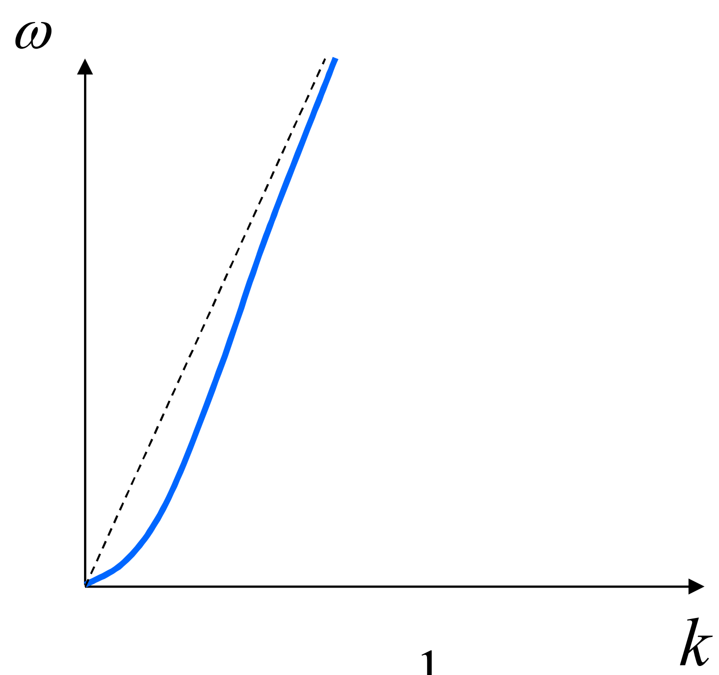
- Dispersion relations $\omega = \omega(k)$. Neglect loss, conductivity for simplicity (real permittivity, permeability)

- Left-hand wave (ordinary wave)

$$k_-^2 = k_0^2 \varepsilon \mu_- = \frac{\varepsilon}{c^2} \frac{(\omega_H + \omega_M + \omega)\omega^2}{\omega_H + \omega}$$

$$\omega \ll \omega_H \rightarrow \omega \sim \frac{k_- c}{\sqrt{\varepsilon(1+\omega_M/\omega_H)}}$$

$$\omega \gg \omega_H + \omega_M \rightarrow \omega \sim \frac{k_- c}{\sqrt{\varepsilon}}$$



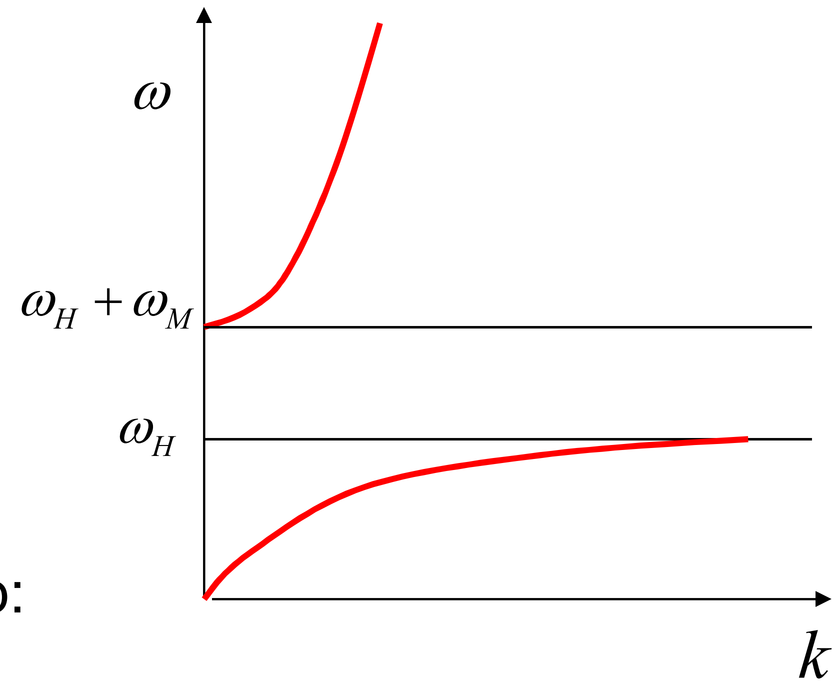
$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

(iv) *Uniform plane waves: Longitudinal magnetization*

- Right-hand wave
  (extraordinary wave)

$$k_+^2 = k_0^2 \varepsilon \mu_+$$

$$= \frac{\varepsilon}{c^2} \frac{\left(\omega_H + \omega_M - \omega\right)\omega^2}{\omega_H - \omega}$$



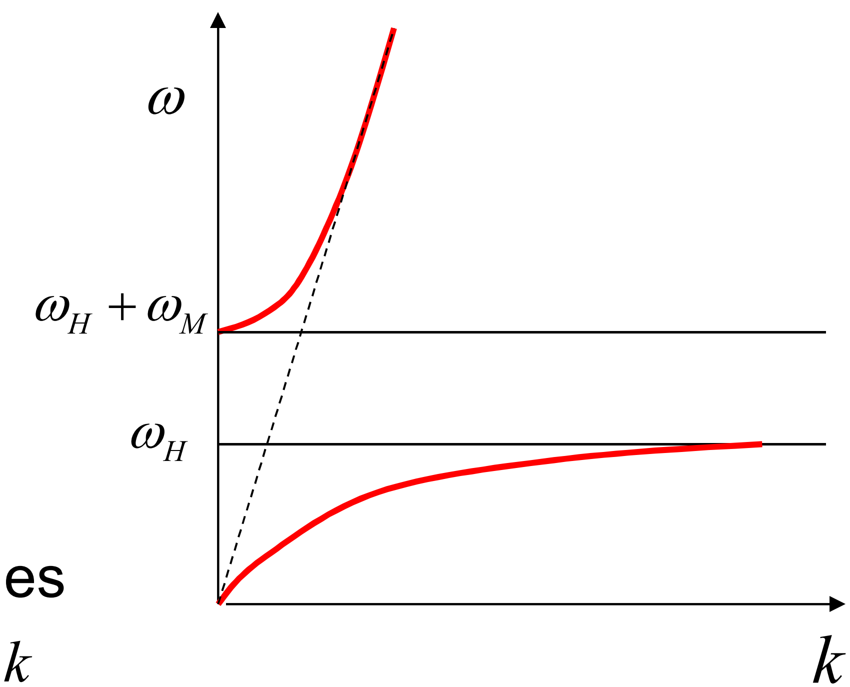
- There are two branches;
  separated by a propagation gap:

$$\omega_H < \omega < \omega_H + \omega_M$$

- If losses are included, waves
  exist in gap, but highly damped

(iv) *Uniform plane waves: Longitudinal magnetization*

- Upper branch resembles ordinary mode, but starts at higher frequency.
- For large values of k, it resembles an ordinary electromagnetic wave
- Lower branch: frequency reaches a constant (ω_H) with growing k
- Phase and group velocities both approach zero in this limit



$$\omega \ll \omega_H \rightarrow \omega \sim \frac{k\_c}{\sqrt{\varepsilon\left(1+\omega_M/\omega_H\right)}}$$

$$\omega \gg \omega_H + \omega_M \rightarrow \omega \sim \frac{k\_c}{\sqrt{\varepsilon}}$$

(iv) *Uniform plane waves: Faraday effect*

- General wave solution (for waves moving along z)

$$\mathbf{h}_\perp(z) = \begin{bmatrix} \bar{\mathsf{h}}_x(z) \\ \bar{\mathsf{h}}_y(z) \end{bmatrix} = \begin{pmatrix} 1 \\ -j \end{pmatrix} \left[A_+ \exp(-jk_+ z) + B_+ \exp(jk_+ z) \right]$$

$$+ \begin{pmatrix} 1 \\ j \end{pmatrix} \left[A_- \exp(-jk_- z) + B_- \exp(jk_- z) \right]$$

$$\mathbf{e}_\perp(z) = \begin{bmatrix} \bar{\mathsf{e}}_x(z) \\ \bar{\mathsf{e}}_y(z) \end{bmatrix} = -\frac{jk_+}{\omega\varepsilon_0\varepsilon} \begin{pmatrix} 1 \\ -j \end{pmatrix} \left[A_+ \exp(-jk_+ z) - B_+ \exp(jk_+ z) \right]$$

$$+ \frac{jk_-}{\omega\varepsilon_0\varepsilon} \begin{pmatrix} 1 \\ j \end{pmatrix} \left[A_- \exp(-jk_- z) - B_- \exp(jk_- z) \right]$$

(iv) *Uniform plane waves: Faraday effect*

- Circular polarization defined with respect to positive z-axis. Independent of direction of wave motion → Sense of rotation (left or right) independent of propagation direction
- Now, consider general solution consisting of waves moving towards +z-direction:

$$\mathbf{h}_\perp(z) = \begin{pmatrix} 1 \\ -j \end{pmatrix} A_+ \exp(-jk_+ z) + \begin{pmatrix} 1 \\ j \end{pmatrix} A_- \exp(-jk_- z)$$

$$\mathbf{e}_\perp(z) = -\frac{jk_+}{\omega\varepsilon_0\varepsilon} \begin{pmatrix} 1 \\ -j \end{pmatrix} A_+ \exp(-jk_+ z) + \frac{jk_-}{\omega\varepsilon_0\varepsilon} \begin{pmatrix} 1 \\ j \end{pmatrix} A_- \exp(-jk_- z)$$

(iv) *Uniform plane waves: Faraday effect*

- Consider a point on z-axis (here z=0). Let us look as a particular choice of the coefficients such that e_y=0 on this point (linearly polarized electric field):

$$k_+ A_+ + k_- A_- = 0$$

$$\mathbf{e}_\perp(0) = -\frac{2jk_+ A_+}{\omega\varepsilon_0\varepsilon}\begin{pmatrix}1\\0\end{pmatrix} \qquad \mathbf{h}_\perp(0) = \begin{pmatrix}A_+ + A_-\\ -jA_+ + jA_-\end{pmatrix}$$

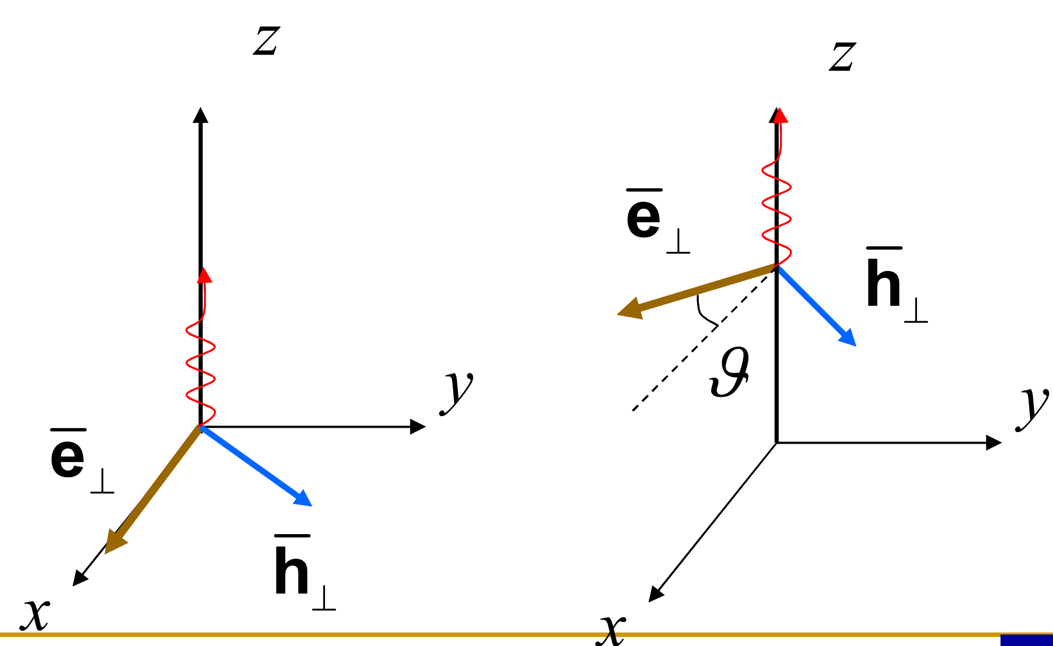
- The two fields are not generally orthogonal. (Why is this the case? Didn’t we state that we had TEM modes?)

(iv) *Uniform plane waves: Faraday effect*

- At a certain distance *l* from this point:

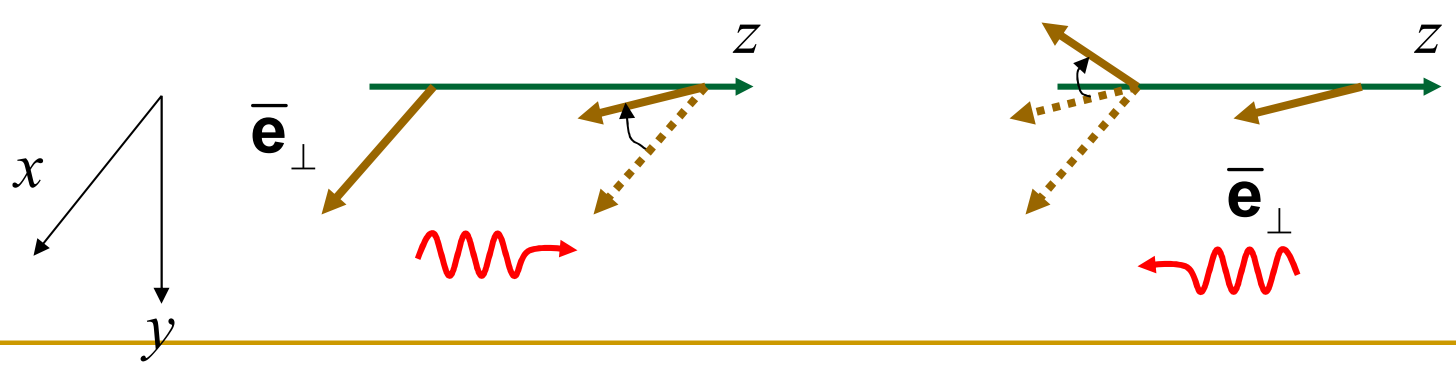
$$\mathbf{e}_\perp(l) = \left(-\frac{2jk_+A_+}{\omega\varepsilon_0\varepsilon}\right)\exp\left[-j(k_+ + k_-)l/2\right]\begin{bmatrix}\cos\vartheta \\ -\sin\vartheta\end{bmatrix}$$

$$\vartheta = \frac{1}{2}\left(k_+ - k_-\right)l$$



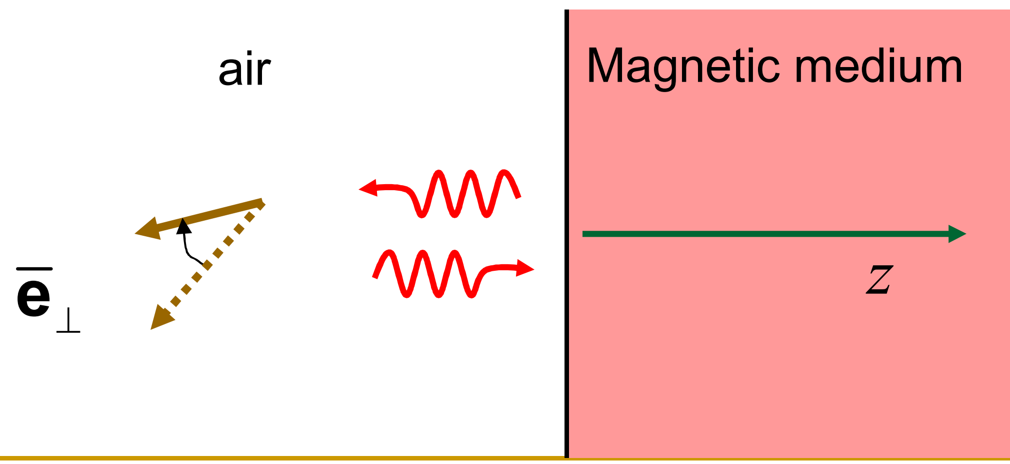
(iv) *Uniform plane waves: Faraday effect*

- The turn of polarization in a longitudinally magnetized medium is called the **Faraday effect**
- Caused by non-diagonal elements of permeability tensor
- If waves were moving in –z-direction: polarization will rotate **in the same direction**!
- The angle is doubled when a wave traverses a distance in forward and reverse



(iv) *Uniform plane waves: Kerr effect*

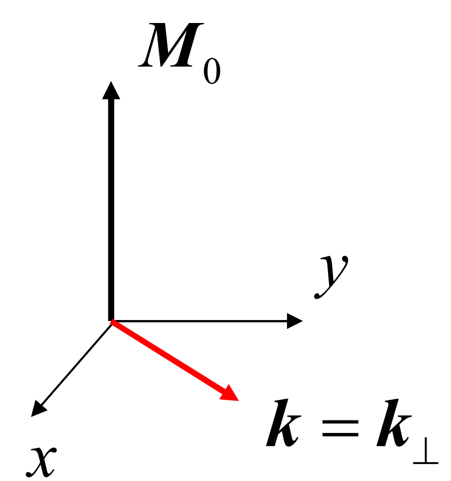
- Reflection from and transmission through interfaces with longitudinally magnetized media solved by decomposing fields into right- and left-hand circularly polarized waves.
- Kerr effect: turn of the polarization axis of elliptically polarized waves reflected from a longitudinally magnetized layer



(v) *Uniform plane waves: Transverse magnetization*

- Now, we focus on waves propagating perpendicular to the magnetization. To analyze such waves we put k_z=0

$$
\begin{cases}
\boldsymbol{k}_\perp \overline{\mathrm{e}}_z = \omega\mu_0 \left(\mu \, \hat{z} \times \overline{\mathbf{h}}_\perp + j\mu_a \overline{\mathbf{h}}_\perp \right) \\
\boldsymbol{k}_\perp \overline{\mathrm{h}}_z = -\omega\varepsilon_0\varepsilon \, \hat{z} \times \overline{\mathbf{e}}_\perp \\
\left(\boldsymbol{k}_\perp \times \overline{\mathbf{e}}_\perp \right)_z = \omega\mu_0\mu_{\|} \, \overline{\mathrm{h}}_z \\
\left(\boldsymbol{k}_\perp \times \overline{\mathbf{h}}_\perp \right)_z = -\omega\varepsilon_0\varepsilon \, \overline{\mathrm{e}}_z
\end{cases}
$$



$$
\boldsymbol{k}_\perp \cdot \overline{\mathbf{e}}_\perp = 0 \qquad \mu \, \boldsymbol{k}_\perp \cdot \overline{\mathbf{h}}_\perp + j\mu_a \left(\boldsymbol{k}_\perp \times \overline{\mathbf{h}}_\perp \right)_z = 0
$$

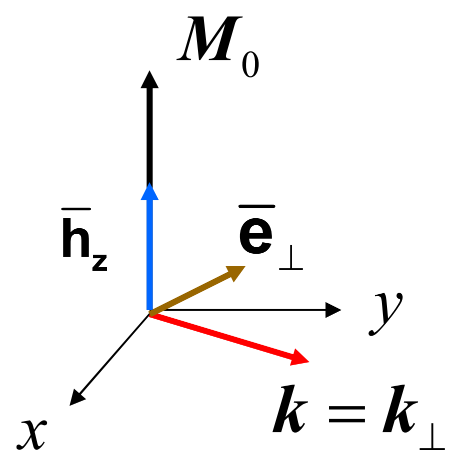
(v) *Uniform plane waves: Transverse magnetization*

- Two sets of independent equations:

TEM

$$k_\perp \bar{\mathrm{h}}_z = -\omega\varepsilon_0\varepsilon\, \hat{z}\times\bar{\mathbf{e}}_\perp$$

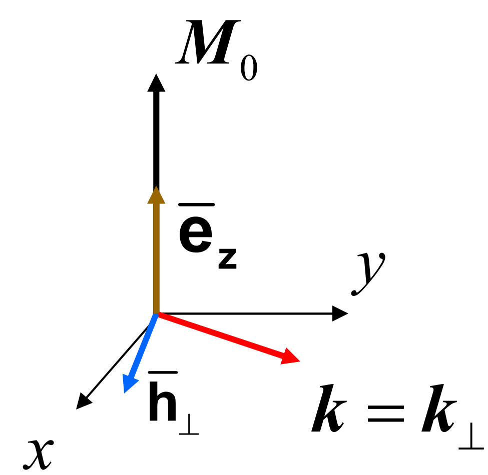
$$\left(k_\perp \times \bar{\mathbf{e}}_\perp\right)_z = \omega\mu_0\mu_\parallel\, \bar{\mathrm{h}}_z$$



TE

$$k_\perp \bar{\mathrm{e}}_z = \omega\mu_0\left(\mu\, \hat{z}\times\bar{\mathbf{h}}_\perp + j\mu_a\bar{\mathbf{h}}_\perp\right)$$

$$\left(k_\perp \times \bar{\mathbf{h}}_\perp\right)_z = -\omega\varepsilon_0\varepsilon\, \bar{\mathrm{e}}_z$$



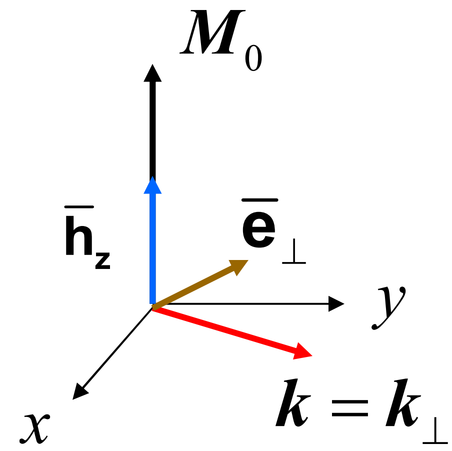
(v) *Uniform plane waves: Transverse magnetization*

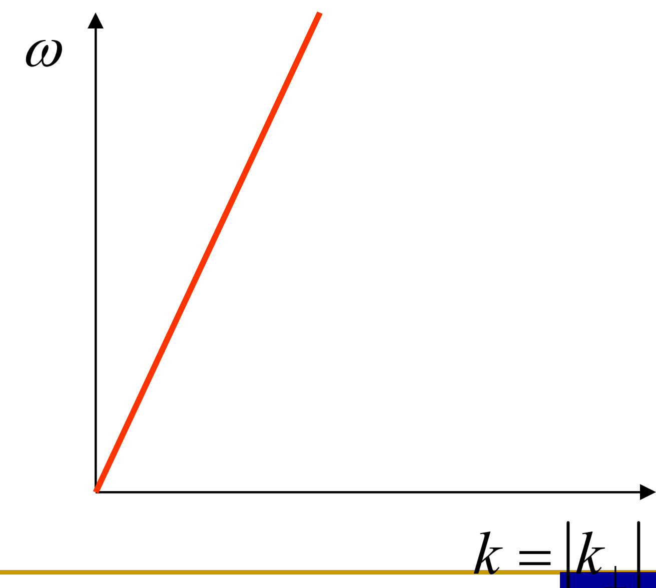
- First set leads to

$$k_\perp^2 = \omega^2 \varepsilon_0 \varepsilon \mu_0 \mu_{||} = k_0^2 \varepsilon \mu_{||}$$

- This is identical to a TEM plane wave propagating in a **non-magnetic** medium
- It has a simple, linear dispersion relation

$$\omega = \frac{|k_\perp|}{\sqrt{\varepsilon_0 \varepsilon \mu_0 \mu_{||}}}$$





(v) *Uniform plane waves: Transverse magnetization*

- Second set gives:

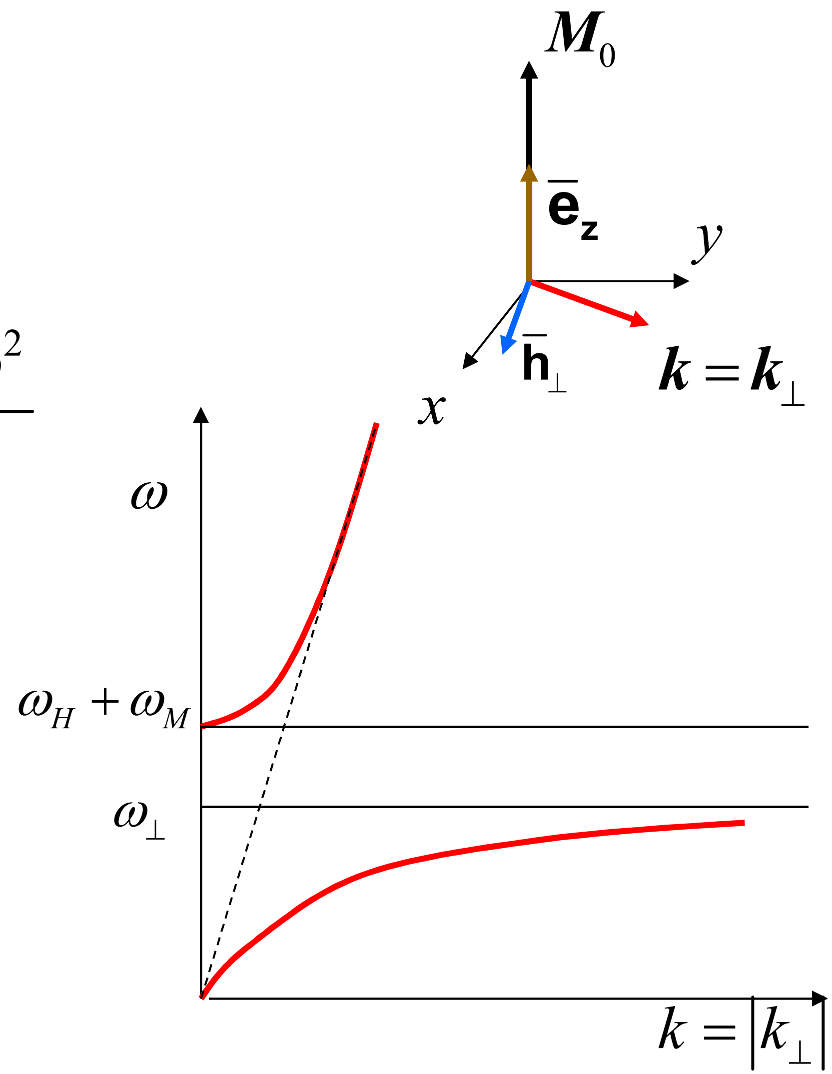
$$k_\perp^2 = \omega^2 \varepsilon_0 \varepsilon \mu_0 \mu_\perp = k_0^2 \varepsilon \mu_\perp$$

$$\mu_\perp = \mu - \frac{\mu_a^2}{\mu} = \frac{(\omega_H + \omega_M)^2 - \omega^2}{\omega_\perp^2 - \omega^2}$$

$$\omega_\perp = \sqrt{\omega_H(\omega_H + \omega_M)}$$

- Dispersion similar to extra-ordinary mode in longitudinal case, but

$$\omega_H \to \omega_\perp$$



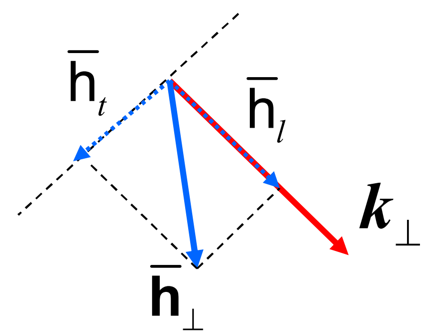
(v) *Uniform plane waves: Transverse magnetization*

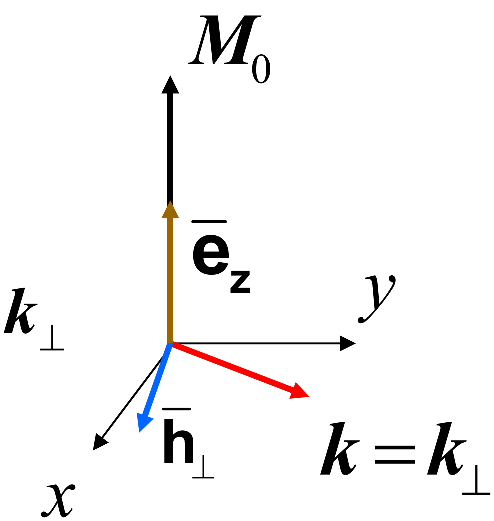
- Field distribution in the second case

$$\boldsymbol{k}_{\perp} \cdot \overline{\mathbf{h}}_{\perp} = j\omega\varepsilon_0\varepsilon \frac{\mu_a}{\mu} \overline{\mathbf{e}}_z$$

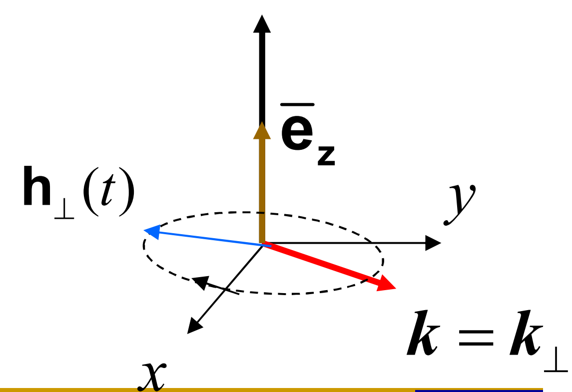
$$\left(\boldsymbol{k}_{\perp} \times \overline{\mathbf{h}}_{\perp}\right)_z = -\omega\varepsilon_0\varepsilon\, \overline{\mathbf{e}}_z$$

$$\frac{\overline{\mathrm{h}}_l}{\overline{\mathrm{h}}_t} = -j\frac{\mu_a}{\mu} = -j\frac{\omega_M\omega}{\omega_{\perp}^2 - \omega^2}$$



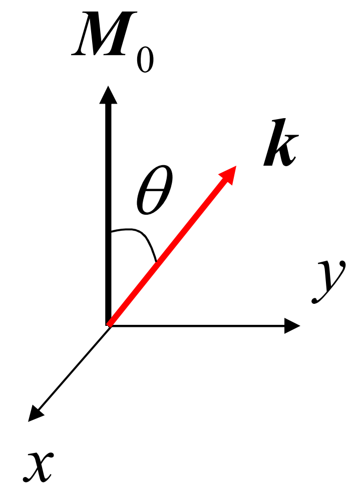


- Magnetic field now elliptically polarized in the x-y plane; rotates around static magnetization



(vi) *Uniform plane waves: general case*

- General case: propagation with an angle θ with respect to magnetization:



$$\left(k^2 - k_0^2 \varepsilon \mu_\perp\right)\bar{\mathrm{e}}_z + jk_z\left(\omega\mu_0 \frac{\mu_a \mu_\parallel}{\mu}\right)\bar{\mathrm{h}}_z = 0$$

$$-jk_z\left(\omega\varepsilon_0\varepsilon \frac{\mu_a}{\mu}\right)\bar{\mathrm{e}}_z + \left(k_\perp^2 + \frac{\mu_\parallel}{\mu} k_z^2 - k_0^2 \varepsilon \mu_\parallel\right)\bar{\mathrm{h}}_z = 0$$

$$k_z = k\cos\theta \qquad k_\perp = k\sin\theta$$

(vi) *Uniform plane waves: general case*

- It follows that $$k^2 = k_0^2 \varepsilon \mu_{eff}(\theta)$$

$$\left(\mu_{eff} - \mu_{\perp}\right)\left(\eta\mu_{eff} - 1\right) = \mu_{eff}\left(\frac{\mu_a}{\mu}\right)^2 \cos^2\theta$$

$$\eta = \frac{1}{\mu_{||}}\sin^2\theta + \frac{1}{\mu}\cos^2\theta$$

$$\mu_{eff}(\theta) = \frac{1 + \frac{1}{2}\left(\frac{\mu_{\perp}}{\mu_{||}} - 1\right)\sin^2\theta \pm \sqrt{\frac{1}{4}\left(\frac{\mu_{\perp}}{\mu_{||}} - 1\right)^2 \sin^4\theta + \left(\frac{\mu_a}{\mu}\right)^2 \cos^2\theta}}{\frac{1}{\mu_{||}}\sin^2\theta + \frac{1}{\mu}\cos^2\theta}$$

(vi) *Uniform plane waves: general case*

- Let us check this solution for longitudinal (θ=0) and transversal (θ=π/2) situations discussed previously:

$$\mu_{eff}(0) = \frac{1 \pm \sqrt{(\mu_a/\mu)^2}}{1/\mu} = \mu \pm \mu_a \quad \Rightarrow \quad k^2 = k_0^2 \varepsilon (\mu \pm \mu_a)$$

$$\mu_{eff}(\pi/2) = \frac{\frac{1}{2}\left(\frac{\mu_\perp}{\mu_\|}+1\right) \pm \sqrt{\frac{1}{4}\left(\frac{\mu_\perp}{\mu_\|}-1\right)^2}}{1/\mu_\|} = \begin{cases} \mu_\perp \\ \mu_\| \end{cases} \quad \Rightarrow \quad k^2 = k_0^2 \varepsilon \begin{cases} \mu_\perp \\ \mu_\| \end{cases}$$

(vi) *Uniform plane waves: general case*

- In general, if permeabilities are real (losses neglected), this equation always has two real roots
- The product of these roots is

$$\frac{\mu_\perp}{\eta} = \frac{\mu_\perp}{\frac{1}{\mu_{||}}\sin^2\theta + \frac{1}{\mu}\cos^2\theta}$$

- Now, neglect losses and take

$$\mu_\perp = \frac{(\omega_H + \omega_M)^2 - \omega^2}{\omega_\perp^2 - \omega^2} \qquad \mu_{||} = 1 \qquad \mu = \frac{\omega_\perp^2 - \omega^2}{\omega_H^2 - \omega^2}$$

(vi) *Uniform plane waves: general case*

- Then

$$\frac{\mu_\perp}{\eta} = \frac{(\omega_H + \omega_M)^2 - \omega^2}{\omega_\perp^2 \sin^2\theta + \omega_H^2 \cos^2\theta - \omega^2}$$

- There is always a frequency range where one solution is negative. Only one mode propagates; the other mode has a propagation 'gap'

$$\sqrt{\omega_\perp^2 \sin^2\theta + \omega_H^2 \cos^2\theta} < \omega < \omega_H + \omega_M$$

- Or

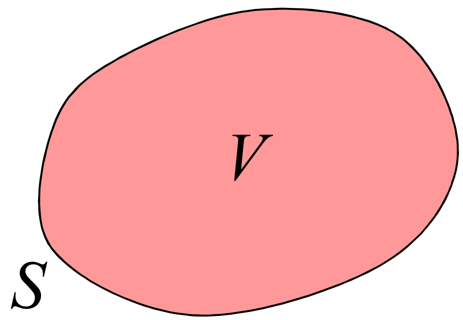
$$\sqrt{\omega_H^2 + \omega_H \omega_M \sin^2\theta} < \omega < \omega_H + \omega_M$$

(vii) *Reciprocity theorem for magnetic media*

- Reciprocity theorem: often useful in circuit theory and microwave engineering. It states that the response of an output of the system to a signal at another input does not change if the input and output are switched.
- It is based on the general reciprocity theorem of Lorentz which is applicable to devices/circuits built from "conventional" electric materials
- To derive this theory for magnetic media, we return to Maxwell equations, and consider two different solutions of those equations inside a certain volume

(vii) *Reciprocity theorem for magnetic media*

- Consider two different solutions of those equations inside a certain volume of a magnetic material



$$\nabla \times \mathbf{e}_1 = -j\omega\mu_0\bar{\bar{\mu}}\cdot\mathbf{h}_1 \qquad \nabla \times \mathbf{e}_2 = -j\omega\mu_0\bar{\bar{\mu}}\cdot\mathbf{h}_2$$

$$\nabla \times \mathbf{h}_1 = j\omega\varepsilon_0\varepsilon\mathbf{e}_1 + \mathbf{j}_{\mathbf{ext,1}} \qquad \nabla \times \mathbf{h}_2 = j\omega\varepsilon_0\varepsilon\mathbf{e}_2 + \mathbf{j}_{\mathbf{ext,2}}$$

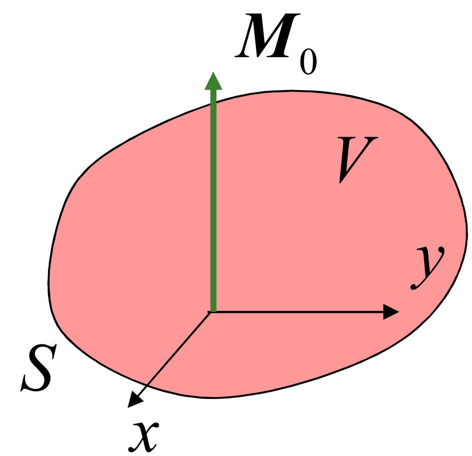
$$\oint_S (\mathbf{e}_1 \times \mathbf{h}_2 - \mathbf{e}_2 \times \mathbf{h}_1)\cdot d\boldsymbol{s} =$$

$$\int_V (\mathbf{j}_{\mathbf{ext,1}} \cdot \mathbf{e}_2 - \mathbf{j}_{\mathbf{ext,2}} \cdot \mathbf{e}_1)\,dV + j\omega\mu_0 \int_V (\mathbf{h}_1 \cdot \bar{\bar{\mu}} \cdot \mathbf{h}_2 - \mathbf{h}_2 \cdot \bar{\bar{\mu}} \cdot \mathbf{h}_1)\,dV$$

(vii) *Reciprocity theorem for magnetic media*

- Again choose a constant permeability tensor magnetized in the z-direction. Then

$$\mathbf{h_1} \cdot \bar{\bar{\mu}} \cdot \mathbf{h_2} - \mathbf{h_2} \cdot \bar{\bar{\mu}} \cdot \mathbf{h_1} = j\mu_a \hat{z} \cdot (\mathbf{h_1} \times \mathbf{h_2})$$



- Reciprocity theorem

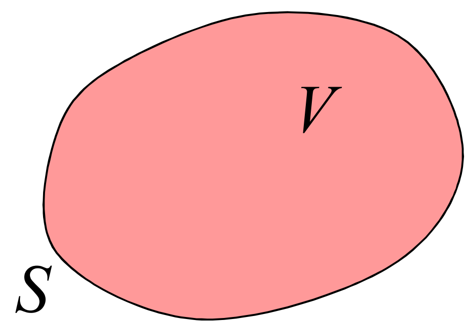
$$\oint_S (\mathbf{e_1} \times \mathbf{h_2} - \mathbf{e_2} \times \mathbf{h_1}) \cdot d\boldsymbol{s} =$$

$$\int_V (\mathbf{j}_{\mathbf{ext,1}} \cdot \mathbf{e_2} - \mathbf{j}_{\mathbf{ext,2}} \cdot \mathbf{e_1}) dV - \omega\mu_0\mu_a \int_V \hat{z} \cdot (\mathbf{h_1} \times \mathbf{h_2}) dV$$

(viii) *Energy relations*

- We formulate the energy balance equation in frequency domain, assuming harmonic signals
- Once more, consider the Maxwell equations

$$\nabla \times \mathbf{e} = -j\omega\mu_0 \bar{\bar{\mu}} \cdot \mathbf{h} \qquad \nabla \times \mathbf{h} = j\omega\varepsilon_0\varepsilon \mathbf{e} + \mathbf{j}_{\mathbf{ext}}$$



- It leads to the equation

$$-\frac{1}{2}\oint_S \left(\mathbf{e} \times \mathbf{h}^*\right) \cdot d\boldsymbol{s} - \frac{1}{2}\int_V \mathbf{j}^*_{\mathbf{ext}} \cdot \mathbf{e}\, dV =$$

$$\frac{j\omega\mu_0}{2}\int_V \mathbf{h}^* \cdot \bar{\bar{\mu}} \cdot \mathbf{h}\, dV - \frac{j\omega\varepsilon_0\varepsilon^*}{2}\int_V \mathbf{e}^* \cdot \mathbf{e}\, dV$$

(viii) *Energy relations*

- Total delivered complex power:

$$P_c = -\frac{1}{2}\oint_S \left(\mathbf{e}\times\mathbf{h}^*\right)\cdot d\boldsymbol{s} - \frac{1}{2}\int_V \mathbf{j}_{\mathbf{ext}}^* \cdot \mathbf{e}\; dV$$

Complex power flowing into the volume through its surface

Power delivered by the impressed source

- Then $\operatorname{Im} P_c = 2j\omega\left(W_M - W_H\right)$

$$W_M = \frac{\mu_0}{4}\operatorname{Re}\int_V \mathbf{h}^* \cdot \bar{\bar{\mu}} \cdot \mathbf{h}\; dV \qquad W_E = \frac{\varepsilon_0 \operatorname{Re}\left(\varepsilon^*\right)}{4}\int_V \mathbf{e}^* \cdot \mathbf{e}\; dV$$

Not generally the stored energy due to dispersion!

(viii) *Energy relations*

$$W_M = \frac{\mu_0}{4}\mathrm{Re}\int_V \mathbf{h}^* \cdot \bar{\bar{\boldsymbol{\mu}}} \cdot \mathbf{h}\, dV =$$

$$\frac{\mu_0}{4}\left[\mathrm{Re}(\mu)\mathbf{h}_\perp^* \cdot \mathbf{h}_\perp + \mathrm{Re}(\mu_{||})\left|\mathrm{h}_z\right|^2 - 2\,\mathrm{Re}(\mu_a)\,\mathrm{Im}\left(\mathrm{h}_x^*\mathrm{h}_y\right)\right]$$

- It only depends on the real part of the elements of the permeability tensor

$$\bar{\bar{\boldsymbol{\mu}}} = \begin{bmatrix} \mu & j\mu_a & 0 \\ -j\mu_a & \mu & 0 \\ 0 & 0 & \mu_{||} \end{bmatrix} = \begin{bmatrix} \mu' - j\mu'' & j\mu_a' + \mu_a'' & 0 \\ -j\mu_a' - \mu_a'' & \mu' - j\mu'' & 0 \\ 0 & 0 & \mu_{||}' - j\mu_{||}'' \end{bmatrix}$$

(viii) *Energy relations*

- Real delivered power

$$\mathrm{Re}\, P_c = -\frac{\omega\mu_0}{2}\mathrm{Im}\int_V \mathbf{h}^* \cdot \bar{\bar{\boldsymbol{\mu}}} \cdot \mathbf{h}\, dV + \frac{\omega\varepsilon_0\, \mathrm{Im}\left(\varepsilon^*\right)}{2}\int_V \mathbf{e}^* \cdot \mathbf{e}\, dV$$

- Power dissipated due to losses in magnetic material

$$-\frac{\omega\mu_0}{2}\mathrm{Im}\int_V \mathbf{h}^* \cdot \bar{\bar{\boldsymbol{\mu}}} \cdot \mathbf{h}\, dV =$$

$$\frac{\omega\mu_0}{2}\left[\mu''\mathbf{h}_\perp^* \cdot \mathbf{h}_\perp + \mu_\parallel''\left|\mathrm{h}_z\right|^2 - 2\mu_a''\,\mathrm{Im}\left(\mathrm{h}_x^*\mathrm{h}_y\right)\right]$$