Microwave Magnetics

Graduate Course
Electrical Engineering (Communications)
2nd Semester, 1389-1390
Sharif University of Technology
General information

- Contents of lecture 10:
  - Y-junction microwave circulators
    - Field analysis
    - Resonances of an isolated system
    - The Y-junction circulator
    - The Green’s function
    - The impedance matrix
    - The circulation condition
    - Basic circulator design
    - Field profile
    - Remarks
Y-junction microwave circulators

- Most widely used microwave component based on magnetic (ferrite) materials are Y-junction circulators.
- Circulators based on Faraday effect in circular waveguides were discussed before.
- The device passes the signal to the next port in a cyclic fashion, without the reverse being true.
- Planar circulators: 3-port junctions are the most popular.
Y-junction microwave circulators

- In terms of scattering parameters, a cyclic symmetric, 3-port network (or device) is described by

\[
\bar{S} = \begin{bmatrix}
\alpha & \gamma & \beta \\
\beta & \alpha & \gamma \\
\gamma & \beta & \alpha
\end{bmatrix}
\]

- If the device is perfectly matched:

\[
\alpha = 0 \quad \Rightarrow \quad \bar{S} = \begin{bmatrix}
0 & \gamma & \beta \\
\beta & 0 & \gamma \\
\gamma & \beta & 0
\end{bmatrix}
\]
Y-junction microwave circulators

- If the device is lossless then

\[
\begin{bmatrix}
0 & 0 & \beta \\
\beta & 0 & 0 \\
0 & \beta & 0
\end{bmatrix}
\]

\( \gamma = 0 \rightarrow \bar{S} = \begin{bmatrix} 0 & 0 & \beta \\ \beta & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix} \quad |\beta| = 1 \)

- We will study the realization of this device using vertically magnetized ferrite disks as in the paper:

Y-junction microwave circulators

- Planar realizations based on a magnetic (ferrite) disk magnetized perpendicular to the disk
- Both stripline and microstrip versions have been in use, but we limit ourselves to the microstrip version
(i) Field analysis

- Maxwell equations in cylindrical coordinates:

\[
\begin{align*}
1 \frac{\partial e_z}{r \partial \phi} - \frac{\partial e_\phi}{\partial z} &= -j\omega b_r \\
\frac{\partial e_r}{\partial z} - \frac{\partial e_z}{\partial r} &= -j\omega b_\phi \\
\frac{1}{r} \frac{\partial}{\partial r} \left( re_\phi \right) - \frac{1}{r} \frac{\partial e_r}{\partial \phi} &= -j\omega b_z \\
1 \frac{\partial h_z}{r \partial \phi} - \frac{\partial e_\phi}{\partial z} &= j\omega \varepsilon_0 \varepsilon e_r \\
\frac{\partial h_r}{\partial z} - \frac{\partial h_z}{\partial r} &= j\omega \varepsilon_0 \varepsilon e_\phi \\
1 \frac{\partial}{r} \left( rh_\phi \right) - \frac{1}{r} \frac{\partial h_r}{\partial \phi} &= j\omega \varepsilon_0 \varepsilon e_z
\end{align*}
\]
(i) Field analysis

- We again assume uniformity of fields inside the disk in the vertical (z-) direction which is a good assumption if the disk is thin compared to its lateral dimensions:

\[
\frac{1}{r} \frac{\partial e_z}{\partial \phi} = -j \omega b_r
\]

\[
\frac{\partial e_z}{\partial r} = j \omega b_\varphi
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( re_\varphi \right) - \frac{1}{r} \frac{\partial e_r}{\partial \phi} = -j \omega b_z
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( rh_\varphi \right) - \frac{1}{r} \frac{\partial h_r}{\partial \phi} = j \omega \varepsilon_0 \varepsilon e_z
\]
(i) Field analysis

Inside the magnetic material we have

\[
\begin{bmatrix}
    b_r \\
    b_\varphi \\
    b_z
\end{bmatrix} = \mu_0 \begin{bmatrix}
    \mu & j\mu_a & 0 \\
    -j\mu_a & \mu & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    h_r \\
    h_\varphi \\
    h_z
\end{bmatrix}
\]
(i) Field analysis

- This leads to two decoupled sets

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial \varphi} e_z &= -j \omega b_r \\
\frac{\partial}{\partial r} e_z &= j \omega b_\varphi \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r h_\varphi \right) - \frac{1}{r} \frac{\partial h_r}{\partial \varphi} &= j \omega \varepsilon_0 \varepsilon e_z \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r e_\varphi \right) - \frac{1}{r} \frac{\partial e_r}{\partial \varphi} &= -j \omega b_z
\end{align*}
\]
(i) **Field analysis**

- The 2\textsuperscript{nd} set has no solution which can satisfy boundary conditions on the metal disk and ground plane.

- 1\textsuperscript{st} set leads to

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial e_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 e_z}{\partial \varphi^2} + k_0^2 \varepsilon \mu_\perp e_z = 0
\]

- Solutions have to be periodic functions of the angle $\varphi$.
  There, try solutions of the type

\[
e_z(r, \varphi) = f_n(r) \exp(-jn\varphi)
\]
(i) Field analysis

- Result: \( r^2 \frac{\partial^2 f_n(r)}{\partial r^2} + r \frac{\partial f_n(r)}{\partial r} + \left( k_0^2 \varepsilon \mu_\perp r^2 - n^2 \right) f_n(r) = 0 \)

- \( f_n(r) = AJ_n(k_\perp r) + BY_n(k_\perp r) \quad k_\perp = k_0 \sqrt{\varepsilon \mu_\perp} \)

- Finite field at the disk center requires \( B = 0 \)

- \( e_z(r, \varphi) = A_n J_n(k_\perp r) \exp(-jn\varphi) \)
(i) Field analysis

- Magnetic field:

\[
h_r = \frac{1}{\omega \mu_0 \mu_\perp} \left( \frac{j}{r} \frac{\partial e_z}{\partial \phi} - \frac{\mu_a}{\mu} \frac{\partial e_z}{\partial r} \right)
= \frac{A_n k_\perp}{\omega \mu_0 \mu_\perp} \left( \frac{n J_n(k_\perp r)}{k_\perp r} - \frac{\mu_a}{\mu} J'_n(k_\perp r) \right) \exp(-jn\phi)
\]

\[
h_\phi = \frac{1}{j\omega \mu_0 \mu_\perp} \left(-j \frac{\mu_a}{\mu} \frac{1}{r} \frac{\partial e_z}{\partial \phi} + \frac{\partial e_z}{\partial r} \right)
= \frac{A_n k_\perp}{j\omega \mu_0 \mu_\perp} \left(-n \frac{\mu_a}{\mu} \frac{J_n(k_\perp r)}{k_\perp r} + J'_n(k_\perp r) \right) \exp(-jn\phi)
\]
(i) Field analysis

- Note that in each mode $n$ all the fields contain the exponential factor

$$\exp(-jn\varphi)$$

- If we return to time domain, we observe that each mode represent a field rotating in the counter clockwise ($n>0$) or clockwise ($n<0$) direction

- Example:

$$e_z (r, \varphi) = J_n(k_\perp r) \exp(-jn\varphi)$$

$$e_z (r, \varphi, t) = \text{Re} \left[ \exp(j\omega t) e_z \right]$$

$$= J_n (k_\perp r) \cos(j\omega t - jn\varphi)$$
(ii) Resonances of an isolated system

- Let us first consider an isolated system. We again use the magnetic-wall boundary conditions by assuming that
  - The ferrite disk is thin compared to its diameter
  - There is negligible electric current on the top metal disk flowing perpendicular to the outer edge of the disk

\[
h_\varphi (R, \varphi) = 0
\]
(ii) Resonances of an isolated system

- The solutions to this problem (without any sources or connections to the external ‘world’) define the resonances of the system.
- Let us now impose the magnetic wall boundary condition on the solutions found.

\[ h_\varphi (R, \varphi) = 0 \]

\[ n \frac{\mu_a}{\mu} = \frac{(k_\perp R) J'_n(k_\perp R)}{J_n(k_\perp R)} \]
(ii) Resonances of an isolated system

- 1\textsuperscript{st} case: \( \omega < \omega_\perp \) or \( \omega > \omega_M + \omega_H \) \( \rightarrow \mu_\perp > 0 \rightarrow k_\perp: \text{real} \)

\[
k_\perp = k_0 \sqrt{\varepsilon \mu_\perp} = \omega \sqrt{\varepsilon_0 \mu_0 \varepsilon} \sqrt{\frac{(\omega_M + \omega_H)^2 - \omega^2}{\omega_\perp^2 - \omega^2}}
\]
(ii) Resonances of an isolated system

- Solution for the rotationally symmetric non-rotating mode \( n=0 \):

\[
J'_0(k_\perp R) = 0 \rightarrow k_\perp R = \zeta_{0p} : \text{zero's of } J'_0(z)
\]
(ii) Resonances of an isolated system

- To each zero correspond two resonances
- Electric field distribution is same at both

\[ e_z^{0p} = J_0 \left( \frac{\zeta_0 r}{R} \right) \]

- Magnetic field polarization will be different for the two frequencies since \( \mu_\perp \) and \( \mu_a / \mu \) will be different

\[ h_r^{0p} = -\frac{k_\perp}{\omega \mu_0 \mu_\perp} \frac{\mu_a}{\mu} J'_0 \left( \frac{\zeta_0 r}{R} \right) \]
\[ h_\varphi^{0p} = \frac{k_\perp}{j\omega \mu_0 \mu_\perp} J'_0 \left( \frac{\zeta_0 r}{R} \right) \]
(ii) Resonances of an isolated system

- But $n=0$ is not the most interesting case. For other values of $n$, however, we have to solve

$$n \frac{\mu_a}{\mu} = \frac{(k_\perp R)J'_n(k_\perp R)}{J_n(k_\perp R)}$$

- For each $n$, we have a number of solutions (as before indexed by $p$)

- But $+n$ and $-n$ will not resonate at the same frequency because the above equation depends on the sign of $n$

- Note: right hand side does not depend on the sign of $n$.

$$J_{-n}(z) = (-1)^n J_n(z)$$
(ii) Resonances of an isolated system

- If the ratio $\frac{\mu_a}{\mu}$ would have been zero (no off-diagonal permeability element), the resonance frequencies would have been the same.

- For a non-zero, but small $\frac{\mu_a}{\mu}$, the two modes will have different but close resonance frequencies.

- Then the roots of the two equations are

$$k_{\perp}^\pm R \approx \zeta_{np} \pm n \frac{\mu_a}{\mu} \frac{\zeta_{np}}{n^2 - \zeta_{np}^2} \quad J_n'\left(\zeta_{np}\right) = 0$$

- When $\frac{\mu_a}{\mu} = 0$ the roots are: $k_{\perp}^0 R = \zeta_{np}$.
(ii) Resonances of an isolated system

- To find the resonance frequencies let us expand the equation

\[ k_\perp(\omega)R = \zeta_{np} \pm n \frac{\mu_a(\omega)}{\mu(\omega)} \frac{\zeta_{np}}{n^2 - \zeta_{np}^2} \]

\[ k_\perp(\omega) = \frac{\omega}{c} \sqrt{\varepsilon \mu_\perp(\omega)} \]

- We expand around the solution when \( \mu_a / \mu = 0 \). That solution is defined by

\[ k_\perp(\omega_{res}^0) = \frac{\zeta_{np}}{R} \]
(ii) Resonances of an isolated system

- Expansion:

\[
\frac{dk_{\perp}}{d\omega} \bigg|_{\omega_{\text{res}}^{0}} \left( \omega - \omega_{\text{res}}^{0} \right) = \pm \frac{n}{R} \frac{\zeta_{np}}{n^2 - \zeta_{np}^2} \left[ \kappa(\omega_{\text{res}}^{0}) + \frac{d\kappa}{d\omega} \bigg|_{\omega_{\text{res}}^{0}} \left( \omega - \omega_{\text{res}}^{0} \right) \right]
\]

\[
k_{\perp}(\omega) = \frac{\omega}{c} \sqrt{\varepsilon \mu_{\perp}(\omega)} \quad \kappa(\omega) = \frac{\mu_{a}(\omega)}{\mu(\omega)}
\]

\[
\omega_{\text{res}}^{\pm} = \omega_{\text{res}}^{0} \pm \frac{c_{np} \kappa(\omega_{\text{res}}^{0})}{\left( \frac{dk_{\perp}}{d\omega} + c_{np} \frac{d\kappa}{d\omega} \right)\bigg|_{\omega_{\text{res}}^{0}}}
\]

\[
c_{np} = \frac{n}{R} \frac{\zeta_{np}}{n^2 - \zeta_{np}^2}
\]
(ii) Resonances of an isolated system

- Manipulating the terms:

\[ \frac{d k_\perp}{d \omega} = \frac{k_\perp}{\omega} \left[ 1 + \frac{\omega}{2 \mu_\perp} \frac{d \mu_\perp}{d \omega} \right] \]

\[
\frac{\omega}{2 \mu_\perp} \frac{d \mu_\perp}{d \omega} = \frac{(\omega_M + \omega_H) \omega^2}{(\omega_\perp - \omega^2) \left[ (\omega_M + \omega_H)^2 - \omega^2 \right]} 
\]

\[
= \frac{\mu_a}{\mu} \frac{(\omega_M + \omega_H) \omega}{(\omega_M + \omega_H)^2 - \omega^2} 
\]

\[
\omega \frac{d \kappa}{d \omega} = \frac{(\omega_\perp^2 + \omega^2) \omega_M \omega}{(\omega_\perp^2 - \omega^2)^2} = \frac{\mu_a}{\mu} \frac{(\omega_\perp^2 + \omega^2)}{(\omega_\perp^2 - \omega^2)} 
\]
(ii) Resonances of an isolated system

- Numerical example: magnetic disk with

\[ f_H = 2 \text{ GHz}, \ f_M = 6 \text{ GHz}, \ \varepsilon = 4 \]
(ii) Resonances of an isolated system

- 2nd case: $\omega_\perp < \omega < \omega_M + \omega_H \rightarrow \mu_\perp < 0 \rightarrow k_\perp$: imaginary

$$k_\perp = j q_\perp \quad n \frac{\mu_a}{\mu} = \frac{(k_\perp R) J'_n(k_\perp R)}{J_n(k_\perp R)} = \frac{(q_\perp R) I'_n(q_\perp R)}{I_n(q_\perp R)}$$

- The right hand side is always positive. The left hand side is negative for positive $n$ and positive for negative $n$.

- Therefore, only solutions with $n < 0$ are possible: these are modes rotating in the clockwise sense
(ii) Resonances of an isolated system

- Besides, fields drop exponentially towards the disk centre: they are concentrated near the edge of the disk

\[ e_z(r, \varphi) = A_n I_n(q \sqrt{r}) \exp(-jn\varphi) \]

- These are similar to the edge-guided modes in a microstrip built on a vertically magnetized ferrite substrate (why?)
(iii) The Y-junction circulator

- In order to see how such a device can be built, we should now add the junctions to the system.
- These are three microstrip lines connected to the metal disk on top of the magnetic (ferrite) disk.
- Electric currents will now flow through these microstrips.
(iii) The Y-junction circulator

- To analyze this problem we again impose magnetic wall boundary conditions on the vertical wall of the ferrite disk, but only outside the microstrip junctions.
- Below the microstrips we assume a given distribution of $h_\phi$ on the ferrite surface.
(iii) The Y-junction circulator

- Returning to the general solution:

\[ e_z(r, \varphi) = \sum_{n=-\infty}^{\infty} A_n J_n(k r) \exp(-jn\varphi) \]

\[ h_r(r, \varphi) = \frac{k_\perp}{\omega \mu_0 \mu_\perp} \sum_{n=-\infty}^{\infty} A_n \left[ \frac{nJ_n(k r)}{k r} - \frac{\mu_a}{\mu} J'_n(k r) \right] \exp(-jn\varphi) \]

\[ h_\varphi(r, \varphi) = \frac{k_\perp}{j \omega \mu_0 \mu_\perp} \sum_{n=-\infty}^{\infty} A_n \left[ -n \frac{\mu_a}{\mu} \frac{J_n(k r)}{k r} + J'_n(k r) \right] \exp(-jn\varphi) \]
(iii) The Y-junction circulator

- We next impose the following boundary condition:

\[
    h_\phi(R, \phi) = \begin{cases} 
        h^1_\phi(\phi) & -\Psi < \phi < \Psi \\
        h^2_\phi(\phi) & \frac{2\pi}{3} - \Psi < \phi < \frac{2\pi}{3} + \Psi \\
        h^3_\phi(\phi) & -\frac{2\pi}{3} - \Psi < \phi < -\frac{2\pi}{3} + \Psi \\
        0 & \text{elsewhere} 
    \end{cases}
\]

\( \equiv h^b_\phi(\phi) \)
(iii) The Y-junction circulator

- Fourier analysis:

\[
A_n = \frac{j \omega \mu_0 \mu_\perp}{2\pi k_\perp} \int_{-\pi}^{\pi} \exp(jn\varphi')h^b_\varphi(\varphi')d\varphi'
\]

\[
= \frac{J'_n(k_\perp R) - n \frac{\mu_a}{\mu} \frac{J_n(k_\perp R)}{k_\perp R}}{J'_n(k_\perp R) - n \frac{\mu_a}{\mu} \frac{J_n(k_\perp R)}{k_\perp R}}
\]

\[
\mathbf{e}_z(r, \varphi) = \frac{j \omega \mu_0 \mu_\perp}{2\pi k_\perp} \sum_{n=-\infty}^{\infty} \frac{J_n(k_\perp r) \int_{-\pi}^{\pi} \exp[jn(\varphi' - \varphi)]h^b_\varphi(\varphi')d\varphi'}{J'_n(k_\perp R) - n \frac{\mu_a}{\mu} \frac{J_n(k_\perp R)}{k_\perp R}}
\]

Y-junction circulator
(iv) The Green’s function

The electric field at the edge (why do we need this?) can be written as

\[ e_z(R, \varphi) = \int_{-\pi}^{\pi} G(\varphi - \varphi') h^b_\varphi(\varphi') d\varphi' \]

\[ G(\varphi - \varphi') = \frac{j \omega \mu_0 \mu_\perp R}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\exp[jn(\varphi' - \varphi)]}{(k_\perp R) J'_n(k_\perp R) - \frac{\mu_a}{\mu} n} \]

Green’s function
(iv) The Green’s function

- $G(\phi - \phi_0)$: response of the electric field at the disk edge at any angle $\phi$ to a $\delta$-function magnetic source at the angle $\phi_0$

$$h^b_\phi(\phi) = \delta(\phi - \phi_0)$$

$$e_z(R, \phi) = \int_{-\pi}^{\pi} G(\phi - \phi') h^b_\phi(\phi') d\phi' = G(\phi - \phi_0)$$
(iv) The Green’s function

- Consider the frequency dependence of the Green’s function which is brought about by $\mu_\perp, k_\perp, \mu_a, \mu$

$$G(\varphi - \varphi'; \omega) = \frac{j\omega \mu_0 \mu_\perp(\omega)R}{2\pi} \times$$

$$\sum_{n=-\infty}^{\infty} \frac{\exp[jn(\varphi' - \varphi)]}{k_\perp(\omega)R J_n'[k_\perp(\omega)R] - \frac{\mu_a}{\mu} n}$$

- Poles of Green’s function in frequency are exactly the resonances of an isolated disk!
(iv) The Green’s function

- Let us inspect the Green’s function in more detail: $G(\varphi)$ is, in fact, the electric field at the disk edge at an angle $\varphi$, generated by a source at $\varphi_0 = 0$. It can be written as

$$G(\varphi) = \frac{j \omega \mu_0 \mu_\perp R}{2\pi} \left\{ \frac{1}{U_0} + \sum_{n=1}^{\infty} \frac{U_n \cos(n\varphi) - j(n \mu_a / \mu) \sin(n\varphi)}{U_n^2 - (n \mu_a / \mu)^2} \right\}$$

$$U_n \equiv \frac{(k_\perp R) J'_n(k_\perp R)}{J_n(k_\perp R)}$$
(iv) The Green’s function

- In a disk without any gyrotropic properties, excited field will be symmetric with respect to the source at $\varphi_0 = 0$

$$\mu_a = 0 \rightarrow G(\varphi) = \frac{j \omega \mu_0 \mu_{\perp} R}{2\pi} \left\{ \frac{1}{U_0} + 2 \sum_{n=1}^{\infty} \frac{\cos(n\varphi)}{U_n} \right\}$$
(iv) The Green’s function

- But in our case this is not true. The imaginary part of the electric field is symmetric but the real part is anti-symmetric

\[
\text{Im}[G(\varphi)] = \frac{\omega \mu_0 \mu_\perp R}{2\pi} \left\{ \frac{1}{U_0} + 2 \sum_{n=1}^{\infty} \frac{U_n \cos(n\varphi)}{U_n^2 - (n\mu_a / \mu)^2} \right\}
\]

\[
\text{Re}[G(\varphi)] = \frac{\omega \mu_0 \mu_\perp R}{2\pi} \sum_{n=1}^{\infty} \frac{(2n\mu_a / \mu) \sin(n\varphi)}{U_n^2 - (n\mu_a / \mu)^2}
\]
(v) The impedance matrix

- Neglecting the fringe, the magnetic field between the microstrip and the ground plane is approximately constant and transverse to the microstrip

- Hence we assume that $h_\varphi$ is constant in the junction regions (better approximation: use the true magnetic field distribution below the microstrip)
(v) *The impedance matrix*

- We approximate this field by

\[
\mathbf{h}^k_\varphi = -\frac{I_k}{w}, \quad k = 1, 2, 3
\]

\(I_k\): Total current through the \(k\)-th microstrip
(v) The impedance matrix

- Approximation justified as propagation mode of microstrip is quasi TEM
- Electric and magnetic fields then studied separately using electrostatics and magnetostatics
- Magnetic field of microstrip ~ field of a (~uniform) current sheet (microstrip) and its (negative) image in the ground plane

- \( h_{MS} \approx -\frac{J}{2}\hat{x} \)
- \( h_{image} \approx -\frac{J}{2}\hat{x} \)
- \( h_{MS} + h_{image} \approx -J\hat{x} = -\frac{I}{w}\hat{x} \)
(v) The impedance matrix

- Electric field at any point on the edge given by

\[ e_z(R, \varphi) = \]
\[- \frac{I_1}{W} G(\varphi - \varphi') d\varphi' \]
\[- \frac{I_2}{W} G(\varphi - \varphi') d\varphi' \]
\[- \frac{I_3}{W} G(\varphi - \varphi') d\varphi' \]
(v) The impedance matrix

- Considering the complex power, voltage on each port is found to be the “averaged” voltage on the port

\[ V_k = -\frac{b}{2\Psi} \int_{\varphi_{k1}}^{\varphi_{k2}} e_z(R, \varphi) d\varphi \]
(v) *The impedance matrix*

- For simplicity assume that the microstrips are narrow compared to the disk circumference:

\[ w \ll 2\pi R \rightarrow \Psi \ll \pi \]

\[
\int_{-\Psi}^{\Psi} G(\varphi - \varphi') d\varphi' \approx 2\Psi \ G(\varphi)
\]

\[
\int_{\frac{-2\pi}{3}}^{\frac{2\pi}{3} + \Psi} G(\varphi - \varphi') d\varphi' \approx 2\Psi \ G(\varphi - 2\pi / 3)
\]

\[
\int_{\frac{-2\pi}{3}}^{\frac{2\pi}{3} + \Psi} G(\varphi - \varphi') d\varphi' \approx 2\Psi \ G(\varphi + 2\pi / 3)
\]

Y-junction circulator
(v) The impedance matrix

- As a result: (use \( w = 2 \Psi R \))

\[
e_Z (R, \varphi) \approx - \frac{I_1}{R} G(\varphi) - \frac{I_2}{R} G\left(\varphi - \frac{2\pi}{3}\right) - \frac{I_3}{R} G\left(\varphi + \frac{2\pi}{3}\right)
\]

- Applying the same approximation to voltages:

\[
V_1 \approx -b e_Z (R, 0)
\]

\[
V_2 \approx -b e_Z (R, 2\pi / 3)
\]

\[
V_3 \approx -b e_Z (R, -2\pi / 3)
\]
The impedance matrix: note that $G(\varphi + 2m\pi) = G(\varphi)$

\[ V_1 = \frac{b}{R} G(0) I_1 + \frac{b}{R} G\left(-\frac{2\pi}{3}\right) I_2 + \frac{b}{R} G\left(\frac{2\pi}{3}\right) I_3 \]

\[ V_2 = \frac{b}{R} G\left(\frac{2\pi}{3}\right) I_1 + \frac{b}{R} G(0) I_2 + \frac{b}{R} G\left(-\frac{2\pi}{3}\right) I_3 \]

\[ V_3 = \frac{b}{R} G\left(-\frac{2\pi}{3}\right) I_1 + \frac{b}{R} G\left(\frac{2\pi}{3}\right) I_2 + \frac{b}{R} G(0) I_3 \]
(v) The impedance matrix

\[
\bar{Z} = \frac{b}{R} \begin{bmatrix}
G(0) & G(-2\pi/3) & G(2\pi/3) \\
G(2\pi/3) & G(0) & G(-2\pi/3) \\
G(-2\pi/3) & G(2\pi/3) & G(0)
\end{bmatrix}
\]

- We have \( G(0) \): imaginary \( G(-2\pi/3) = -G^*(2\pi/3) \)

\[\bar{Z}_{ki} = -\left(\bar{Z}_{ik}\right)^*\]

- This result is more general and not limited to the approximations made (why?)
Now take 1 to be the input port. Let 2 and 3 be connected to infinite transmission lines. Replace the lines by their characteristic impedances.
(vi) The circulation condition

- For circulator operation we should have
  \[ I_2 \neq 0, \ I_3 = 0 \]

- Use the notation

\[
\begin{pmatrix}
\xi & \eta & \tau \\
\tau & \xi & \eta \\
\eta & \tau & \xi
\end{pmatrix}
\]

\[
\xi = \frac{b}{R} G (0) \\
\eta = \frac{b}{R} G (-2\pi / 3) \\
\tau = \frac{b}{R} G (2\pi / 3)
\]
(vi) The circulation condition

- Circulation condition:
  \[ \frac{\tau^2}{\eta} = \xi + Z_0 \]

- Input impedance:
  \[ Z_{in} = \frac{V_1}{I_1} = \xi - \frac{\eta^2}{\tau} = \frac{\tau^2}{\eta} - \frac{\eta^2}{\tau} - Z_0 \]

- Matching condition: \[ Z_{in} = Z_0 \rightarrow \frac{\tau^2}{\eta} - \frac{\eta^2}{\tau} = 2Z_0 \]
(vi) **The circulation condition**

- Since we consider a lossless system:
  
  \[ \xi = j\nu : \text{imaginary} \]
  
  \[ \tau = -\eta^* \]

\[
\bar{Z} = \begin{bmatrix}
\xi & \eta & \tau \\
\tau & \xi & \eta \\
\eta & \tau & \xi 
\end{bmatrix}
\]

\[
\left( \frac{\eta^*}{\eta} \right)^2 = j\nu + Z_0 \quad \quad \frac{(\eta^*)^2}{\eta} + \frac{\eta^2}{\eta^*} = 2Z_0
\]

- 2\textsuperscript{nd} condition is just the real part of the first: it is automatically satisfied if the 1\textsuperscript{st} condition holds.
(vi) The circulation condition

\[ \eta = |\eta| \exp(j\theta) \quad \Rightarrow \quad \begin{align*} |\eta| \cos(3\theta) &= Z_0 \\ |\eta| \sin(3\theta) &= -\nu \end{align*} \]

- Remember that

\[ \nu = -j \frac{b}{R} G(0) \quad \eta = \frac{b}{R} G(-2\pi/3) \]

\[ G(\varphi) = \frac{\omega \mu_0 \mu_\perp R}{2\pi} \sum_{n=1}^{\infty} \frac{\left(2n\mu_a / \mu\right) \sin(n\varphi)}{U_n^2 - \left(n\mu_a / \mu\right)^2} + \]

\[ j \frac{\omega \mu_0 \mu_\perp R}{2\pi} \left\{ \frac{1}{U_0} + 2 \sum_{n=1}^{\infty} \frac{U_n \cos(n\varphi)}{U_n^2 - \left(n\mu_a / \mu\right)^2} \right\} \]
(vi) The circulation condition

\[ \nu = \frac{\omega \mu_0 \mu \perp b}{2\pi} \left\{ \frac{1}{U_0} + \sum_{n=1}^{\infty} \frac{2U_n}{U_n^2 - (n\mu_a / \mu)^2} \right\} \]

\[ \eta = -\frac{\omega \mu_0 \mu \perp b}{2\pi} \sum_{n=1}^{\infty} \frac{(2n\mu_a / \mu) \sin \left(\frac{2n\pi}{3}\right)}{U_n^2 - (n\mu_a / \mu)^2} + \]

\[ j \frac{\omega \mu_0 \mu \perp b}{2\pi} \left\{ \frac{1}{U_0} + \sum_{n=1}^{\infty} \frac{2U_n \cos \left(\frac{2n\pi}{3}\right)}{U_n^2 - (n\mu_a / \mu)^2} \right\} \]

\[ U_n = \frac{(k \perp R) J_n'(k \perp R)}{J_n(k \perp R)} \]
(vii) *Basic circulator design*

- The circulation (and matching) conditions satisfied if

\[ |\eta| \cos(3\theta) = Z_0 \quad |\eta| \sin(3\theta) = -\nu \]

- A possible solution to this problem provided by

\[ \eta : \text{real} \quad \rightarrow \quad \theta = 0 \text{ or } \pi \quad \rightarrow \quad \nu = 0, \quad \eta = \pm Z_0 \]

- As a result

\[ \xi = j\nu = 0 \quad \Rightarrow \quad \bar{Z} = \begin{bmatrix} 0 & \eta & -\eta \\ -\eta & 0 & \eta \\ \eta & -\eta & 0 \end{bmatrix} \]

\[ \tau = -\eta^* = -\eta \]
(vii) Basic circulator design

- But how can we realize this situation? Consider

\[
\nu = \frac{\omega \mu_0 \mu \perp b}{2\pi} \left\{ \frac{1}{U_0} + \sum_{n=1}^{\infty} \frac{2U_n}{U_n^2 - (n\mu_a / \mu)^2} \right\}
\]

\[
\eta = -\frac{\omega \mu_0 \mu \perp b}{2\pi} \sum_{n=1}^{\infty} \frac{(2n\mu_a / \mu) \sin(2n\pi/3)}{U_n^2 - (n\mu_a / \mu)^2} +
\]

\[
j \frac{\omega \mu_0 \mu \perp b}{2\pi} \left\{ \frac{1}{U_0} + \sum_{n=1}^{\infty} \frac{2U_n \cos(2n\pi/3)}{U_n^2 - (n\mu_a / \mu)^2} \right\}
\]

\[
U_n = \frac{(k \perp R) J_n'(k \perp R)}{J_n(k \perp R)}
\]
(vii) **Basic circulator design**

- Each quantity is written as a sum over contributions by different modes \( n \). Each term itself is a function of frequency. For example \((n > 0)\): 

\[
\frac{2U_n}{U_n^2 - \left( n\mu_a / \mu \right)^2} = \frac{1}{U_n - n\mu_a / \mu} + \frac{1}{U_n + n\mu_a / \mu}
\]

\[
= \frac{1}{(k_R \, J'_n(k_R))} \frac{1}{J_n(k_R)} - n\mu_a / \mu + \frac{1}{(k_R \, J'_n(k_R))} \frac{1}{J_n(k_R)} + n\mu_a / \mu
\]

- Poles of the right and left term: resonances of the \( +n \) and \( -n \) modes, for different values of \( p \)
(vii) Basic circulator design

- We consider the case where both resonances corresponding to the left- and right turning fields exist.
- This is only possible if
  \[
  \mu_\perp > 0 \rightarrow \omega < \omega_\perp \text{ or } \omega > \omega_M + \omega_H
  \]
- We restrict ourselves to this situation.
- Since the lower range is limited in frequency, usually the second region is used. In this range, note that
  \[
  \omega > \omega_M + \omega_H \rightarrow \frac{\mu_a}{\mu} < 0
  \]
Basic circulator design

- Assume again that $\mu_a / \mu$ is small and that the two resonances are relatively close.
- Then consider one particular value of $p$. (Usually the lowest value 1).
- Let us call the corresponding resonance frequencies

$$\omega = \omega_{res}^{\pm} \rightarrow \frac{(k_{\perp} R) J'_n (k_{\perp} R)}{J_n (k_{\perp} R)} \mp n \mu_a / \mu = 0$$

- In the absence of gyrotropic term resonance occurs when

$$\omega = \omega_{res}^0 \rightarrow k_{\perp} (\omega_{res}^0) = \frac{\zeta_{np}}{R} \quad J'_n (\zeta_{np}) = 0$$
(vii) Basic circulator design

- Expansion in frequency:

\[ F_{\pm} = \frac{(k_R J_n(k_R)'}{J_n(k_R)} \mp n \frac{\mu_a}{\mu} \]

\[ F_{\pm}(\omega) \approx F_{\pm}(\omega_{res}^0) + \frac{dF_{\pm}}{d\omega}\bigg|_{\omega_{res}^0} (\omega - \omega_{res}^0) \]

\[ F_{\pm}(\omega_{res}^0) = \mp n\kappa(\omega_{res}^0) \]

\[ \frac{dF_{\pm}}{d\omega}\bigg|_{\omega_{res}^0} = \frac{n^2 - \zeta_{np}^2}{\zeta_{np}} R \frac{dk_{\perp}}{d\omega}\bigg|_{\omega_{res}^0} \mp n \frac{d\kappa}{d\omega}\bigg|_{\omega_{res}^0} \]

\[ \kappa \equiv \frac{\mu_a}{\mu} \]
(vii) Basic circulator design

- Expansion in frequency:

\[
F_\pm(\omega) \approx \mp n\kappa(\omega_{\text{res}}^0) + \left[ \frac{n^2 - \zeta_{np}^2}{\zeta_{np}} R \frac{dk_\perp}{d\omega} \right]_{\omega_{\text{res}}^0} \mp n \left[ \frac{d\kappa}{d\omega} \right]_{\omega_{\text{res}}^0} (\omega - \omega_{\text{res}}^0)
\]

- At resonance:

\[
F_+^+\left(\omega_{\text{res}}^+\right) = 0 \rightarrow \left[ \frac{n^2 - \zeta_{np}^2}{\zeta_{np}} R \frac{dk_\perp}{d\omega} \right]_{\omega_{\text{res}}^0} \left[ -n \frac{d\kappa}{d\omega} \right]_{\omega_{\text{res}}^0} = \frac{n\kappa(\omega_{\text{res}}^0)}{\left(\omega_{\text{res}}^+ - \omega_{\text{res}}^0\right)}
\]

\[
F_+^+ (\omega) \approx \frac{n\kappa(\omega_{\text{res}}^0)}{\omega_{\text{res}}^+ - \omega_{\text{res}}^0} (\omega - \omega_{\text{res}}^+)
\]
(vii) Basic circulator design

- Similarly:

\[ F_-(\omega_{res}) = 0 \rightarrow \]

\[ F_-(\omega) \approx \frac{n\kappa(\omega^0_{res})}{\omega^0_{res} - \omega_{res}}(\omega - \omega_{res}) \]

- Bear in mind that these approximations are only valid when the frequency is close to the two resonances which, are also close to each other.
(vii) Basic circulator design

- Collecting the results:

\[
\frac{2U_n}{U_n^2 - \left( n \mu_a / \mu \right)^2} = \frac{1}{F_+} + \frac{1}{F_-}
\]

\[
\approx \frac{1}{n \kappa \left( \omega_{res}^0 \right)} \left[ \frac{\omega_{res}^+ - \omega_{res}^0}{\omega - \omega_{res}^+} + \frac{\omega_{res}^0 - \omega_{res}^-}{\omega - \omega_{res}^-} \right]
\]

- Note that in the range \( \omega > \omega_M + \omega_H \) we have

\[
\omega_{res}^+ - \omega_{res}^0 > 0 \quad \text{and} \quad \omega_{res}^0 - \omega_{res}^- > 0
\]
(vii) **Basic circulator design**

- Let us plot this function as function of frequency

\[
\frac{1}{nK(\omega_{res}^0)} \left[ \frac{\omega_{res}^+ - \omega_{res}^0}{\omega - \omega_{res}^+} + \frac{\omega_{res}^0 - \omega_{res}^-}{\omega - \omega_{res}^-} \right]
\]

- Between the resonances, it becomes zero when

\[\omega = \omega_{res}^0\]
(vii) Basic circulator design

- In a similar fashion:

\[
\frac{2n\mu_a / \mu}{U_n^2 - (n\mu_a / \mu)^2} = \frac{1}{F_+} - \frac{1}{F_-}
\]

\[
\approx \frac{1}{n\kappa(\omega_{res}^0)} \left[ \frac{\omega^+_{res} - \omega^0_{res}}{\omega - \omega^+_{res}} - \frac{\omega^0_{res} - \omega^-_{res}}{\omega - \omega^-_{res}} \right]
\]

- This term remain nearly flat when

\[
\omega = \omega^0_{res} \rightarrow \frac{1}{F_+} - \frac{1}{F_-} \approx -\frac{2}{n\kappa(\omega^0_{res})}
\]
(vii) Basic circulator design

- Now consider the mode $n = 1$ as an example. This is the mode usually used for Y-junction circulator design.
- Assume that we are operating close to the (first) left- and right resonances of this mode.
- Also assume that the other resonances (other $n$’s) are far away in frequency and do not contribute significantly.

\[
\nu \approx \frac{\omega \mu_0 \mu_\perp b}{2\pi} \frac{2U_1}{U_1^2 - (\mu_a / \mu)^2}
\]

\[
\eta \approx -\frac{\omega \mu_0 \mu_\perp b}{2\pi} \frac{(2\mu_a / \mu) \sin (2\pi / 3)}{U_1^2 - (\mu_a / \mu)^2} + j \frac{\omega \mu_0 \mu_\perp b}{2\pi} \frac{2U_1 \cos (2\pi / 3)}{U_1^2 - (\mu_a / \mu)^2}
\]
(vii) **Basic circulator design**

- Now, when \( \omega = \omega_{res} \rightarrow \)

\[
\nu \approx 0, \quad \eta \approx \frac{\omega \mu_0 \mu_\parallel b}{2\pi} \frac{2}{\kappa} \sin \left(\frac{2\pi}{3}\right)
\]

- Circulation condition satisfied when

\[
\eta = \frac{\omega \mu_0 \mu_\parallel (\omega_{res})^0 b}{\pi \kappa (\omega_{res})^0} \sin \left(\frac{2\pi}{3}\right) = -Z_0
\]

- Satisfied by choosing right ferrite thickness \( (b) \), or changing material parameters in order to tune \( \mu_\parallel \) or \( \kappa \).
(vii) Basic circulator design

- Disk radius is mainly determined by operation frequency

\[ R = \frac{\zeta_{np}}{k_\perp (\omega_{res}^0)} \]

- General remark: since we considered the crossing through zero of the parameters, it is not strictly correct to neglect the terms due to other values of \( n \).

- They can be included in the calculation, but the overall picture does not change much.
(viii) Field profile

- How do fields look like at the circulation frequency?
- From the circulation condition it follows that

\[ I_1 = I_2 = I, \quad I_3 = 0 \]

- We next calculate the fields using the general expressions in terms of the (Fourier series) solution
(viii) **Field profile**

\[
A_n = \frac{j\omega \mu_0 \mu_\perp}{2\pi k_\perp} \int_{-\pi}^{\pi} \exp(jn\phi') h_\phi (\phi') d\phi' \\
\approx -\frac{j\omega \mu_0 \mu_\perp}{2\pi k_\perp} \sum_{n=1}^{\infty} \frac{\exp(j2n\pi/3)}{J'_n(k_\perp R) - n \frac{\mu_a}{\mu} \frac{J_n(k_\perp R)}{k_\perp R}}
\]

\[
e_z (r, \phi) = \sum_{n=-\infty}^{\infty} A_n J_n (k_\perp r) \exp(-jn\phi)
\]
(viii) *Field profile*

- As before, assume we are working close to the resonances of the $n = \pm 1$ modes. Neglecting other terms

\[
e_z(r, \varphi) \approx A_1 J_1(k_r r) \exp(-j\varphi) + A_{-1} J_{-1}(k_r r) \exp(j\varphi)
\]

\[
A_1 = -\frac{j\omega\mu_0\mu_\perp I}{2\pi k_\perp R} \frac{1+\exp(j2\pi/3)}{J'_1(k_\perp R) - \frac{\mu_a}{\mu} \frac{J_1(k_\perp R)}{k_\perp R}}
\]

\[
A_{-1} = -\frac{j\omega\mu_0\mu_\perp I}{2\pi k_\perp R} \frac{1+\exp(-j2\pi/3)}{J'_{-1}(k_\perp R) + \frac{\mu_a}{\mu} \frac{J_{-1}(k_\perp R)}{k_\perp R}}
\]
(viii) Field profile

\[ e_z(r, \varphi) \approx -\frac{j\omega\mu_0\mu_\perp I}{2\pi} \frac{J_1(k_\perp r)}{J_1(k_\perp R)} \times \]

\[ \left\{ \frac{[1 + \exp(j2\pi/3)] \exp(-j\varphi)}{(k_\perp R) J'_1(k_\perp R) - \frac{\mu_a}{\mu}} \frac{J_1(k_\perp R)}{J_1(k_\perp R)} + \frac{[1 + \exp(-j2\pi/3)] \exp(j\varphi)}{(k_\perp R) J'_1(k_\perp R) + \frac{\mu_a}{\mu}} \frac{J_1(k_\perp R)}{J_1(k_\perp R)} \right\} \]

\[ F_+ \approx \frac{\kappa(\omega_{res}^0)}{\omega_+ - \omega_{res}^0} (\omega - \omega_+^0) \]

\[ F_- \approx \frac{\kappa(\omega_{res}^0)}{\omega_- - \omega_{res}^0} (\omega - \omega_-^0) \]
(viii) **Field profile**

\[
e_z(r, \varphi) \approx -\frac{j \omega \mu_0 \mu_{\perp} I}{2\pi \kappa \left( \omega_{res}^0 \right)} \frac{J_1(k_{\perp} r)}{J_1(k_{\perp} R)} \times \left\{ \frac{\omega_{res}^+ - \omega_{res}^0}{\omega - \omega_{res}^+} [1 + \exp(j2\pi / 3)] \exp(-j\varphi) + \frac{\omega_{res}^0 - \omega_{res}^-}{\omega - \omega_{res}^-} [1 + \exp(-j2\pi / 3)] \exp(j\varphi) \right\}
\]

\[
\omega = \omega_{res} \rightarrow
\]

\[
e_z(r, \varphi) \approx \frac{\omega \mu_0 \mu_{\perp} I}{\pi \kappa \left( \omega_{res}^0 \right)} \frac{J_1(k_{\perp} r)}{J_1(k_{\perp} R)} \times \left[ \sin(\varphi) + \sin(\varphi - 2\pi / 3) \right]
\]
(viii) **Field profile**

\[
e_z(r, \varphi) \approx \frac{\omega \mu_0 \mu_\perp I}{\pi \kappa \left( \omega_{res}^0 \right)} \frac{J_1(k_\perp r)}{J_1(k_\perp R)} \times \left[ \sin(\varphi) + \sin\left(\varphi - \frac{2\pi}{3}\right) \right]
\]

Due to current at port 1

Due to current at port 2

\[
e_z(r, \varphi) \approx -\frac{\omega \mu_0 \mu_\perp I}{\pi \kappa \left( \omega_{res}^0 \right)} \frac{J_1(k_\perp r)}{J_1(k_\perp R)} \sin\left(\varphi - \frac{\pi}{3}\right)
\]
(viii) Field profile

\[
\begin{align*}
\mathbf{h}_r &= \frac{1}{\omega \mu_0 \mu_\perp} \left( \frac{j \partial \mathbf{e}_z}{r \partial \varphi} - \frac{\mu_a \partial \mathbf{e}_z}{\mu \partial r} \right) \\
&\approx -\frac{k_\perp I}{\pi \kappa} \frac{1}{J_1(k_\perp R)} \times \\
&\left[ j \frac{J_1(k_\perp r)}{k_\perp r} \cos (\varphi - 2\pi / 3) - \kappa J_1'(k_\perp r) \sin (\varphi - 2\pi / 3) \right]
\end{align*}
\]

\[
\begin{align*}
\mathbf{h}_\varphi &= \frac{1}{j \omega \mu_0 \mu_\perp} \left( -j \frac{\mu_a}{\mu} \frac{1}{r \partial \varphi} + \frac{\partial \mathbf{e}_z}{\partial r} \right) \\
&\approx \frac{jk_\perp I}{\pi \kappa} \frac{1}{J_1(k_\perp R)} \times \\
&\left[ -j\kappa \frac{J_1(k_\perp r)}{k_\perp r} \cos (\varphi - 2\pi / 3) + J_1'(k_\perp r) \sin (\varphi - 2\pi / 3) \right]
\end{align*}
\]
(ix) Remarks

- Note that to arrive at the final design we made a number of assumptions and approximations
  - We assumed uniformity of fields in the vertical direction
  - We adopted the magnetic wall boundary condition
  - We approximated the magnetic field of at the microstrip ports by a constant field (can be improved by using the true distribution)
  - We assumed narrow microstrip lines (not a big issue, can be improved by performing the actual integrals)
(ix) *Remarks*

- We derived the circulation condition, but just looked at a particular solution (this is restrictive, in fact some papers have shown that choosing a different solution leads to higher bandwidth)
- We choose frequencies at which $\mu_\perp > 0$ so that there are two close resonances corresponding to right- and left-turning fields (This is, in fact, not necessary: designs can be made also when $\mu_\perp < 0$, then there is just one resonance, but different n’s can be combined. The resulting designs have a much wider bandwidth.)
(ix) **Remarks**

- We just considered the mode $n=1$. Better calculations should include all modes.