بسمه تعالی

درس فرایندهای تصادفی

تمرین شماره ۵

1) Find mean and variance of the random variable

$$\mathbf{n}_{\mathrm{T}} = \frac{1}{2T}\int_{-T}^{T} \boldsymbol{x}(t)dt \quad \text{where} \qquad \boldsymbol{x}(t) = 10 + \boldsymbol{v}(t)$$

2) Show that if a process is normal and the two random varaiables $x(t)$ and $x(t+\tau)$ are uncorrelated for large τ then it is also mean-ergodic.

3) Show that if $\boldsymbol{x}(t)$ is normal with $\boldsymbol{\eta}_x = 0$ and $R_x(\tau) = 0$ for $|\tau| > a$, then it is correlation-ergodic.

4) Show that the process $\boldsymbol{a}e^{j(\omega t+\boldsymbol{\varphi})}$ is not correlation-ergodic. Here a and φ are two independent random variables.

5) The process $\boldsymbol{x}(t)$ is cyclostationary with period T, means $\eta(t)$, and correlation $R(t_1, t_2)$.

Show that $R(t+\tau, t) \to \eta^2(t)$ as $|\tau| \to \infty$, then

$$\lim_{c\to\infty} \frac{1}{2c}\int_{-c}^{c} \boldsymbol{x}(t)dt = \frac{1}{T}\int_{0}^{T} \eta(t)dt$$

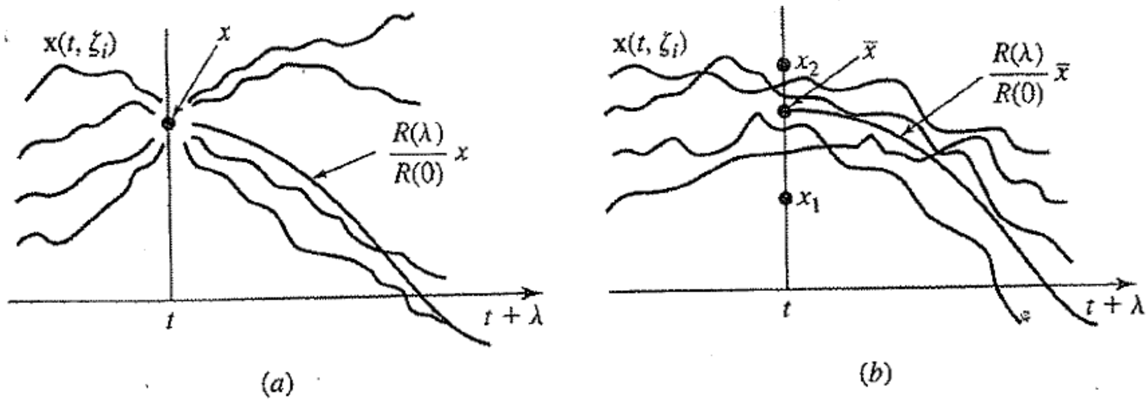
Hint: The process $\overline{\boldsymbol{x}}(t) = \boldsymbol{x}(t-\boldsymbol{\theta})$ is mean-ergodic.

6) Show that if

$$C(t+\tau, t) \xrightarrow[t\to\infty]{} 0$$

uniformly in t; then $\boldsymbol{x}(t)$ is mean-ergodic.

7) The process $\boldsymbol{x}(t)$ is normal with 0 mean and WSS. (a) Show that (Fig. a)



$$E\{\boldsymbol{x}(t+\lambda)|\boldsymbol{x}(t)=x\}=\frac{R(\lambda)}{R(0)}x$$

b) Show that if D is an arbitrary set of real numbers x_i and $\bar{x}=E\{\boldsymbol{x}(t)|\boldsymbol{x}(t)\epsilon D\}$, then (fig. b)

$$E\{\boldsymbol{x}(t+\lambda)|\boldsymbol{x}(t)\epsilon D\}=\frac{R(\lambda)}{R(0)}\bar{x}$$

c) Using this, design an analog correlometer for normal processes.

8) The process $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are jointly normal with zero mean. Show that: (a) if $\boldsymbol{w}(t)=\boldsymbol{x}(t+\lambda)\boldsymbol{y}(t)$ then

$$C_{ww}(\tau)=C_{xy}(\lambda+\tau)C_{xy}(\lambda-\tau)+C_{xx}(\tau)C_{yy}(\tau)$$

(b) If the functions $C_{xx}(\tau)$, $C_{yy}(\tau)$, and $C_{xy}(\tau)$ tend to 0 as $\tau\to\infty$ then the processes $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are cross-variance ergodic.

9) We wish to estimate the mean η of process $\boldsymbol{x}(t)=\eta+\boldsymbol{v}(t)$, where $R_{vv}(\tau)=5\delta(\tau)$.

a) Find the 0.95 confidence interval of η. (b) Improve the estimate if $\boldsymbol{v}(t)$ is a normal process.

10) (a) Show that if we use as estimate of the power spectrum $S(w)$ of a discrete-time process $\boldsymbol{x}[n]$ the function

$$S_w(w)=\sum_{m=-N}^{N}w_m R[m]e^{-jmwT}$$

Then

$$S_w(w)=\frac{1}{2\sigma}\int_{-\sigma}^{\sigma}S(y)W(w-y)dy \qquad W(w)=\sum_{-N}^{N}w_n e^{-jnwT}$$

(b) Find $W(w)$ if $N = 10$ and $w_n = 1 - {|n|}/_{11}$.

11) Show that if $\boldsymbol{x}(t)$ is zero-mean normal process with sample spectrum

$$\boldsymbol{S}_T(w) = \frac{1}{2T}\left|\int_{-T}^{T} \boldsymbol{x}(t)e^{-jwt}dt\right|^2$$

and $S(\omega)$ is sufficiently smooth, then

$$E^2\{\boldsymbol{S}_T(w)\} \leq Var\boldsymbol{S}_T(w) \leq 2E^2\{\boldsymbol{S}_T(w)\}$$

The right side is an equality if $\omega = 0$. The left side is an approximate equality if $T \gg {1}/_{\omega}$.

12) Given a normal process $\boldsymbol{x}(t)$ with zero mean and power spectrum $S(\omega)$, we form its sample autocorrelation $R_{\boldsymbol{T}}(\tau)$. Show that for large T,

$$Var\ R_{\boldsymbol{T}}(\tau) \cong \frac{1}{4\pi T}\int_{-\infty}^{\infty}(1 + e^{j2\lambda\omega})S^2(\omega)d\omega$$

13) Show that if

$$\boldsymbol{R}_{\boldsymbol{T}}(\tau) = \frac{1}{2T}\int_{-T+{|\tau|}/_2}^{T-{|\tau|}/_2} \boldsymbol{x}\left(t+\frac{\tau}{2}\right)\boldsymbol{x}\left(t-\frac{\tau}{2}\right)dt$$

is the estimate of $R(\tau)$ of a zero-mean normal process, then

$$\sigma_{R_T}^2 = \frac{1}{2T}\int_{-2T+|\tau|}^{2T+|\tau|}[R^2(\alpha) + R(\alpha+\tau)R(\alpha-\tau)]\left(1 - \frac{|\tau| + |\alpha|}{2T}\right)d\alpha$$