Consider a transmission line with length $x$ excited with an arbitrary input $v_s$ through a source impedance $Z_s$ and terminated with a load impedance $Z_l$, as shown in Fig.1. A general transmission line can be considered as a distributed $zy$ (in contrast with distributed $r\ell c$) where $z(s)$ is the impedance per unit length and $y(s)$ is the admittance per unit length of the line as shown in Fig. 2.

1. Show that the transfer function of a T-line can be written as:

$$H(s) = \frac{v_o(s)}{v_i\pi(s)} = k_1 \left(1 + k_3\right) \frac{e^{-\gamma x}}{1 - k_2 k_3 e^{-2\gamma x}}$$

where

$$k_1 = \frac{Z_0}{Z_s + Z_0} \; ; \; k_2 = \frac{Z_s - Z_0}{Z_s + Z_0} \; ; \; k_3 = \frac{Z_l - Z_0}{Z_l + Z_0} \; ; \; \gamma = \sqrt{2y} \; ; \; Z_0 = \sqrt{z/y}$$

2. As we discussed in the course, normalizing a line ($z(s) = r + sl, y(s) = sc$) to a line with unity time-of-flight ($T_F$) and characteristic impedance makes all equations shorter and comparisons easier. Suppose an $r\ell c$ line with length $x$ is driven by a source with an impedance of $R_s + sL_s$ and is terminated by a load capacitance $C_l$. The concepts that are used for the normalization are as follows:

i. An $r\ell c$ line with length $x$, can be replaced by $(xr)(xl)(xc)$ line with unity length

ii. Multiplication of all impedances of any linear circuit by a certain value does not change the voltage-voltage transfer function of the system; therefore, all impedances can be multiplied by the reciprocal of characteristic impedance of the line, $Z_0$.

iii. Finally, time-frequency duality implies that shrinking time by $T_F$ is the same as multiplying all frequencies by the same factor ($T_F$).

Show that, by using these three concepts, an $r\ell c$ line with length $x$ can be replaced by and $r\ell c$ line with unity length. The following relations between timing in the original and normalized cases, however, should be considered (all normalized parameters are marked with a tilde sign):

$$\tilde{t} = t/T_F \; ; \; \tilde{f} = fT_F$$

Other parameters should be also normalized as:

$$\tilde{R}_s = R_s/Z_0 \; ; \; \tilde{L}_s = L_s/T_F Z_0 \; ; \; \tilde{C}_l = C_l Z_0/T_F \; ; \; \tilde{r} = r x/Z_0$$
3. Considering skin effect as \( z(s) = r + r' \sqrt{s} + sl \), show that normalized \( \tilde{r}' \) can be written as

\[
\tilde{r}' = r' x /(Z_0 \sqrt{T_F})
\]

4. For n-tier design, 2 adjacent levels with orthogonal metal levels are called pair, and a tier is a collection of pairs with the same pitch \( p \). For an IC designed in 7.5nm Technology based on [1], fill out Table 1. For \( C_i \) consider it can be anything from 5 times \( C_0 \) up to 20 times \( C_0 \) (as given in Table I of [1], \( C_0 = 3.45 \alpha F \))

<table>
<thead>
<tr>
<th>Pair</th>
<th>Pitch[nm]</th>
<th>( L_{\text{max}} )[GP]</th>
<th>( \tilde{r}'_{\text{min}} )</th>
<th>( \tilde{r}'_{\text{max}} )</th>
<th>( \tilde{r}'_\text{max} )</th>
<th>( \tilde{C}_i ),min</th>
<th>( \tilde{C}_i ),MAX</th>
</tr>
</thead>
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<tr>
<td>Pair1</td>
<td>15</td>
<td>42</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>15</td>
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<td></td>
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</tr>
<tr>
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</tr>
</tbody>
</table>

5. Consider an \( rlc \) line (with no skin effect) is driving with an ideal source \( (Z_x = 0) \) and is open ended \( (Z_I = \infty) \). Rewrite the transfer function with the normalized parameters and plot the root locus of the normalized TF for different values of \( \tilde{r} \). What is the minimum value for \( \tilde{r} \) such that the T-line acts like an \( rlc \) line (with real roots)? How can this be considered as the minimum length in the original line?

6. **(Extra credit)** Redo 5, now considering skin effect \( (\tilde{r}' \neq 0) \). You may plot the root locus for changes in \( \tilde{r}' \) for different values for \( \tilde{r} \). Use data in part 4 for choosing reasonable range for \( \tilde{r} \) and \( \tilde{r}' \).

7. Insert optimal buffer insertion for longest line in [1]:
   A. Considering Bakoglu buffer insertion
   B. Considering Cascaded buffer insertion

8. **(Extra credit)** Use SPICE model (taken from PTM website for 7.5nm [2]) compare your delay calculation with HSPICE results for 7.B. Sweep over the optimal distance between buffers and find \( x_{opt} \) based on HSPICE results and compare it with 7.B.

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