IDA-PBC of Mechanical Systems

Problem Formulation

- Model
$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u$$
where $H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q)$, $rank(G) = m < n$.

- Desired energy is parameterized
  - $H_d(q,p) = \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q)$, $M_d(q) = M_d^\top(q) > 0$
  - $q_\star = \arg\min V_d(q)$.

- Desired interconnection and damping matrices

$$J_d(q,p) = \begin{bmatrix} 0 & M^{-1}(q) M_d(q) \\ -M_d(q) M^{-1}(q) & J_2(q,p) \end{bmatrix} = -J_d^\top(q,p)$$

$$\mathcal{R}_d(q) = \begin{bmatrix} 0 & 0 \\ 0 & G(q) K_v G^\top(q) \end{bmatrix} \geq 0, \quad K_v > 0$$

Proposition

Assume there is $M_d(q) = M_d^\top(q) \in \mathbb{R}^{n\times n}$ and a function $V_d(q)$ that satisfy the PDEs

$$
\begin{aligned}
G^\perp \left\{\nabla_q(p^\top M^{-1}p) - M_d M^{-1}\nabla_q(p^\top M_d^{-1}p) + 2J_2 M_d^{-1}p\right\} &= 0 \\
G^\perp\{\nabla V - M_d M^{-1}\nabla V_d\} &= 0,
\end{aligned}
$$

for some $J_2(q,p) = -J_2^\top(q,p) \in \mathbb{R}^{n\times n}$ and a full rank left annihilator $G^\perp(q) \in \mathbb{R}^{(n-m)\times n}$ of G, i.e., $G^\perp G = 0$ and $\mathtt{rank}(G^\perp) = n-m$. Then, the system in closed–loop with

$$
u = (G^\top G)^{-1}G^\top(\nabla_q H - M_d M^{-1}\nabla_q H_d + J_2 M_d^{-1}p) - K_v G^\top \nabla_p H_d,
$$

takes the desired Hamiltonian form. Further, if $M_d > 0$ (in a neighborhood of $q^\star$) and

$$
q^\star = \arg\min V_d(q),
$$

then $(q^\star, 0)$ is a stable equilibrium point with Lyapunov function H_d.

Proof

$$
\begin{aligned}
& \begin{bmatrix} G^{\perp} \\ G^{\top} \end{bmatrix} \dot{p} = \\
= & \begin{bmatrix} G^{\perp} \\ G^{\top} \end{bmatrix} (-\nabla_q H + Gu) \\
= & \begin{bmatrix} G^{\perp} \\ G^{\top} \end{bmatrix} (-\frac{1}{2}\nabla_q (p^{\top} M^{-1} p) - \nabla V + Gu) \\
\equiv & \begin{bmatrix} G^{\perp} \\ G^{\top} \end{bmatrix} (-M_d M^{-1} \nabla_q H_d + (J_2 - G K_v G^{\top}) \nabla_p H_d) \\
= & \begin{bmatrix} G^{\perp} \\ G^{\top} \end{bmatrix} (-M_d M^{-1} [\frac{1}{2}\nabla_q (p^{\top} M_d^{-1} p) + \nabla V_d] + (J_2 - G K_v G^{\top}) M_d^{-1} p.
\end{aligned}
$$

To prove stability: H_d is positive definite and

$$
\dot{H}_d \quad \leq \quad -\lambda_{min}\{K_v\}|G^{\top} M_d^{-1} p|^2 \leq 0.
$$

Connection with Controlled Lagrangians

- PDE's (with $J_2(q,p) = \frac{1}{2}\sum_{k=1}^{n} U_k(q)p_k, \quad U_k = -U_k^\top$)

$$
\begin{aligned}
G^\perp\{\frac{\partial^\top}{\partial q}(M^{-1}_{(\cdot,k)}) - M_d M^{-1}\frac{\partial^\top}{\partial q}(M_d^{-1})_{(\cdot,k)} + U_k M_d^{-1}\} &= 0 \\
G^\perp\{\frac{\partial V}{\partial q} - M_d M^{-1}\frac{\partial V_d}{\partial q}\} &= 0
\end{aligned}
$$

- If $J_2(q,p) = M_d M^{-1}\left\{[\nabla_q(MM_d^{-1}p)]^\top - \nabla_q(MM_d^{-1}p)\right\} M^{-1}M_d$, we recover the controlled–Lagrangian method

- All matrices that "preserve mechanical structure" (arbitrary $Q(q)$)

$$J_2(q,p) = \text{"}J_2\text{ above"} + M_d M^{-1}\left[[\nabla_q Q]^\top - \nabla_q Q\right] M^{-1} M_d$$

- Gyroscopic (intrinsic) terms are added to the Lagrangian

$$L_c(q,\dot q) = \frac{1}{2}\dot q^\top M(q) M_d^{-1}(q) M(q)\dot q + \dot q^\top Q(q) - V_d(q)$$

Constructive Solution (Co-dimension 1)

- Identification of a class of underactuation degree one mechanical systems for which the PDEs are explicitly solved.
- The KE–PDE becomes an algebraic equation and we give a set of solutions.
- Assume that the inertia matrix and the force induced by the potential energy (on the unactuated coordinate) are independent of the unactuated coordinate.
- One condition for stability—an algebraic inequality—that measures our ability to influence, through the modification of the inertia matrix, the unactuated component of the force induced by potential energy.
- Suitable parametrization of assignable energy functions—via two free functions and a gain matrix—to address transient performance and robustness issues.

Parametrization of the Kinetic Energy PDE

Assumption A.1 Underactuation degree one: $m = n - 1$.

Assumption A.2 $G^{\perp}\nabla_q(p^\top M^{-1}p) = 0$.

Then, kinetic energy PDE becomes

$$\sum_{i=1}^{n} \gamma_i(q)\frac{\partial M_d}{\partial q_i} = -[\mathcal{J}(q)\mathcal{A}^\top(q) + \mathcal{A}(q)\mathcal{J}^\top(q)],$$

where $\mathcal{J}(q)$ is free,

$$\gamma \triangleq M^{-1}M_d(G^{\perp})^\top, \quad \mathcal{A} \triangleq \left[W_1\left(G^{\perp}\right)^\top, W_2\left(G^{\perp}\right)^\top, \ldots, W_{n_o}\left(G^{\perp}\right)^\top\right]$$

with $n_0 \triangleq \frac{n}{2}(n-1)$ and $W_i = -W_i^\top$, e.g., for $n = 3$

$$W_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ W_2 \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ W_3 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solving the Kinetic Energy PDE

- The expression above characterizes all solutions to the KE-PDE.
  Assumption A.3 G is function of a single element of q, say q_r, $r \in \{1, \ldots, n\}$.
- A.3 satisfied (for partially–linearized systems) if the column of M corresponding to the unactuated coordinate depends only on q_r.
- A subset, for which KE–PDE becomes algebraic and we can find explicit solutions, is

$$\gamma_r \frac{dM_d}{dq_r} = -2\mathcal{A}\mathcal{J}^\top \Rightarrow \frac{dM_d}{dq_r} e_i \in \text{Im } \mathcal{A}$$

- Furthermore, $G^\perp \mathcal{A} = 0 \Leftrightarrow \mathcal{A} \in \text{Im } G$, suggesting $\frac{dM_d}{dq_r} e_i \in \text{Im } G$
- Proposition For all desired (locally) positive definite inertia matrices

$$M_d(q_r) = \int_{q_r^\star}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0$$

  where $\Psi = \Psi^\top$ and $M_d^0 = (M_d^0)^\top > 0$, may be arbitrarily chosen, there exists J_2 such that the kinetic energy PDE holds.

Solving the Potential Energy PDE

- Recalling PE–PDE:

$$G^{\perp}\{\nabla V - M_d M^{-1}\nabla V_d\} = 0$$

- Can be written as

$$\gamma^{\top}(q)\nabla V_d = s(q), \quad s \triangleq G^{\perp}\nabla V.$$

- Remarks concerning s:
  - For all admissible equilibria $\bar{q}$, we have $s(\bar{q}) = 0$.
  - $G^{\perp}\nabla V$ are forces that cannot be (directly) affected by the control.
- Since G, M_d depends on q_r it is reasonable:
  Assumption A.4 γ, s are functions of q_r only. $(\Leftarrow M = M(q_r))$
- A generic condition is needed to ensure that the PDE admits a well–defined solution:
  Assumption A.5 $\gamma_r(q_r^\star) \neq 0$.

Proposition

Under Assumptions A.1–A.5 and $M_d(q_r) = \int_{q_r^\star}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0$ all solutions of the PE–PDE are given by

$$V_d(q) = \int_0^{q_r} \frac{s(\mu)}{\gamma_r(\mu)}d\mu + \Phi(z(q)),$$

with $z(q) \triangleq q - \int_0^{q_r} \frac{\gamma(\mu)}{\gamma_r(\mu)}d\mu$ the characteristic of the PE–PDE, and Φ an arbitrary differentiable function.

Remarks

- Identify a set of assignable energy functions parameterized by $\{\Psi, M_d^0, \Phi\}$.
- γ_r is the element of the "coupling term", $G^\perp M^{-1} M_d$, through which we can modify the (unactuated coordinates of the) open–loop potential energy.
- For stability, since $\Phi(z)$ is arbitrary, restrictions will only be imposed on $\int \frac{s}{\gamma_r}$. Namely, that its second derivative, evaluated at $q_r^\star$, is positive.

Main Stabilization Result

Assumption A.6 $\gamma_r(q_r^\star)\frac{ds}{dq_r}(q_r^\star) > 0$ ensures $(q^\star, 0)$ is a locally stable equilibrium with Lyapunov function $H_d(q,p)$.

Assumption A.7 $|G^\top M^{-1} e_r(q_r^\star)| \neq 0$, makes it asymptotically stable.

Furthermore, if we select

$$\Phi(z(q)) = \frac{1}{2}\left[z(q) - z(q^\star)\right]^\top P\left[z(q) - z(q^\star)\right]$$

with $P = P^\top > 0$, the control law is of the form

$$u = A_1(q)PS(q - q^\star) + \begin{bmatrix} p^\top A_2(q_r)p \\ \vdots \\ p^\top A_n(q_r)p \end{bmatrix} + A_{n+1}(q_r) - K_v A_{n+2}(q_r)p$$

where $K_v = K_v^\top > 0$ is free, $S \in \mathbb{R}^{(n-1)\times n}$ is obtained removing the r–th row from the n–dimensional identity matrix.

Summarizing

- Identification of a class of mechanical systems for which the PDEs are explicitly solved.
- A.1 $m = n - 1$.
- A.2 M and V do not depend on the unactuated coordinate. The former can be enforced with Spong's partial feedback linearization.
- A.3 G and M are functions of a single element of q, say q_r.

An explicit solution of the PDE's is given by

$$M_d(q_r) = \int_{q_r^\star}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0$$

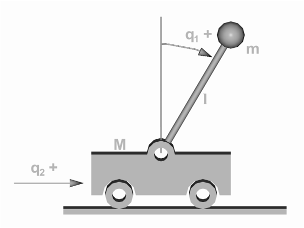
$$V_d(q) = \int_0^{q_r} \frac{G^\perp \nabla V(\mu)}{\gamma_r(\mu)} d\mu + \Phi(z(q)),$$

where

$$\gamma := M^{-1}M_d(G^\perp)^\top,\ z(q) := q - \int_0^{q_r} \frac{\gamma(\mu)}{\gamma_r(\mu)} d\mu$$

and $\Psi = \Psi^\top$, $M_d^0 = (M_d^0)^\top > 0$ and Φ may be arbitrarily chosen.

Pendulum on a Cart



Model

$$
\begin{aligned}
ml^2\ddot{q}_1 + ml\cos q_1\ddot{q}_2 - mgl\sin q_1 &= 0\\
(M+m)\ddot{q}_2 + ml\cos q_1\ddot{q}_1 - ml\sin q_1\dot{q}_1^{\,2} &= v.
\end{aligned}
$$

Can be transformed into

$$
\begin{aligned}
\dot{q} &= p\\
\dot{p} &= a\sin q_1 e_1 + \begin{bmatrix} -b\cos q_1 \\ 1 \end{bmatrix} u
\end{aligned}
$$

- Notice that $G^{\perp}(q_1) = [1,\ b\cos q_1]$.
- It can be shown that $\psi(q_1)$ cannot be a constant. Propose $\psi = -k\sin q_1$.
- Independently of $\{\Psi, M_d^0, \Phi\}$ assumptions cannot be satisfied outside $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Stability Result

The IDA–PBC

$$u = A_1(q_1)P(q_2 - q_{2*}) + p^\top A_2(q_1)p - K_v A_3(q_1)p + A_4(q_1)$$

ensures asymptotic stability of the desired equilibrium $(0, q_{2*}, 0, 0)$ with a domain of attraction containing the set $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^3$ and Lyapunov function $H_d(q, p)$ where

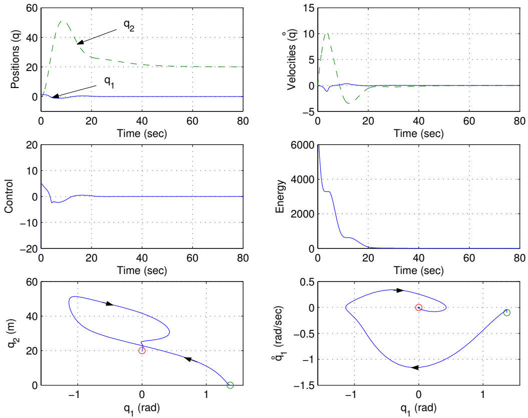
$$M_d(q_1) = \begin{bmatrix} \frac{kb^2}{3}\cos^3 q_1 & -\frac{kb}{2}\cos^2 q_1 \\ -\frac{kb}{2}\cos^2 q_1 & k(\cos q_1 - 1) + m_{22}^0 \end{bmatrix},$$

$$V_d(q) = \frac{3a}{kb^2\cos^2 q_1} + \frac{P}{2}\left[q_2 - q_{2*} + \frac{3}{b}\ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^0}{kb}\tan q_1\right]^2,$$

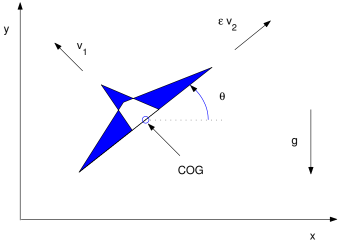
(which is radially unbounded on the set $(-\frac{\pi}{2}, \frac{\pi}{2})$.)

Simulations

- Trajectories with $[q(0), p(0)] = [\pi/2 - 0.2, -0.1, 0.1, 0]$—pendulum starting near the horizontal ♡



Strongly Coupled VTOL Aircraft



Model ($\epsilon \neq 0$, possibly large)

$$\ddot{x} = -\sin\theta v_1 + \epsilon \cos\theta v_2$$
$$\ddot{y} = \cos\theta v_1 + \epsilon \sin\theta v_2 - g$$
$$\ddot{\theta} = v_2$$

Can be transformed into

$$\dot{q} = p$$
$$\dot{p} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\epsilon}\cos\theta & \frac{1}{\epsilon}\sin\theta \end{bmatrix} u + \frac{g}{\epsilon}\sin\theta e_3$$

- Objective: Characterize assignable energy functions with $(x_*, y_*, 0, 0, 0, 0)$ asymptotically stable.

Proposition

A set of assignable energy functions is characterized by

$$M_d(q_3) = \begin{bmatrix} k_1\epsilon\cos^2 q_3 + k_3 & k_1\epsilon\cos q_3 \sin q_3 & k_1 \cos q_3 \\ k_1\epsilon\cos q_3 \sin q_3 & -k_1\epsilon\cos^2 q_3 + k_3 & k_1 \sin q_3 \\ k_1 \cos q_3 & k_1 \sin q_3 & k_2 \end{bmatrix}$$

with $k_1 > 0$ and

$$k_3 > 5k_1\epsilon, \frac{k_1}{\epsilon} > k_2 > \frac{k_1}{2\epsilon}$$

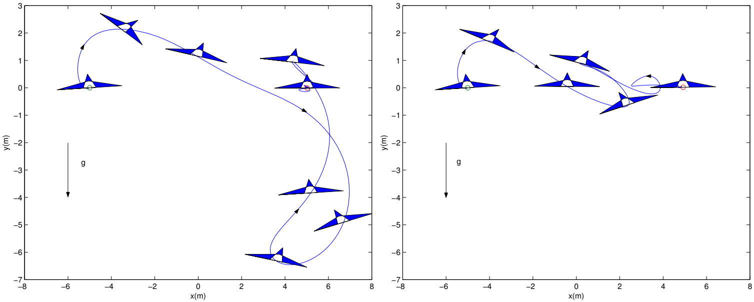
and the potential energy function

$$V_d(q) = -\frac{g}{k_1 - k_2\epsilon}\cos q_3 + \frac{1}{2}\| \begin{bmatrix} q_1 - q_{1*} - \frac{k_3}{k_1 - k_2\epsilon}\sin q_3 \\ q_2 - q_{2*} + \frac{k_3 - k_1\epsilon}{k_1 - k_2\epsilon}(\cos q_3 - 1) \end{bmatrix} \|_P.$$

Moreover, the IDA–PBC law ensures almost global asymptotic stability of the desired equilibrium $(q_{1*}, q_{2*}, 0, 0, 0, 0)$.

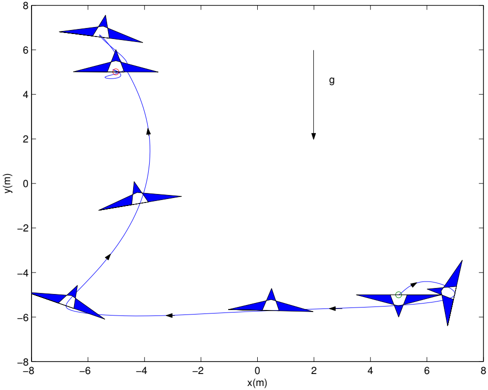
Simulations

- Effect of tuning (matrix P)



cont'd

- Upside down simulation. ♡



Simplifying the PDEs via Coordinate Changes

- The KE-PDE is nonlinear and nonhomogeneous. The presence of the forcing term introduces a quadratic term in M_d that renders very difficult its solution—even with the help of the free skew–symmetric matrix J_2.
- Perform a coordinate change $(q,p) \hookrightarrow (q,\tilde{p})$, with $p = T(q)\tilde{p}$, where $T \in \mathbb{R}^{n\times n}$ is full rank. This yields:

$$\tilde{\Sigma} : \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & T^{-\top} \\ -T^{-1} & -T^{-1}(S - S^\top)T^{-\top} \end{bmatrix} \begin{bmatrix} \nabla_q \tilde{H} \\ \nabla_{\tilde{p}} \tilde{H} \end{bmatrix} + \begin{bmatrix} 0 \\ T^{-1}G \end{bmatrix} u,$$

where $\tilde{H}(q,\tilde{p}) = \frac{1}{2}\tilde{p}^\top T^\top(q) M^{-1}(q) T(q)\tilde{p} + V(q)$, and $S(q,\tilde{p}) = \nabla_q(T(q)\tilde{p})$. • Define new target dynamics, in the coordinates $(q,\tilde{p})$, as

$$\tilde{\Sigma}_d : \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}(q)T(q)\tilde{M}_d(q) \\ -\tilde{M}_d T^\top(q)(q)M^{-1}(q) & \tilde{J}_2(q,\tilde{p}) \end{bmatrix} \begin{bmatrix} \nabla_q \tilde{H}_d \\ \nabla_{\tilde{p}} \tilde{H}_d \end{bmatrix},$$

where $\tilde{H}_d(q,\tilde{p}) = \frac{1}{2}\tilde{p}^\top \tilde{M}_d^{-1}(q)\tilde{p} + \tilde{V}_d(q)$ and $\tilde{J}_2 = -\tilde{J}_2^\top$ is free.

Obtaining an Homogeneous KE–PDE

Proposition T is such that

$$\sum_{i=1}^{n}\left[T^\top M^{-1} e_i G_k^\perp \frac{\partial T}{\partial q_i} + \frac{\partial T^\top}{\partial q_i}(e_i G_k^\perp)^\top M^{-1} T + G_k^\perp e_i T^\top \frac{\partial M^{-1}}{\partial q_i} T\right] = 0.$$

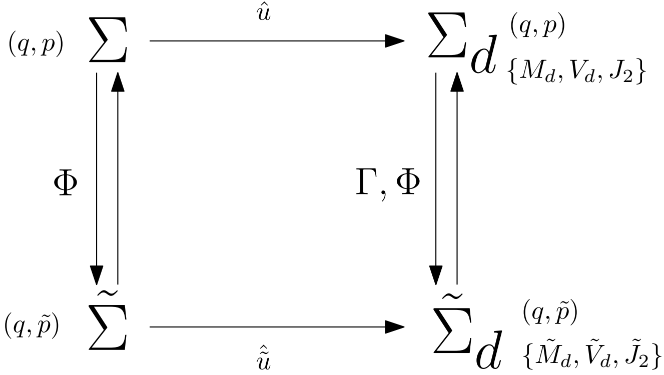
The PDEs become

$$\begin{aligned} G^\perp T\left[\tilde{M}_d T^\top M^{-1}\nabla_q(\tilde{p}^\top \tilde{M}_d^{-1}\tilde{p}) - 2\tilde{J}_2\tilde{M}_d^{-1}\tilde{p}\right] &= 0 \\ G^\perp T\tilde{M}_d T^\top M^{-1}\nabla\tilde{V}_d &= G^\perp\nabla V, \end{aligned}$$

Remarks

- $T = M$ solves the new PDE if and only if $G^\perp(q)C(q,\dot{q})\dot{q} = 0$, where $C \in \mathbb{R}^{n\times n}$ is the matrix of Coriolis and centrifugal forces of the system.
- Solving the new PDEs is, in principle, simpler: it has been possible for several practical examples, including the pendulum of Furuta ♡.
- For the Acrobot ♡ first proof of smooth stabilization with domain of attraction including the lower half plane.

Relationship Between New and Original Problem



$\Gamma : \{M_d, V_d, J_2\} \to \{\tilde{M}_d, \tilde{V}_d, \tilde{J}_2\}$ is one–to–one.

$\Phi : (q, p) \to (q, \tilde{p})$ is the coordinate transformation.

LTI (Conservative) Mechanical Systems

- IDA for LTI systems: Find $u(x)$ such that $Ax + Bu(x) \equiv F\nabla H_d$ with $H_d(x) = \frac{1}{2}x^\top P x,\ P > 0$ and $F + F^\top \leq 0$.
- (Prajna, et al., SCL'02) IDA if and only if (A, B) is stabilizable.
- IDA for mechanical systems: Given $H(q,p) = \frac{1}{2}|p|^2 + \frac{1}{2}q^\top C q$ find $u(q,p)$ such that
$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ G \end{bmatrix} u(q,p) \equiv \begin{bmatrix} 0 & M_d \\ -M_d & 0 \end{bmatrix} \nabla H_d.$$
where $H_d(q,p) = \frac{1}{2}p^\top M_d^{-1} p + \frac{1}{2}q^\top C_d q, \quad M_d > 0, C_d > 0.$
- Differences with general IDA is that H_d is separable and the structure of F is fixed.
- (Liu, et al., IJC'12), (Zenkov, MTNS'02) IDA applicable if and only if the matrix associated to the uncontrollable part of the pair $(-C, G)$ is diagonalizable and has negative real eigenvalues.
- Stabilizability is not enough.

Asymptotic Stabilization via Sign–indefinite Damping

- A motivating example

$$\ddot{q} + \sin q = u.$$

- With $u = 0$ has a stable equilibrium at zero. Can be rendered GAS with $u = -k_{di}\dot{q}$.
- Almost GAS with the sign–indefinite damping

$$u = -k_{di}(\cos q)\dot{q}.$$

- Applying the partial change of coordinates $z = \dot{q} + k_{di}\sin q$, yields

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -k_{di} & 1 \\ -1 & 0 \end{bmatrix} \nabla W(q, z),$$

$$W(q, z) = \frac{1}{2}z^2 + (1 - \cos q).$$

- The derivative yields

$$\dot{W} = -k_{di}(\sin q)^2 \leq 0.$$

Systems with Constant Inertia

Proposition (Sarras, et al., WLHM'12) Consider the system $M\ddot{q} + R(q)\dot{q} + \nabla V(q) = u$, where $q_* = \arg\min V(q)$, and the minimum is unique and isolated and $r_M \geq R(q) \geq 0$. The sign–indefinite damping injection

$$u = -k_{di}[\nabla^2 V(q)]\dot{q}, \quad M > \frac{k_{di}}{4} r_M$$

ensures $(q, \dot{q}) = (q_*, 0)$ is almost GAS.

Proof Let $z = M\dot{q} + k_{di}\nabla V(q)$. Then,

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -k_{di}M^{-1} & I_n \\ -[I_n + k_{di}R(q)M^{-1}] & -R(q) \end{bmatrix} \nabla W(q, z),$$

with $W(q, z) := \frac{1}{2} z^T M^{-1} z + V(q)$. Now, with $R(q) = T^\top(q) T(q)$, we have

$$\begin{aligned} \dot{W} &= -k_{di} \begin{bmatrix} \nabla V^\top & z^\top M^{-1} T^\top \end{bmatrix} \begin{bmatrix} M^{-1} & \frac{1}{2} M^{-1} T^\top \\ \frac{1}{2} T M^{-1} & \frac{1}{k_{di}} I_n \end{bmatrix} \begin{bmatrix} \nabla V \\ T M^{-1} z \end{bmatrix} \\ &\leq -\epsilon |\nabla V|^2. \end{aligned}$$

IDA–PBC with Generalized Gyroscopic Forces

- Classical IDA–PBC Given $H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + V(q)$, find $u(q,p)$ such that

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u(q,p)$$
$$\equiv \begin{bmatrix} 0 & M^{-1}(q)M_d(q) \\ -M_d(q)M^{-1}(q) & J_2(q,p) \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix},$$

with $H_d(q,p) = \frac{1}{2}p^\top M_d^{-1}(q)p + V_d(q)$, where $J_2(q,p) = -J_2^\top(q,p)$ is free.

- IDA–PBC: Generalized Forces (Chang, MCCA'10)

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} + \begin{bmatrix} 0 \\ C(q,p) \end{bmatrix},$$

where $p^\top M_d^{-1} C(q,p) \leq 0$, to ensure $\dot{H}_d \leq 0$, otherwise is free.

- Proposition (Crasta, et al., IJC'15) The number of PDEs to be solved in both cases is $\frac{1}{6}(n-m)(n-m+1)(n-m+2)$.