Electromagnetic scattering

Graduate Course
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Contents of lecture 4:

- Scattering from cylindrical objects
  - Introduction
  - Scalar waves in cylindrical coordinates
  - Vector wave equation
  - Vector solutions in cylindrical coordinates
  - Expansion of plane waves
  - Scattering by perfectly conducting infinitely long cylinder
  - Scattering width
  - Scattering by a dielectric cylinder
  - Finite length approximation
Introduction

- Scattering from flat, layered media is an example of an exactly solvable problem.
- This problem was solved by considering the reflection and transmission of TE and TM waves and matching the solutions at the interfaces.
- Another class of exactly solvable systems is the scattering by cylinders of infinite length.
- In case of dielectric cylinders, they are solvable for a constant dielectric constant.
Introduction

- The system has rotational symmetry but, of course, the incident plane wave is not rotationally symmetric
- Nonetheless, we can still analyze the problem using the ‘natural’ solutions of the wave equation in cylindrical coordinates
- First, consider scalar waves in a homogeneous medium

\[
\left( \nabla^2 + k^2 \right) \psi = 0
\]

\[
k^2 = \omega^2 \mu \epsilon
\]

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0
\]

In cylindrical coordinates
Scalar waves in cylindrical coordinates

- Try solutions of the type

\[ \psi(\rho, \phi, z) = f(\rho) \exp(-jm\phi - jk_z z) \]

\[ \rho^2 \frac{d^2 f(\rho)}{d\rho^2} + \rho \frac{df(\rho)}{d\rho} + \left( k^2 \rho^2 - m^2 \right) f(\rho) = 0 \]

\[ k_\rho = \sqrt{k^2 - k_z^2} \]

- General solution:

\[ f_{m,k_z}(\rho) = AJ_m(k_\rho \rho) + BY_m(k_\rho \rho) \rightarrow \]

\[ \psi_{m,k_z}(\rho, \phi, z) = \left[ AJ_m(k_\rho \rho) + BY_m(k_\rho \rho) \right] \exp(-jm\phi - jk_z z) \]
Scalar waves in cylindrical coordinates

- Alternatively, one can represent the solution in terms of Hankel functions of the 1\textsuperscript{st} and 2\textsuperscript{nd} kind

\[ H_m^{(1,2)}(k_\rho \rho) = J_m(k_\rho \rho) \pm jY_m(k_\rho \rho) \]

\[ \psi_{m,k_z}(\rho, \phi, z) = \left[ AH_m^{(1)}(k_r \rho) + BH_m^{(2)}(k_r \rho) \right] \exp(-jm\phi - jk_z z) \]

- Hankel functions preserve the ‘wave’ picture. For large radial distances they represent inward and outward moving waves

\[ \rho \rightarrow \infty : H_m^{(1,2)}(k_\rho \rho) \sim (\mp j)^m \sqrt{\frac{2}{\pi k_\rho \rho}} \exp\left[ \pm\left( jk_\rho \rho - j\pi / 4 \right) \right] \]
Vector wave equation

- These were the solutions in a homogeneous medium
- At the first sight the scattering problem for a cylinder can now be easily solved: just solve the Bessel and Hankel waves outside and inside the cylinder (if dielectric) and match them at the interface
- But there are two problems:
  - We have to deal with a vector problem and a vector wave equation: EM fields are vector fields and have a polarization.
  - Incident wave is not a ‘cylindrical’ wave, but a normal plane wave
- These facts make the analysis ‘a bit’ more complicated!
Vector wave equation

- Remember vector wave equation in a homogeneous medium

\[ \nabla \times (\nabla \times \mathbf{E}) - k^2 \mathbf{E} = -\nabla^2 \mathbf{E} + \nabla \left( \nabla \cdot \mathbf{E} \right) - k^2 \mathbf{E} = 0 \]

- Instead of trying to move to cylindrical coordinates and directly solve the equation, consider the following general theorem

- Assume that \( \psi(r) \) is a solution of the ‘scalar’ wave equation solved in any coordinate system

- Also consider a known vector field \( a(r) \), such that

\[ \nabla \times a(r) = 0 \]
**Vector wave equation**

- Now, let us introduce the new vector fields

\[
M(r) = \frac{1}{k} \nabla \times [\psi(r) \vec{a}(r)] \quad \quad N(r) = \frac{1}{k} \nabla \times M(r)
\]

- It can then be shown that the 1st vector field is a solution of the vector wave equation if

\[
\nabla \times [\nabla \psi \nabla \cdot \vec{a} - 2(\nabla \psi \cdot \nabla) \vec{a}] = 0
\]

- For the derivation use

\[
\nabla \times \nabla \times (\psi \vec{a}) = \nabla \times (\nabla \psi \times \vec{a})
\]

\[
= \nabla \psi \nabla \cdot \vec{a} - \vec{a} \nabla^2 \psi + (\vec{a} \cdot \nabla) \nabla \psi - (\nabla \psi \cdot \nabla) \vec{a}
\]

\[
= \nabla \psi \nabla \cdot \vec{a} - \vec{a} \nabla^2 \psi + \nabla (\vec{a} \cdot \nabla \psi) - 2(\nabla \psi \cdot \nabla) \vec{a}
\]
Vector wave equation

- The 2\textsuperscript{nd} vector field is then automatically a solution as well.
- These two vector fields are linearly independent from each other.
- They both have zero divergence.
- Now, every solution of the vector wave equation in a homogeneous medium can be written as a combination of these vectors, generated by different solutions of the scalar wave equation.
Vector wave equation

- Example: choose the scalar function to be a solution of the wave equation (Helmholtz equation) in the Cartesian system
- This would be a simple scalar plane wave
  \[
  \left( \nabla^2 + k^2 \right) \varphi = 0 \rightarrow \varphi = \exp(-jk \cdot r)
  \]
- Take, as an example, a constant \(a(r)\), i.e., \(a(r) = \hat{x}\)
  \[
  M(r) = -jk \times \hat{x} \exp(-jk \cdot r)
  \]
  \[
  N(r) = -\hat{k} \times (\hat{k} \times \hat{x}) \exp(-jk \cdot r)
  \]
- These correspond to two independent polarizations
Vector waves in cylindrical coordinates

- For cylindrical objects, a proper choice is to use the constant vector field

\[ a(r) = \hat{z} \]

- This field satisfies all the requirements and results in

\[
M = \frac{1}{k} \nabla \times [\psi \hat{z}] = \frac{1}{k} \nabla \psi \times \hat{z} \\
N = \frac{1}{k^2} \nabla \times [\nabla \psi \times \hat{z}] = \frac{1}{k^2} \left[ -\nabla^2 \psi \hat{z} + \frac{\partial}{\partial z} \nabla \psi \right] = \frac{1}{k^2} \left[ k^2 \psi \hat{z} + \frac{\partial}{\partial z} \nabla \psi \right]
\]
Vector waves in cylindrical coordinates

- We next use cylindrical coordinates

\[
\nabla \psi_{m,kz} = \hat{\rho} \frac{\partial \psi_{m,kz}}{\partial \rho} + \hat{\phi} \frac{\partial \psi_{m,kz}}{\rho \partial \phi} + \hat{z} \frac{\partial \psi_{m,kz}}{\partial z}
\]

\[
M_{m,kz} = \frac{1}{k} \nabla \psi_{m,kz} \times \hat{z} = -\hat{\phi} \frac{\partial \psi_{m,kz}}{k \partial \rho} + \hat{\rho} \frac{\partial \psi_{m,kz}}{k \rho \partial \phi}
\]

\[
= -\hat{\rho} \frac{jm}{k \rho} \psi_{m,kz} - \hat{\phi} \frac{\partial \psi_{m,kz}}{k \partial \rho}
\]

\[
N_{m,kz} = \frac{1}{k^2} \left[ -\hat{\rho} j k_z \frac{\partial \psi_{m,kz}}{\partial \rho} - \hat{\phi} \frac{mk_z}{\rho} \psi_{m,kz} + \hat{z} k^2 \psi_{m,kz} \right]
\]
Vector waves in cylindrical coordinates

- Let us see what kind of fields they represent. 1\textsuperscript{st} solution:

\[
E = M_{m,k_z} = \exp(-jm\phi - jk_z z) \left[ -\hat{\rho} \frac{j}{k\rho} f_{m,k_z}(\rho) - \hat{\phi} \frac{df_{m,k_z}(\rho)}{kd\rho} \right]
\]

- This field has no vertical component (TE$^z$ wave). It has an elliptic polarization in the horizontal plane.

- Its corresponding magnetic field is

\[
H = -\frac{1}{j\omega\mu} \nabla \times E = \frac{j}{\eta} N_{m,k_z} = \frac{j}{\eta} \exp(-jm\phi - jk_z z)
\]

\[
\left[ -\hat{\rho} \frac{j}{k^2} \frac{df_{m,k_z}(\rho)}{d\rho} - \hat{\phi} \frac{k}{k^2} \frac{m}{\rho} f_{m,k_z}(\rho) + \hat{z} \frac{k^2}{k^2} f_{m,k_z}(\rho) \right]
\]
Vector waves in cylindrical coordinates

- The 2nd solution for the electric field is:

\[ E = N_{m,k_z} = \exp(-j m \phi - j k_z z) \]

\[ \left[ -\hat{\rho} \frac{j k_z}{k^2} \frac{df_{m,k_z}(\rho)}{d\rho} - \hat{\phi} \frac{k_z}{k^2} \frac{m}{\rho} f_{m,k_z}(\rho) + \hat{z} \frac{k^2}{k^2} f_{m,k_z}(\rho) \right] \]

- Here, the electric field does have a vertical component.

- It again has elliptic polarization, but in a plane rotated around the radial unit vector.
Vector waves in cylindrical coordinates

- The corresponding magnetic field is

\[
H = -\frac{1}{j\omega\mu} \nabla \times E = -\frac{1}{j\omega\mu k} \nabla \times \left( \nabla \times M_{m,k_z} \right) = \frac{j}{\eta} M_{m,k_z}
\]

\[
= \frac{j}{\eta} \exp(-jm\phi - jk_zz) \left[ -\hat{\rho} \frac{jm}{k\rho} f_{m,k_z}(\rho) - \hat{\phi} \frac{df_{m,k_z}(\rho)}{kd\rho} \right]
\]

- Now the magnetic field lies in the horizontal plane. It has no vertical component (TM\textsuperscript{z} wave)

- Note also that in both cases

\[
M_{m,k_z} \cdot N_{m,k_z} = 0 \rightarrow E \cdot H = 0
\]
Vector waves in cylindrical coordinates

- Example: \( m = 0 \) corresponds to the cylindrically symmetric fields

\[
M_{0,k_z} = -\hat{\phi} \exp(-jk_z z) \frac{df_{m,k_z}(\rho)}{kd\rho}
\]

\[
N_{0,k_z} = \exp(-jk_z z) \left[ -\hat{\rho} \frac{j k_z}{k^2} \frac{df_{m,k_z}(\rho)}{d\rho} + \hat{z} \frac{k^2}{k^2} f_{m,k_z}(\rho) \right]
\]

- Example: \( k_z = 0 \) corresponds to solutions uniform along \( z \)

\[
M_{m,0} = \exp(-jm\phi) \left[ -\hat{\rho} \frac{jm}{k\rho} f_{m,0}(\rho) - \hat{\phi} \frac{df_{m,0}(\rho)}{kd\rho} \right]
\]

\[
N_{m,0} = \hat{z} \exp(-jm\phi) f_{m,0}(\rho)
\]

\( k_{\rho} = k \)
Vector waves in cylindrical coordinates

- For later use, let us consider the vector functions when

\[ f_{m, k_z}(r) = H_m^{(2)} \left( k_\rho \rho \right) \]

- This choice is useful for representing scattered wave since it satisfies the radiation condition.

- Consider the functions at a large distance from the center. Asymptotic relation for the Hankel function

\[ k_\rho \rho \gg 1 \rightarrow H_m^{(2)}(k_\rho \rho) \sim j^m \sqrt{\frac{2}{\pi k_\rho \rho}} \exp \left( -j k_\rho \rho + \frac{j \pi}{4} \right) \]
Vector waves in cylindrical coordinates

- The TE solution has the asymptotic behavior: \( k_\rho \rho \gg 1 \rightarrow \)

\[
E = M_{m, k_z} \sim \hat{\phi} \frac{j^m k_\rho}{k} C_0 \frac{\exp(-jk_\rho \rho - jm\phi - jk_z z)}{\sqrt{k_\rho \rho}}
\]

\[
H = \frac{j}{\eta} N_{0, k_z} \sim \left[ -\frac{k_z}{k} \hat{\rho} + \frac{k_\rho}{k} \hat{z} \right] \left( \frac{j^m k_\rho}{\eta k} C_0 \right) \frac{\exp(-jk_\rho \rho - jm\phi - jk_z z)}{\sqrt{k_\rho \rho}}
\]

\[
C_0 = j \sqrt{\frac{2}{\pi}} \exp\left( \frac{j\pi}{4} \right)
\]
Vector waves in cylindrical coordinates

- At large distance these are waves with a conical wave front propagating with the wave vector $k_\rho \hat{\rho} + k_z \hat{z}$ for every angle $\phi$. They all make an angle $\sin \theta = k_z / k$ with the z-axis.

- They behave as TEM waves: fields are perpendicular to each other and to the wave vector in all directions.
Vector waves in cylindrical coordinates

- Magnetic field has no $\phi$ component, electric field has only a $\phi$ component

- Again the electric and magnetic fields are orthogonal

- At a large distance these wave behave as TEM waves

- *Remember*: the above analysis of the asymptotic behavior is only valid for solutions based on Hankel function of the 2$^{nd}$ kind
Vector waves in cylindrical coordinates

- The behavior of the TM solution at large distance is similar to TE but now the electric field has no $\phi$ component and magnetic field only has a $\phi$ component

$$E = \mathbf{N}_{0,k_z} \sim (-j) \left[ -\frac{k_z}{k} \hat{\rho} + \frac{k}{k} \hat{z} \right] \left( \frac{j^m k_{\rho}}{k} C_0 \right) \frac{\exp(-jk_{\rho} \rho - jm\phi - jk_z z)}{\sqrt{k_{\rho} \rho}}$$

$$H = \frac{j}{\eta} \mathbf{M}_{m,k_z} \sim \frac{j}{\eta} \hat{\phi} \left( \frac{j^m k_{\rho}}{k} C_0 \right) \frac{\exp(-jk_{\rho} \rho - jm\phi - jk_z z)}{\sqrt{k_{\rho} \rho}}$$