So far we have analyzed the ‘natural’ solutions to the vector wave equation in cylindrical coordinates. We expect them to be useful when treating scattering by cylindrically symmetric objects.

We have seen that asymptotically, these solutions behave as TEM waves at large distance.

But the actual problem we are interested is not the scattering of these waves, but the scattering of a simple plane wave such as

\[ E_i(r) = E_i^0 \exp(-j \mathbf{k_i} \cdot \mathbf{r}) \]

\[ |\mathbf{k_i}| = k \]
Expansion of a plane wave

- To be able to use the cylindrical vector solutions, we first have to expand the plane wave into these functions.

- Let us write

\[
\mathbf{k}_i = (k_{i,x}, k_{i,y}, k_{i,z}) = (k_{i,\rho} \cos \phi_i, k_{i,\rho} \sin \phi_i, k_{i,z})
\]

\[
k_{i,\rho}^2 + k_{i,z}^2 = k^2
\]
Expansion of a plane wave

- Remember that
  \[ r = (x, y, z) = (\rho \cos \phi, \rho \sin \phi, z) \]

- We next use the following relationship from the theory of Bessel functions
  \[ \exp(-jz \cos \theta) = \sum_{m=-\infty}^{\infty} (-j)^m J_m(z) \exp(-jm \theta) \]

- \[ \exp(-jk_i \cdot r) = \exp(-jk_{i, z}z) \sum_{m=-\infty}^{\infty} (-j)^m J_m(k_{i, \rho} \rho) \exp[-jm(\phi - \phi_i)] \]
Expansion of a plane wave

- Remember that

\[ \psi_{m,k_{i,z}}^J (\rho, \phi, z) = J_m (k_{i,\rho} \rho) \exp (-jm\phi - jk_{i,z} z) \]

- is one solution of the scalar wave equation with the Bessel function of the first kind

\[ \exp (-j k_i \cdot r) = \sum_{m=-\infty}^{\infty} (-j)^m \psi_{m,k_{i,z}}^J (\rho, \phi, z) \exp (jm\phi_i) \]

- This is the expansion of a scalar plane wave for an incident wave with \( k_{i,z}, \phi_i \). What about the vector plane wave?
Expansion of a plane wave

- 1st result:

\[
\frac{1}{k} \nabla \times \left[ \hat{z} \exp(-jk_i \cdot r) \right] = \sum_{m=-\infty}^{\infty} (-j)^m M^J_{m,k,i,z}(\rho, \phi, z) \exp(jm\phi)
\]

\[
M^J_{m,k,i,z} = \frac{1}{k} \nabla \times \left( \psi^J_{m,k,i,z} \hat{z} \right) = -\hat{\rho} \frac{jm}{k\rho} \psi^J_{m,k,i,z} - \hat{\phi} \frac{\partial \psi^J_{m,k,i,z}}{k\partial \rho}
\]

\[
= \left[ -\hat{\rho} \frac{jm}{k\rho} J_m(k_{i,\rho} \rho) - \hat{\phi} \frac{k_{i,\rho}}{k} J'_m(k_{i,\rho} \rho) \right] \exp(-jm\phi - jk_{i,z} z)
\]

\[
J'_m(z) \equiv \frac{dJ_m(z)}{dz}
\]
Expansion of a plane wave

- The first vector relation now follows:

\[
(\hat{z} \times \hat{k}_i) \exp(-jk_i \cdot r) = -j \sum_{m=-\infty}^{\infty} (-j)^m M^J_{m,k_i,z}(\rho, \phi, z) \exp(jm\phi_i)
\]

- For the 2\textsuperscript{nd} vector relation take the curl of the above equation

\[
\frac{1}{k} \nabla \times \left[ (\hat{z} \times \hat{k}_i) \exp(-jk_i \cdot r) \right] = -j \sum_{m=-\infty}^{\infty} (-j)^m N^J_{m,k_i,z}(\rho, \phi, z) \exp(jm\phi_i)
\]

\[
N^J_{m,k_i,z} = \frac{1}{k} \nabla \times M^J_{m,k_i,z}
\]
Expansion of a plane wave

- Explicit form of $N$:

$$N_{m,k_i,z}^J = \exp(-jm \phi - jk_{i,z} z)$$

$$\left[-\hat{\rho} \frac{jk_{i,z}k_i,\rho}{k^2} J_m'(k_{i,\rho},\rho) - \hat{\phi} \frac{mk_{i,z}}{k^2 \rho} J_m(k_{i,\rho},\rho) + \hat{z} \frac{k_{i,\rho}^2}{k^2} J_m(k_{i,\rho},\rho)\right]$$

- The 2nd vector relation now becomes

$$\left(\hat{z} \times \hat{k}_i\right) \times \hat{k}_i \exp(-jk_i \cdot r) = -\sum_{m=-\infty}^{\infty} (-j)^m N_{m,k_i,z}^J(\rho, \phi, z) \exp(jm \phi_i)$$
Expansion of a plane wave

- Now, we define the unit vector system

\[ \hat{h}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|} \]

\[ \hat{v}_i = \hat{h}_i \times \hat{k}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|} \times \hat{k}_i \]
Let us see what these vectors are:

\[ \mathbf{k}_i = (k_{i,x}, k_{i,y}, k_{i,z}) = (k_i, \rho \cos \phi_i, k_i, \rho \sin \phi_i, k_{i,z}) \]

\[ \hat{k}_i = \frac{1}{k}(k_i, \rho \cos \phi_i, k_i, \rho \sin \phi_i, k_{i,z}) \]

\[ \hat{h}_i = \frac{\hat{z} \times \hat{k}_i}{\mathbf{\hat{z} \times \hat{k}_i}} = (-\sin \phi_i, \cos \phi_i, 0) = \hat{\phi}_i \]

\[ \hat{\nu}_i = \hat{h}_i \times \hat{k}_i = \hat{\phi}_i \times \hat{k}_i = \frac{k_{i,z}}{k} \hat{\rho}_i - \frac{k_{i,\rho}}{k} \hat{z} \]

\[ \hat{\rho}_i = (\cos \phi_i, \sin \phi_i, 0) \]

These cylindrical unit vectors are defined with respect to \( k_i \). They have a subscript \( i \). Do not confuse them with the cylindrical unit vectors for position!
Expansion of a plane wave

- We thus have the relations

\[
\hat{h}_i \exp(-jk_i \cdot r) = \hat{\phi}_i \exp(-jk_i \cdot r) = \]
\[
- j \frac{k}{k_{i,\rho}} \sum_{m=-\infty}^{\infty} (-j)^m M^J_{m,k_i,z} (\rho, \phi, z) \exp(jm\phi_i)
\]

\[
\hat{v}_i \exp(-jk_i \cdot r) = \left( \frac{k_{i,z}}{k} \hat{\rho}_i - \frac{k_{i,\rho}}{k} \hat{z} \right) \exp(-jk_i \cdot r) =
\]
\[
- \frac{k}{k_{i,\rho}} \sum_{m=-\infty}^{\infty} (-j)^m N^J_{m,k_i,z} (\rho, \phi, z) \exp(jm\phi_i)
\]
Expansion of a plane wave

For any polarization of the incident plane wave:

\[ E_i(r) = E_i^0 \exp(-j k_i \cdot r) = \left[ (E_i^0 \cdot \hat{h}_i) \hat{h}_i + (E_i^0 \cdot \hat{v}_i) \hat{v}_i \right] \exp(-j k_i \cdot r) \]

\[ E_i(r) = \sum_{m=\infty}^{\infty} u_m M_{m,k_i,z}^J (\rho, \phi, z) + \sum_{m=\infty}^{\infty} v_m N_{m,k_i,z}^J (\rho, \phi, z) \]

\[ u_m = -j \left( E_i^0 \cdot \hat{h}_i \right) \frac{k}{k_{i,\rho}} (-j)^m \exp(j m \phi_i) \]

\[ v_m = - \left( E_i^0 \cdot \hat{v}_i \right) \frac{k}{k_{i,\rho}} (-j)^m \exp(j m \phi_i) \]

Related to TE: horizontal electric field

Related to TM: horizontal magnetic field

Scattering from cylindrical objects
Exact solution of the problem

- We now consider the scattering of a plane wave by an infinitely long, perfectly conducting cylinder.

- When the incident wave hits the cylinder, surface currents (and charges) are induced.

- These currents create the ‘scattered’ field. At any point, the total electric field is

\[ E(r) = E_i(r) + E_s(r) \]

\[ E_i(r) = E_i^0 \exp(-jk_i \cdot r) \]
Exact solution of the problem

- We saw how the incident plane wave can be represented in terms of cylindrical vector solutions.
- The scattered field (outside the cylinder) can also be expanded in terms of those solutions.
- But: for the scattered field we should use vectors with the right condition at the infinity \( \rho \to \infty \).
Exact solution of the problem

- We should use the Hankel function of the 2nd kind for these waves which satisfy the radiation condition (behave as outgoing waves at infinity)

- Also, since the system is uniform in z-direction, $k_z$ is the same as that of the incident wave: $k_z = k_{i,z}$. Of course the wave will be scattered along different angles (different $\phi$’s)

$$E_s(r) = \sum_{m=-\infty}^{\infty} a_m M_{m,k_{i,z}}^H(\rho, \phi, z) + \sum_{m=-\infty}^{\infty} b_m N_{m,k_{i,z}}^H(\rho, \phi, z)$$

$$M_{m,k_{i,z}}^H = \frac{1}{k} \nabla \times \left[ \psi_{m,k_{i,z}}^H \hat{z} \right] \quad N_{m,k_{i,z}}^H = \frac{1}{k} \nabla \times M_{m,k_{i,z}}^H$$
Exact solution of the problem

- The vector functions are explicitly given by

\[
M_{m,k_i,z}^H = \exp\left(-jm\phi - jk_{i,z}z\right)
\]
\[
\begin{bmatrix}
-\hat{\rho} \frac{jm}{k\rho} H_m^{(2)}(k_{i,\rho}\rho) - \hat{\phi} \frac{k_{i,\rho}}{k} H_m^{(2)'}(k_{i,\rho}\rho)
\end{bmatrix}
\]

\[
N_{m,k_i,z}^H (r) = \exp\left(-jm\phi - jk_{i,z}z\right)
\]
\[
\begin{bmatrix}
-\hat{\rho} \frac{j k_{i,z}k_{i,\rho}}{k^2} H_m^{(2)'}(k_{i,\rho}\rho) - \hat{\phi} \frac{mk_{i,z}}{k^2\rho} H_m^{(2)}(k_{i,\rho}\rho) + \hat{\mathbf{z}} \frac{k_{i,\rho}^2}{k^2} H_m^{(2)}(k_{i,\rho}\rho)
\end{bmatrix}
\]

- Note that we have used

\[
k_{\rho} = \sqrt{k^2 - k_{z}^2} = \sqrt{k^2 - k_{i,z}^2} = k_{i,\rho}
\]
Exact solution of the problem

- The next step is to choose the coefficients of the expansion such that the condition at the surface of the cylinder is satisfied.
- Zero tangential component of total electric field

\[ E_z(R, \phi, z) = E_\phi(R, \phi, z) = 0 \]

- 1st Result:

\[ b_m = -v_m \frac{J_m(k_{i,\rho}R)}{H^{(2)}_m(k_{i,\rho}R)} \]

\[ = \left( E_i^0 \cdot \hat{v}_i \right) \frac{k}{k_{i,\rho}} \frac{J_m(k_{i,\rho}R)}{H^{(2)}_m(k_{i,\rho}R)} (-j)^m \exp(jm\phi_i) \]
Exact solution of the problem

2\textsuperscript{nd} result:

\[ a_m = -u_m \frac{J'_m(k_{i,\rho} R)}{\left[H_{m}^{(2)}\right]'(k_{i,\rho} R)} \]

\[ = j \left( E_i^0 \cdot \hat{n}_i \right) \frac{k}{k_{i,\rho}} \frac{J'_m(k_{i,\rho} R)}{H_{m}^{(2)'}}(k_{i,\rho} R)(-j)^m \exp(jm\phi_i) \]

Note that a TE incident wave excites TE polarized waves (horizontal polarization), and a TM wave excites TM polarized waves (vertical polarization)
Exact solution of the problem (TE polarization)

- TE scattering (horizontal electric field):

\[
E_{s}^{TE}(r) = \sum_{m=-\infty}^{\infty} a_{m} M_{m,k_{i},z}^{H} (\rho, \phi, z)
\]

\[
= - \sum_{m=-\infty}^{\infty} u_{m} J'_{m}(k_{i,\rho} R) M_{m,k_{i},z}^{H} (\rho, \phi, z)
\]

\[
u_{m} = - j \left( E_{i}^{0} \cdot \hat{h}_{i} \right) \frac{k}{k_{i,\rho}} (-j)^{m} \exp(j m \phi_{i})
\]
Exact solution of the problem (TE polarization)

- Explicit form in components

\[ E_{s,\rho}^{TE}(r) = \left( E_i^0 \cdot \hat{h}_i \right) \frac{\exp(-jk_{i,z}z)}{k_{i,\rho}\rho} \left( \sum_{m=-\infty}^{\infty} (-j)^m J'_m(k_{i,\rho}R) \right) \frac{H_m^{(2)}(k_{i,\rho}\rho)}{H_m^{(2)'}(k_{i,\rho}R)} \exp[-jm(\phi-\phi_i)] \]

\[ E_{s,\phi}^{TE}(r) = -j \left( E_i^0 \cdot \hat{h}_i \right) \exp(-jk_{i,z}z) \left( \sum_{m=-\infty}^{\infty} (-j)^m J'_m(k_{i,\rho}R) \right) \frac{H_m^{(2)'}(k_{i,\rho}\rho)}{H_m^{(2)'}(k_{i,\rho}R)} \exp[-jm(\phi-\phi_i)] \]
Exact solution of the problem (TE polarization)

- The magnetic field in the TE case

\[ H_{s,TE}^r(r) = -\frac{j}{\eta} \sum_{m=-\infty}^{\infty} \frac{u_m J'_m(k_{i,\rho} R)}{H_m^{(2)'}(k_{i,\rho} R)} N_{m,k_{i,z}}^H(\rho,\phi,z) \]

- In components:

\[ H_{s,\rho}^{TE}(r) = \left( E_i^0 \cdot \hat{h}_i \right) \left( \frac{jk_{i,z}}{k\eta} \right) \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho} R)}{H_m^{(2)'}(k_{i,\rho} R)} H_m^{(2)'}(k_{i,\rho} \rho) \exp[-jm(\phi - \phi_i)] \]
Exact solution of the problem (TE polarization)

\[ H^{TE}_{s,\phi}(r) = \left( E^0_i \cdot \hat{n}_i \right) \left( \frac{k_{i,z}}{\eta k} \right) \exp \left( -jk_{i,z}z \right) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho}R)}{H^{(2)'}_m(k_{i,\rho}R)} \frac{H^{(2)}_m(k_{i,\rho}\rho)}{H_{m}^{(2)'}(k_{i,\rho}R)} \exp \left[ -jm(\phi - \phi_i) \right] \]

\[ H^{TE}_{s,z}(r) = -\left( E^0_i \cdot \hat{n}_i \right) \left( \frac{k_{i,\rho}}{\eta k} \right) \exp \left( -jk_{i,z}z \right) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho}R)}{H^{(2)'}_m(k_{i,\rho}R)} \frac{H^{(2)}_m(k_{i,\rho}\rho)}{H_{m}^{(2)'}(k_{i,\rho}R)} \exp \left[ -jm(\phi - \phi_i) \right] \]
For the TM case we have

\[ E_{s}^{TM}(r) = \sum_{m=-\infty}^{\infty} b_{m} N^{H}_{m,k_{i,z}}(\rho, \phi, z) \]

\[ = -\sum_{m=-\infty}^{\infty} v_{m} \frac{J_{m}(k_{i,\rho} R)}{H_{m}^{(2)}(k_{i,\rho} R)} N^{H}_{m,k_{i,z}}(\rho, \phi, z) \]

\[ v_{m} = -\left(\mathbf{E}_{i}^{0} \cdot \hat{\mathbf{v}}_{i}\right) \frac{k}{k_{i,\rho}} (-j)^{m} \exp(jm\phi_{i}) \]
Exact solution of the problem (TM polarization)

- Explicit form in components

\[
E_{s,\rho}^{TM}(r) = \left( E_i^0 \cdot \hat{n}_i \right) \left( -\frac{jk_i,z}{k} \right) \exp(-jk_i,z z)
\]

\[
\sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m(k_i,\rho R)}{H_m^{(2)}(k_i,\rho R)} H_m^{(2)'}(k_i,\rho \rho) \exp[-jm(\phi - \phi_i)]
\]

\[
E_{s,\phi}^{TM}(r) = \left( E_i^0 \cdot \hat{n}_i \right) \left( -\frac{k_i,z}{k} \right) \frac{\exp(-jk_i,z z)}{k_i,\rho \rho}
\]

\[
\sum_{m=-\infty}^{\infty} m \frac{(-j)^m J_m(k_i,\rho R)}{H_m^{(2)}(k_i,\rho R)} H_m^{(2)}(k_i,\rho \rho) \exp[-jm(\phi - \phi_i)]
\]
Exact solution of the problem (TM polarization)

- Explicit form in components

\[
E_{s,z}^{TM}(r) = \left( E_i^0 \cdot \hat{v}_i \right) \left( \frac{k_{i,\rho}}{k} \right) \exp(-jk_{i,z}z)
\]

\[
\sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(k_{i,\rho}R)}{H_m^{(2)}(k_{i,\rho}R)} H_m^{(2)}(k_{i,\rho}\rho) \exp[-jm(\phi-\phi_i)]
\]

- The magnetic field

\[
H_{s}^{TM}(r) = \sum_{m=-\infty}^{\infty} \frac{J_m(k_{i,\rho}R)}{\eta H_m^{(2)}(k_{i,\rho}R)} M_{m,k_{i,z}}^H(\rho, \phi, z)
\]
Exact solution of the problem (TM polarization)

\[
H_{s, \rho}^{TM} (r) = \left( E_i^0 \cdot \hat{v}_i \right) \frac{\exp(-jk_{i,z}z)}{\eta k_{i,\rho} \rho} \\
\sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m (k_{i,\rho} R)}{H_m^{(2)} (k_{i,\rho} R)} H_m^{(2)} (k_{i,\rho} \rho) \exp[-jm(\phi - \phi_i)]
\]

\[
H_{s, \phi}^{TM} (r) = -\left( E_i^0 \cdot \hat{v}_i \right) \frac{j}{\eta} \exp(-jk_{i,z}z) \\
\sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m (k_{i,\rho} R)}{H_m^{(2)} (k_{i,\rho} R)} H_m^{(2)'} (k_{i,\rho} \rho) \exp[-jm(\phi - \phi_i)]
\]