Scattering cross section (scattering width)

- We saw in the beginning how a scattering cross section is defined for a finite scatterer in terms of the scattered power.
- An infinite cylinder, however, is not a finite object.
- The field radiated by sources inside the cylinder (scattered field) does not drop as $1/r$, but as $1/\sqrt{\rho}$.
- We have to reformulate our definition of scattering cross section for infinite cylindrical objects.
Scattering cross section (scattering width)

- Again consider an incident wave

\[ E_i = e_i E_0 \exp(-jk_i \cdot r) \]

- Now, the scattered far field is always of the type

\[ E_s = e_s E_0 \frac{f(\hat{k}_s, \hat{k}_i)}{\sqrt{\rho}} \exp(-jk_{i,\rho} \rho - jk_{i,z} z) \]

Note: scattering amplitude may depend on the polarization of the incident wave!
Scattering cross section (scattering width)

- Corresponding far field Poynting vectors are
  \[ S_i = \frac{|E_i^0|^2}{2\eta} \hat{k_i} \]
  \[ S_s = \frac{|E_s^0|^2}{2\eta} \hat{k_s} = \frac{f(\hat{k_s}, \hat{k_i})^2}{\rho} \frac{|E_i^0|^2}{2\eta} \hat{k_s} \]

- It is important to bear in mind that
  \[ \hat{k_s} = \frac{k_{i,\rho}}{k} \hat{\rho} + \frac{k_{i,z}}{k} \hat{z} \]
Scattering cross section (scattering width)

- The Poynting vector is independent of $z$, it is the same for all vertical coordinates.
- We, therefore, cannot define the total scattered power since it is infinite.
- But, we can define the scattered power per unit length by integration over the angle:

$$P_L = \rho \int_0^{2\pi} \mathbf{S}_s \cdot \hat{\rho} d\phi = \frac{k_{i,\rho}}{k} |S_i| \int_0^{2\pi} \left| f(\phi, \phi_i) \right|^2 d\phi$$
Remember that

\[ |\mathbf{k}_i| = k \]

\[ \frac{k_{i,\rho}}{k} = \cos \theta_i \]

Total scattering width is now defined as

\[ \sigma_w = \frac{P_L}{|S_i|} = \frac{k_{i,\rho}}{k} \int_0^{2\pi} |f(\phi, \phi_i)|^2 \, d\phi \]
Scattering cross section (scattering width)

- As an example consider the TE case

\[ E_i = \hat{h}_i E_0 \exp(-jk_i \cdot r) \quad \hat{h}_i = \hat{\phi}_i \]

- The scattered far field:

\[
E_{s}^{TE} \to \hat{\phi} \left( E_i^0 \cdot \hat{h}_i \right) \frac{jC_0 \exp(-jk_{i,\rho}\rho - jk_z z)}{\sqrt{k_{i,\rho}\rho}} \sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho}R)}{H^{(2)}_m(k_{i,\rho}R)} \exp[jm(\phi_i - \phi)]
\]
Scattering cross section (scattering width)

- The scattering amplitude

\[ |f^{TE}(\phi, \phi_i)|^2 = \frac{2}{\pi k_{i,\rho}} \left| \sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho}R)}{H^{(2)'}_m(k_{i,\rho}R)} \exp\left[ jm(\phi_i - \phi) \right] \right|^2 \]

- Scattering width for TE waves:

\[ \sigma_{W}^{TE} = \frac{k_{i,\rho}}{k} \int_{0}^{2\pi} |f^{TE}(\phi_s, \phi_i)|^2 d\phi_s = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J'_m(k_{i,\rho}R)}{H^{(2)'}_m(k_{i,\rho}R)} \right|^2 \]
Scattering cross section (scattering width)

- **TM case:**
  
  \[ E_i = \hat{v}_i E_0 \exp(-jk_i \cdot r) \]
  
  \[ \hat{v}_i = \hat{\phi}_i \times \hat{k}_i = \frac{k_{i,z}}{k} \hat{\rho}_i - \frac{k_{i,\rho}}{k} \hat{z} \]

- **Scattered field:**

  \[ E_{s}^{TM} \rightarrow \left[ -\frac{k_{i,z}}{k} \hat{\rho} + \frac{k_{i,\rho}}{k} \hat{z} \right] (E_i^0 \cdot \hat{v}_i) \left( -jC_0 \right) \exp(-jk_{i,\rho} \rho - jk_{i,z} z) \]

  \[
  \sum_{m=-\infty}^{\infty} \frac{J_m(k_{i,\rho} R)}{H_m^{(2)}(k_{i,\rho} R)} \exp[jm(\phi_i - \phi)]
  \]
Scattering cross section (scattering width)

- Scattering amplitude:

\[ \left| f_{\text{TM}}^{\phi, \phi_i} \right|^2 = \frac{2}{\pi k_{i,\rho}} \left| \sum_{m=-\infty}^{\infty} \frac{J_m(k_{i,\rho} R)}{H_m^{(2)}(k_{i,\rho} R)} \exp\left[jm(\phi_i - \phi_s)\right] \right|^2 \]

- Scattering width:

\[ \sigma_{\text{W}}^{\text{TM}} = \frac{k_{i,\rho}}{k} 2\pi \int_0^{2\pi} \left| f_{\text{TM}}^{\phi, \phi_i} \right|^2 d\phi = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J_m(k_{i,\rho} R)}{H_m^{(2)}(k_{i,\rho} R)} \right|^2 \]
Scattering cross section (scattering width)

- Example: thin cylinder

\[
\sigma_{w}^{TE} = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J'_{m}(k_{i,\rho}R)}{H^{(2)'}_{m}(k_{i,\rho}R)} \right|^2 = \frac{3\pi^2}{4k} \left(k_{i,\rho}R\right)^4 + O\left(k_{i,\rho}R\right)^8
\]

\[
\sigma_{w}^{TM} = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J_{m}(k_{i,\rho}R)}{H^{(2)}_{m}(k_{i,\rho}R)} \right|^2 = \frac{\pi^2}{k} \left[\frac{1}{\ln(k_{i,\rho}R)}\right]^2 + O\left(k_{i,\rho}R\right)^4
\]
Scattering cross section (scattering width)

- Numerical results for the TE case

\[ k \sigma_{TE} \]

\[ k_{i,\rho} R \]
Scattering cross section (scattering width)

- Plotted in a different way

\[ \frac{\sigma_{W}^{TE}}{2R} \]

\[ k_{i,\rho}R \]
Scattering cross section (scattering width)

- For the TM case

\[ k \sigma_{W}^{TM} \]

\[ k_{i,\rho} R \]
Scattering by an infinitely long dielectric cylinder

- So far we discussed a (perfectly) conducting cylinder
- What about a dielectric cylinder? We can actually use the same machinery to solve the problem
- Incident plane wave:

\[ E_i(r) = \sum_{m=-\infty}^{\infty} u_m M^J_{m,k_i,z}(\rho,\phi,z) \]

\[ + \sum_{m=-\infty}^{\infty} v_m N^J_{m,k_i,z}(\rho,\phi,z) \]
Scattering by an infinitely long dielectric cylinder

- For the scattered wave (outside cylinder) we use the same expansion as before

\[ E_s(r) = \sum_{m=-\infty}^{\infty} a_m M_{m,k_i,z}^H(\rho, \phi, z) \]

\[ + \sum_{m=-\infty}^{\infty} b_m N_{m,k_i,z}^H(\rho, \phi, z) \]

- Inside the cylinder we use solutions based on Bessel function of 1st kind (why?)

\[ E_c(r) = \sum_{m=-\infty}^{\infty} c_m M_{m,k_i,z}^J(\rho, \phi, z) + \sum_{m=-\infty}^{\infty} d_m N_{m,k_i,z}^J(\rho, \phi, z) \]
Scattering by an infinitely long dielectric cylinder

- Next step is to match the fields at the boundary:

\[ \hat{\phi} \cdot (E_i + E_s) = \hat{\phi} \cdot E_c \]

\[ \hat{z} \cdot (E_i + E_s) = \hat{z} \cdot E_c \]

- This again leads to algebraic equations for coefficients which can be solved

- Note that inside the cylinder

\[ k_{i,\rho} \to k_{c,\rho} = \sqrt{k_d^2 - k_{i,z}^2} \quad k_d^2 = \omega^2 \mu_0 \varepsilon_d \]