Problem 1. Under what condition on the columns of $A$ (which may be rectangular) is $A^T A$ invertible?

Problem 2. Find value(s) for $a$, $b$, and $c$ for which the following matrix will be orthogonal:

\[
Q = \begin{bmatrix}
0 & -\frac{2}{3} & a \\
\frac{1}{\sqrt{5}} & \frac{2}{3} & b \\
-\frac{2}{\sqrt{5}} & \frac{1}{3} & c
\end{bmatrix}
\]

Problem 3. If $u$ is a unit vector, show that $Q = I - 2uu^T$ is a symmetric orthogonal matrix. (It is a reflection, also known as a “Householder transformation”.) Compute $Q$ when $u^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Problem 4. Apply the Gram–Schmidt process to

\[
a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \end{bmatrix}
\]

and write the result in the form $A = QR$.

Problem 5. Consider the vector space $\mathcal{P}_2$ with the inner product

\[
\langle p|q \rangle = \int_0^1 p(x)q(x) \, dx.
\]

Apply the Gram–Schmidt process to the basis $\{1, x, x^2\}$ to find an orthogonal basis of $\mathcal{P}_2$. Find also the corresponding orthonormal basis of $\mathcal{P}_2$.

Problem 6. Let $A$ be a $2 \times 2$ matrix with real entries. For $X, Y$ in $\mathbb{R}^{2 \times 1}$, let $f_A(X, Y) = Y^T AX$. Show that $f_A$ is an inner product on $\mathbb{R}^{2 \times 1}$ if and only if $A = A^T$, $A_{11} > 0$, $A_{22} > 0$, and $\det A > 0$.

Problem 7. Show that the formula

\[
\left\langle \sum_j a_j x^j \middle| \sum_k b_k x^k \right\rangle = \sum_{j,k} \frac{a_j b_k}{j+k+1}
\]
defines an inner product on the space \( \mathbb{R}[x] \) of polynomials over the field \( \mathbb{R} \). Let \( \mathcal{W} \) be the subspace of polynomials of degree less than or equal to \( n \). Restrict the above inner product to \( \mathcal{W} \), and find the matrix of this inner product on \( \mathcal{W} \), relative to the ordered basis \( \{1, x, x^2, \ldots, x^n\} \). \textbf{(Hint: To show that the formula defines an inner product, observe that} \( \langle f | g \rangle = \int_0^1 f(t)g(t) \, dt \text{ and work with the integral.}}

\textbf{Problem 8} (Bessel’s Inequality). Let \( \{a_1, \ldots, a_n\} \) be an orthogonal set of nonzero vectors in an inner product space \( \mathcal{V} \). If \( b \) is any vector in \( \mathcal{V} \), then

\[
\sum_k \frac{|\langle b|a_k \rangle|^2}{\|a_k\|^2} \leq \|b\|^2
\]

and equality holds if and only if

\[
b = \sum_k \frac{\langle b|a_k \rangle}{\|a_k\|^2} a_k.
\]

\textbf{Problem 9.} Let \( A \) be an \( n \times n \) matrix with complex entries such that \( A^* = -A \), and let \( B = e^A \). Show that

(a) \( B^* = e^{-A} \),

(b) \( B \) is unitary.

\textbf{Problem 10.} Prove that the trace of \( P = \frac{aa^T}{a^Ta} \), which is the sum of its diagonal entries, always equals 1.