Problem. Let $\psi(x) = x^3 + 1$ be the minimal polynomial of matrix $A \in \mathbb{R}^{4 \times 4}$. Find the minimal polynomial, trace, and determinant of $(I - A^{-1})$.

Solution:

$\psi(A) = 0 \Rightarrow A^3 + I = 0$
$\Rightarrow -A^3 = (A)(-A^2) = (-A^2)(A) = I$
$\Rightarrow A^{-1} = -A^2$

$A$ is a real matrix. Thus, the coefficients of its characteristic polynomial $(\Delta(x))$ are real. This makes complex eigenvalues of $A$ to appear in conjugate pairs.

$\deg(\psi) = 3 \quad \deg(\Delta) = 4$
$\psi(x) = x^3 + 1 \quad \Delta(x) = ?$
$= (x + 1)(x^2 - x + 1)$
$= (x + 1)(x - \alpha)(x - \overline{\alpha})$
$\alpha = \frac{1 + j\sqrt{3}}{2}$

Considering conjugacy, and the fact $\psi(x)|\Delta(x)$, we conclude that

$\Delta(x) = (x + 1)(x^3 + 1) = (x + 1)^2(x^2 - x + 1)$.

So, the Jordan canonical form of $A$ is

$$J = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & \overline{\alpha}
\end{bmatrix} \quad \text{and} \quad A = PJP^{-1},$$

for some nonsingular $P$.

$\Rightarrow I - A^{-1} = I + A^2$
$= I + PJ^2P^{-1}$
$= PP^{-1} + PJ^2P^{-1}$
$= P(I + J^2)P^{-1}$

$\Rightarrow I + J^2 = \begin{bmatrix}
1 + (-1)^2 & 0 & 0 & 0 \\
0 & 1 + (-1)^2 & 0 & 0 \\
0 & 0 & 1 + \alpha^2 & 0 \\
0 & 0 & 0 & 1 + \overline{\alpha}^2
\end{bmatrix}$

Note that $x^2 - x + 1 = 0$ for $x = \alpha$ and $x = -\alpha$. Therefore,

$I + J^2 = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & \overline{\alpha}
\end{bmatrix}$

$= \text{Jordan form of } I - A^{-1}$.

We know that similarity transformation preserves eigenvalues. Therefore, it preserves trace and determinant since

$$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \det(A) = \prod_{i=1}^{n} \lambda_i.$$
Let $B$ denote $I - A^{-1}$. Readily, we have the following:

\[
\psi_B(x) = (x - 2)(x - \alpha)(x - \bar{\alpha}) \\
= (x - 2)(x^2 - x + 1) = x^3 - 3x^2 + 3x - 2
\]

\[
\text{trace}(B) = 2 + 2 + \alpha + \bar{\alpha} \\
= 4 + 2\Re\{\alpha\} = 5
\]

\[
\det(B) = 2 \times 2 \times \alpha \times \bar{\alpha} \\
= 4|\alpha|^2 = 4
\]