Analytical expression of giant Goos–Hänchen shift in terms of proper and improper modes in waveguide structures with arbitrary refractive index profile

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We analytically relate the giant Goos–Hänchen shift, observed at the interface of a high refractive index prism and a waveguide structure with an arbitrary refractive index profile, to the spatial resonance phenomenon. The proximity effect of the high refractive index prism on modal properties of the waveguide is discussed, and the observed shift is expressed in terms of proper and improper electromagnetic modes supported by the waveguide with no prism. The transversely increasing improper modes are shown playing an increasingly important role as the high refractive index prism comes closer to the waveguide. © 2010 Optical Society of America


The lateral shift of totally reflected optical beams from a planar interface has been referred to as the Goos–Hänchen shift (GHS) and is observed in two different regimes. One corresponds to the nonresonant, usually small, GHS, which is due to the branch point singularity in the reflection coefficient of planar structures [1,2]. The other corresponds to the giant GHS, which is caused by the simple pole singularity in the reflection coefficient of planar structures [1,3]. In the latter regime, the structure basically comprises an optical prism with high refractive index \( n_p \) placed at the distance \( d_g \) above a waveguide with arbitrary refractive index profile. The waveguide has a pure guided mode, whose longitudinal propagation constant is \( \beta_g \). The high refractive index prism in this structure transforms the waveguide into a leaky wave structure with a complex propagation constant \( \beta_g^{\ast} \), whose real part is usually close to \( \beta_g \). A number of works have focused on relating the giant GHS to the leaky mode propagation constant of the whole structure, \( \beta_g^{\ast} \), or the unperturbed mode of the waveguide, \( \beta_g \) [1,4,5]. To the best of our knowledge, no formula is yet reported to express the GHS in terms of the electromagnetic modes in the general case. Here, the giant GHS is analytically linked up to the modes of the unperturbed waveguide with an arbitrary refractive index profile. It is found that the giant GHS depends not only on the transversely decreasing proper modes of the waveguide but also on its transversely increasing improper modes. The latter are shown to play an increasingly important role in the strong coupling regime as the optical prism comes nearer to the waveguide.

To obtain an analytic expression for the GHS, the transfer matrix of the whole structure, \( Q_T \), is first written as a multiplication of \( Q \), the transfer matrix of the waveguide with no prism, and \( T \), the transfer matrix of the gap between prism and waveguide having a thickness of \( d_g \) and refractive index of \( n_c \):

\[
Q_T = Q \times T.
\] (1)

where

\[
T = \begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 + f \frac{\kappa_p}{\kappa_g} & \exp(\kappa_g d_g) \left(1 - f \frac{\kappa_p}{\kappa_g} \right) \\
1 - f \frac{\kappa_p}{\kappa_g} & \exp(\kappa_g d_g) \left(1 + f \frac{\kappa_p}{\kappa_g} \right)
\end{bmatrix},
\] (3)

and \( q_{ij} \) denote the element of the transfer matrix of the waveguide. The transfer matrix of the gap, \( T \), can be also written as

\[
Q = \begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{bmatrix}
\] (2)

where \( \kappa_i \) with \( i = p, c \) stands for the \( x \) component of the wave vector in either the prism \((i = p)\) or cover \((i = c)\) region, and the factor \( f \) is 1 for TE and \( n_c^2/n_p^2 \) for TM polarized waves, respectively.

The overall reflection coefficient of the structure, \( r \), can then be straightforwardly written in terms of the elements of the \( Q_T \) matrix:

\[
r = - \frac{Q_{21}}{Q_{22}}.
\] (4)

Here, the numerator and denominator of the overall reflection coefficient of the structure, i.e., \( Q_{21} \) and \( Q_{22} \), are, respectively, the \((2,1)\)th and \((2,2)\)th element of the overall transfer matrix \( Q_T \). Now, insomuch as the incident spatial frequency, \( \beta \), is close to the resonance spatial frequency, \( \beta_g \), i.e., \(|\beta - \beta_g| \ll 1\), \( Q_{21} \) and \( Q_{22} \) can be represented by their first-order Taylor series expansion around \( \beta_g \):

\[
Q_{21}(\beta) = Q_{21}(\beta_g) + \frac{\partial Q_{21}}{\partial \beta} \bigg|_{\beta_g} (\beta - \beta_g),
\] (5a)

\[
Q_{22}(\beta) = Q_{22}(\beta_g) + \frac{\partial Q_{22}}{\partial \beta} \bigg|_{\beta_g} (\beta - \beta_g).
\] (5b)
By writing $Q_{21}(\beta_g), Q_{22}(\beta_g)$ in terms of $q_{ij}$s and $t_{ij}$s and using the fact that $q_{22}(\beta_g) = 0$ (because $\beta_g$ is an eigenmode of the structure without prism), we have

$$Q_{21}(\beta) = \frac{\partial Q_{21}}{\partial \beta} |_{\beta_g} (\beta - \beta_z),$$  

(6a)

$$Q_{22}(\beta) = \frac{\partial Q_{22}}{\partial \beta} |_{\beta_g} (\beta - \beta_p),$$  

(6b)

where $\beta_z$ and $\beta_p$ denote the zero and the pole of the overall reflection of the structure and read as

$$\beta_z = \beta_g - \frac{Q_{21}(\beta_g)}{\frac{\partial Q_{21}}{\partial \beta} |_{\beta_g}} = \beta_g - \frac{Q_{21}(\beta_g)}{\frac{\partial Q_{22}}{\partial \beta} |_{\beta_g} \times \frac{t_{11}/t_{21}}{1 + \left(\frac{t_{11}/t_{21}}{t_{12}/t_{22}} + \frac{t_{12}/t_{22}}{t_{12}/t_{22}}\right)}},$$  

(7a)

$$\beta_p = \beta_g - \frac{Q_{22}(\beta_g)}{\frac{\partial Q_{22}}{\partial \beta} |_{\beta_g}} = \beta_g - \frac{Q_{22}(\beta_g)}{\frac{\partial Q_{22}}{\partial \beta} |_{\beta_g} \times \frac{t_{11}/t_{21}}{1 + \left(\frac{t_{11}/t_{21}}{t_{12}/t_{22}} + \frac{t_{12}/t_{22}}{t_{12}/t_{22}}\right)}},$$  

(7b)

and the prime denotes derivation with respect to $\beta$.

The overall reflection coefficient of the structure, $r$, is thus approximately represented by a zero and a simple pole in the complex plane of spatial frequency:

$$r = \frac{\partial R_{21}}{R_{21}} |_{\beta_g} \times \frac{\beta - \beta_z}{\beta - \beta_p}. $$  

(8)

The obtained pole-zero representation of the overall reflection coefficient in Eq. (8) can now be applied to the well-known Artmann formula [9], and the GHS can be written as

$$\text{GHS} = \frac{\partial \phi}{\partial \beta} = \text{Im} \left[ \frac{d}{d \beta} \ln(r) \right] = \text{Im} \left[ \frac{\beta_z - \beta_p}{(\beta - \beta_p)(\beta - \beta_z)} \right],$$  

(9)

where Im represents the imaginary part of its argument and $\phi$ stands for the phase factor of the reflection coefficient.

It should be noticed that the modulus of the reflection coefficient, $r$, is 1 for lossless waveguides when the incident beam is totally reflected. The pole and zero of the overall reflection coefficient form a complex conjugate pair, and the maximum GHS is observed at $\beta = \text{Re}[\beta_p]$ and amounts to 2/Im[\beta_p]]. For a lossy structure, on the other hand, Re[\beta_p] = Re[\beta_z], and the maximum GHS is observed at $\beta = \text{Re}[\beta_p]$ and amounts to (Im[\beta_z] - Im[\beta_p])/(Im[\beta_p][\text{Im}[\beta_p]]).

The analytical expression given in Eq. (9) then relates the GHS to $\beta_p$ and $\beta_z$, the propagation constants of the proper and improper modes in the whole structure, i.e., waveguide and prism. These propagation constants, $\beta_p$ and $\beta_z$, are also related to the propagation constants of the proper and improper modes in the waveguide structure with no prism. The reflection coefficient at $x = d_p$, i.e., reflection from the cover-waveguide interface, denoted $r_w$, is written in terms of the elements of the Q matrix:

$$r_w = \frac{q_{21}(\beta)}{q_{22}(\beta)}. $$  

(10)

Given that $\beta_g$ is the propagation constant of a proper waveguide mode, $q_{22}(\beta_g) = 0$ and $\beta_g$ is a simple pole whose residue is

$$\text{Res}(\beta_g) = -\frac{q_{21}(\beta_g)}{q_{22}(\beta_g)}. $$  

(11)

Now by assuming that $\beta_0$ is the nearest zero of $r_w$ to $\beta_g$, we have $q_{21}(\beta_0) = 0$, and thus the first-order Taylor series approximation of $q_{21}(\beta)$ calculated around $\beta_g$ and evaluated at $\beta_0$ yields the following equation:

$$\beta_g - \beta_0 = \frac{q_{21}(\beta_g)}{q_{21}(\beta_g)}.$$

(12)

Interestingly, the right-hand side of Eq. (12) is the difference between propagation constants of proper and improper modes. Furthermore, the elements of the T matrix can be written as

$$\frac{t_{11}}{t_{21}} = -r_{pc}^{-1} \exp(-2x\lambda d_g), $$  

(13a)

$$\frac{t_{12}}{t_{22}} = -r_{pc} \exp(-2x\lambda d_g). $$  

(13b)

where $r_{pc}$ denotes the Fresnel reflection coefficient at the interface of the prism and cover, when there is no waveguide in the structure.

It is now possible to apply Eqs. (11)–(13) in Eqs. (7) and approximate the zero and pole of the overall reflection coefficients in terms of $d_p$, of the $x$ component of the wave vector in the cover region calculated at $\beta_p$, i.e., $\kappa_c(\beta_g)$, of proper and improper mode propagation constants in the waveguide with no prism, i.e., $\beta_g$ and $\beta_0$, and of the residue of the reflection coefficient at the cover-waveguide interface:

$$\tilde{\beta}_z = \beta_g - \frac{\text{Res}(\beta_g) \exp(-2x\lambda(\beta_g) d_g)}{r_{pc}(\beta_g) + \text{Res}(\beta_g)/(\beta_g - \beta_0) \exp(-2x\lambda(\beta_g) d_g)}, $$

(14a)

$$\tilde{\beta}_p = \beta_g - \frac{\text{Res}(\beta_g) r_{pc}(\beta_g) \exp(-2x\lambda(\beta_g) d_g)}{1 + r_{pc}(\beta_g) \text{Res}(\beta_g)/(\beta_g - \beta_0) \exp(-2x\lambda(\beta_g) d_g)}. $$

(14b)

In obtaining these expressions, $r_{pc}(\beta)$ and $\kappa_c(\beta)$ have been, respectively, replaced by $r_{pc}(\beta_g)$ and $\kappa_c(\beta_g)$, and
also the derivatives of $t_{11}(\beta)$ and $t_{12}(\beta)$ with respect to spatial frequency $\beta$ have been neglected. These simplifications, valid at the vicinity of singularity point $\beta_g$, do not cause a significant error because $r_{pc}(\beta)$, $\kappa_c(\beta)$, $t_{11}(\beta)$, and $t_{12}(\beta)$ are all slowly varying functions of the spatial frequency $\beta$. It is possible to derive these expressions by using the zero-pole approximation of the waveguide reflection coefficient, $r_p$. The approximate expressions in Eqs. (14), and consequently the GHS obtained by applying them to Eq. (9), are therefore as valid as the zero-pole approximation of the reflection coefficient at the cover-waveguide region.

It is also easy to show that the approximate expressions for $\tilde{\beta}_z$ and $\tilde{\beta}_p$ in Eqs. (14) can be further simplified to

$$\tilde{\beta}_z = \beta_g - \text{Res}(\beta_g) \times r_{pc}(\beta_g) \exp(-2\kappa_c(\beta_g)d_g), \quad (15a)$$

$$\tilde{\beta}_p = \beta_g - \text{Res}(\beta_g) \times r_{pc}(\beta_g) \exp(-2\kappa_c(\beta_g)d_g), \quad (15b)$$

for the weak coupling regime, when $|\exp(-2\kappa_c(\beta_g)d_g)/(\beta_g - \beta_0)| = \delta \ll 1$. The approximate expressions in Eqs. (15) are equivalent to the single pole approximation of the waveguide reflection coefficient. The presence of the improper mode, coinciding with the zero of the reflection coefficient, is therefore neglected in the weak coupling regime. These expressions show that the proximity of pole, $\beta_p$, and zero, $\beta_0$, in determining the coupling strength of the waveguide structure and prism is as essential as the gap width $d_g$.

As an example, a monochromatic $S$ polarized wave is incident upon a graded refractive index profile:

$$n(x) = \left[ n_i^2 - (n_g^2 - n_s^2) \frac{x}{d} \right]^{1/2}, \quad (16)$$

where $n_f = 2.210$, and $n_s = 2.177$, the normalized frequency is $V = 3.78$, and the asymmetry parameter is $a = 2.27$. A high refractive index prism with $n_p = 3$ is placed over the waveguide structure, leaving a gap of $d_g$ between prism-cover and cover-waveguide interfaces. Two different gap widths are considered, and the GHS for wide enough incident beams is plotted versus angle of incidence in Fig. 1. The proposed approximation coincides with Artmann’s formula. A slight amount of loss is then added to the waveguide, and $n_f = 2.210$ is changed to $n_f = 2.210 - 0.0002i$. The GHS is plotted versus angle of incidence in Fig. 2. Once again, the applicability of the proposed expressions is shown. Figure 2 shows that the presented approximations are still valid when the waveguide is lossy and $\beta_g$ is not a real number.

These examples show that the proposed approximations are as accurate as Artmann’s formula for the near resonant GHS. They, however, provide a good physical insight by relating the GHS to the leaky mode characteristics and to the coupling parameters. They also enable us to perceive two different coupling regimes: weak and strong.

References