CHAPTER

SIX

FIRST-ORDER CIRCUITS

Chapters 2 to 5 have been devoted exclusively to circuits made of resistors and independent sources. The resistors may contain two or more terminals and may be linear or nonlinear, time-varying or time-invariant. We have shown that these resistive circuits are always governed by algebraic equations.

In this chapter, we introduce two new circuit elements, namely, two-terminal capacitors and inductors. We will see that these elements differ from resistors in a fundamental way: They are *lossless*, and therefore *energy* is not dissipated but merely *stored* in these elements.

A circuit is said to be *dynamic* if it includes some capacitor(s) or some inductor(s) or both. In general, dynamic circuits are governed by *differential equations*. In this initial chapter on dynamic circuits, we consider the *simplest* subclass described by only *one* first-order differential equation—hence the name first-order circuits. They include all circuits containing *one* 2-terminal capacitor (or inductor), plus resistors and independent sources.

The important concepts of *initial state*, *equilibrium state*, and *time constant* allow us to find the solution of any first-order linear time-invariant circuit driven by dc sources by inspection (Sec. 3.1). Students should master this material before plunging into the following sections where the inspection method is extended to include *linear switching* circuits in Sec. 4 and *piecewise-linear* circuits in Sec. 5. Here, the important concept of a *dynamic route* plays a crucial role in the analysis of piecewise-linear circuits by inspection.

1 TWO-TERMINAL CAPACITORS AND INDUCTORS

Many devices cannot be modeled accurately using only resistors. In this section, we introduce *capacitors* and *inductors*, which, together with *resistors*, Let N be a two-terminal element driven by a voltage source $v(t)$ [respectively, current source $i(t)$] and let $i(t)$ [respectively, $v(t)$] denote the corresponding current (respectively, voltage) response. If we plot the locus $\mathscr{L}(v, i)$ of $(v(t), i(t))$ in the v-i plane and obtain a *fixed* curve *independent of the excitation waveforms*, then N can be modeled as a two-terminal *resistor*. If $\mathscr{L}(v, i)$ changes with the excitation waveform, then N does not behave like a resistor and a different *model* must be chosen. In this case, we can calculate the associated charge waveform $q(t)$ using Eq. (1.2*a*), or the flux waveform $\phi(t)$ using Eq. (1.2*b*), and see whether the corresponding locus $\mathscr{L}(q, v)$ of $(q(t), v(t))$ in the q-v plane, or $\mathscr{L}(\phi, i)$ of $(\phi(t), i(t))$ in the ϕ-i plane, is a fixed curve independent of the excitation waveforms.

Exercises

1. Apply at $t = 0$ a voltage source $v(t) = A \sin \omega t$ (in volts) across a 1-F capacitor. (*a*) Calculate the associated *current* $i(t)$, *flux* $\phi(t)$, and *charge* $q(t)$ for $t \geq 0$, using Eqs. (1.2*a*) and (1.2*b*). Assume $\phi(0) = -1$ Wb and $q(0) = 0$ C. (*b*) Sketch the loci $\mathscr{L}(v, i)$, $\mathscr{L}(\phi, i)$, and $\mathscr{L}(q, v)$ in the v-i plane, ϕ-i plane, and q-v plane, respectively, for the following parameters (A is in volts, ω is in radians per second):

A	1	2	1
ω	1	1	2

(*c*) Does it make sense to describe this element by a v-i characteristic? ϕ-i characteristic? q-v characteristic? Explain.

2. Apply at $t = 0$ a current source $i(t) = A \sin \omega t$ (in amperes) across a 1-H inductor. (*a*) Calculate the associated *voltage* $v(t)$, *charge* $q(t)$, and *flux* $\phi(t)$ for $t \geq 0$, using Eqs. (1.2*a*) and (1.2*b*). Assume $q(0) = -1$ C and $\phi(0) = 0$ W. (*b*) Sketch the loci $\mathscr{L}(v, i)$, $\mathscr{L}(q, v)$, and $\mathscr{L}(\phi, i)$ in the v-i plane, q-v plane, and ϕ-i plane, respectively, for the following parameters (A is in amperes, ω is in radians per second):

A	1	2	1
ω	1	1	2

(*c*) Does it make sense to describe this element by a v-i characteristic? q-v characteristic? ϕ-i characteristic? Explain.

1.1 q-v and ϕ-i characteristics

A two-terminal element whose *charge* $q(t)$ and *voltage* $v(t)$ fall on some *fixed* curve in the q-v plane at any time t is called a *time-invariant capacitor*.[3] This curve is called the

A two-terminal element whose *flux* $\phi(t)$ and *current* $i(t)$ fall on some *fixed* curve in the ϕ-i plane at any time t is called a *time-invariant inductor*.[4] This curve is called the

[3] This definition is generalized to that of a *time-varying* capacitor in Sec. 1.2.

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q-v characteristic of the capacitor. It may be represented by the equation[5]

$$f_C(q, v) = 0 \qquad (1.3a)$$

If Eq. $(1.3a)$ can be solved for v as a single-valued function of q, namely,

$$v = \hat{v}(q) \qquad (1.4a)$$

the capacitor is said to be *charge-controlled*.

If Eq. $(1.3a)$ can be solved for q as a single-valued function of v, namely,

$$q = \hat{q}(v) \qquad (1.5a)$$

the capacitor is said to be *voltage-controlled*.

If the function $\hat{q}(v)$ is differentiable, we can apply the chain rule to express the current entering a *time-invariant voltage-controlled capacitor* in a form similar to Eq. $(1.1a)$:[6]

$$i = C(v)\dot{v} \qquad (1.6a)$$

where

$$C(v) \triangleq \frac{d\hat{q}(v)}{dv} \qquad (1.7a)$$

is called the *small-signal capacitance* at the operating point v.

Example 1*a* (Linear time-invariant parallel-plate capacitor) Figure 1.1*a* shows a familiar device made of two flat parallel metal plates sepa-

ϕ-i characteristic of the inductor. It may be represented by the equation[7]

$$f_L(\phi, i) = 0 \qquad (1.3b)$$

If Eq. $(1.3b)$ can be solved for i as a single-valued function of ϕ, namely,

$$i = \hat{i}(\phi) \qquad (1.4b)$$

the inductor is said to be *flux-controlled*.

If Eq. $(1.3b)$ can be solved for ϕ as a single-valued function of i, namely,

$$\phi = \hat{\phi}(i) \qquad (1.5b)$$

the inductor is said to be *current-controlled*.

If the function $\hat{\phi}(i)$ is differentiable, we can apply the chain rule to express the voltage across a *time-invariant current-controlled inductor* in a form similar to Eq. $(1.1b)$:[8]

$$v = L(i)\dot{i} \qquad (1.6b)$$

where

$$L(i) \triangleq \frac{d\hat{\phi}(i)}{di} \qquad (1.7b)$$

is called the *small-signal inductance* at the operating point i.

Example 1*b* (Linear time-invariant toroidal inductor) Figure 1.2*a* shows a familiar device made of a conducting wire wound around a toroid

[5] This equation is also called the *constitutive relation* of the capacitor.

[6] We will henceforth use the notation $\dot{v} \triangleq dv(t)/dt$.

[7] This equation is also called *the constitutive relation* of the inductor.

[8] We will henceforth use the notation $\dot{i} \triangleq di(t)/dt$.

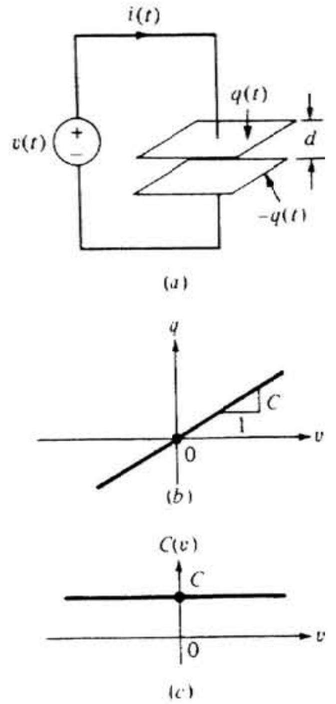


Figure 1.1 Parallel-plate capacitor.

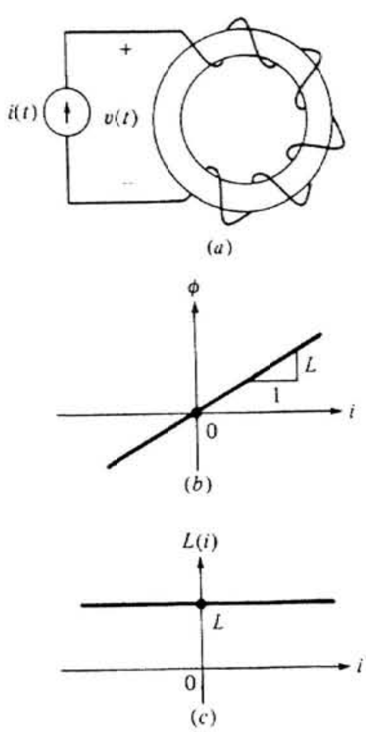


Figure 1.2 Toroidal inductor.

rated in free space by a distance d. When a voltage $v(t) > 0$ is applied, we recall from physics that a charge equal to

$$q(t) = C\,v(t) \qquad (1.8a)$$

is induced at time t on the upper plate, and an equal but opposite charge is induced on the lower plate at time t. The constant of proportionality is given approximately by

$$C = \varepsilon_0 \frac{A}{d} \qquad \text{farad (F)}$$

where $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m is the dielectric constant in free space, A is the plate area in square meters, and d is the separation of the plates in meters.

made of a nonmetallic material such as wood. When a current $i(t) > 0$ is applied, we recall from physics that a flux equal to

$$\phi(t) = Li(t) \qquad (1.8b)$$

is induced at time t and circulates around the interior of the toroid. The constant of proportionality is given approximately by

$$L = \mu_0 \frac{N^2 A}{\ell} \qquad \text{henry (H)}$$

where $\mu_0 = 4 \times 10^{-7}$ H/m is the permeability of the wooden core, N is the number of turns, A is the cross-sectional area in square meters, and ℓ is the midcircumference along the toroid in meters.

Equation (1.8*a*) defines the q-v characteristic of a *linear time-invariant capacitor*, namely, a straight line through the origin with slope equal to C, as shown in Fig. 1.1*b*. Its small-signal capacitance $C(v) = C$ is a constant function (Fig. 1.1*c*). Consequently, Eq. (1.6*a*) reduces to Eq. (1.1*a*).

Example 2*a* (Nonlinear time-invariant parallel-plate capacitor) If we fill the space between the two plates in Fig. 1.1*a* with a nonlinear *ferroelectric* material (such as barium titanate), the *measured* q-v characteristic in Fig. 1.3*a* is no longer a straight line. This *nonlinear* behavior is due to the fact that the dielectric constant of *ferroelectric* materials is not a constant—it changes with the applied *electric* field intensity.

Likewise, the small-signal capacitance shown in Fig. 1.3*b* is a *nonlinear* function of v.

Equation (1.8*b*) defines the ϕ-i characteristic of a *linear time-invariant inductor*, namely, a straight line through the origin with slope equal to L, as shown in Fig. 1.2*b*. Its small-signal inductance $L(i) = L$ is a constant function (Fig. 1.2*c*). Consequently, Eq. (1.6*b*) reduces to Eq. (1.1*b*).

Example 2*b* (Nonlinear time-invariant toroidal inductor) If we replace the wooden core in Fig. 1.2*a* with a nonlinear *ferromagnetic* material (such as superpermalloy) the *measured* ϕ-i characteristic in Fig. 1.4*a* is no longer a straight line. This *nonlinear* behavior is due to the fact that the permeability of *ferromagnetic* materials is not a constant—it changes with the applied *magnetic* field intensity.

Likewise, the small-signal inductance shown in Fig. 1.4*b* is a *nonlinear* function of i.

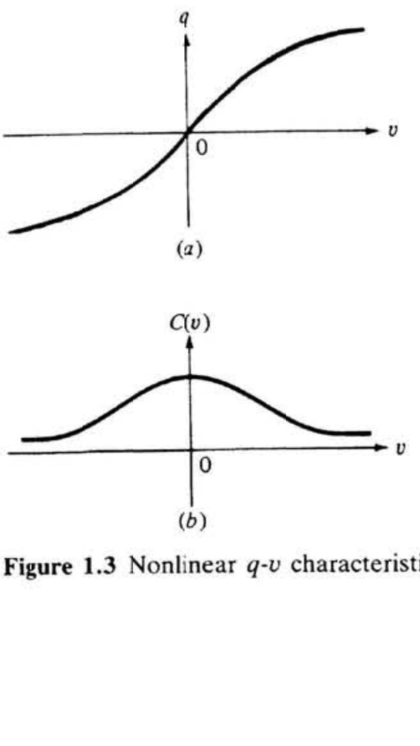


Figure 1.3 Nonlinear q-v characteristic.

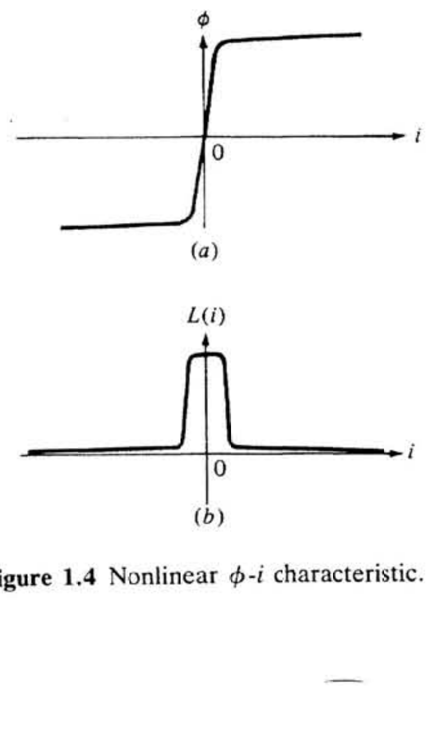


Figure 1.4 Nonlinear ϕ-i characteristic.

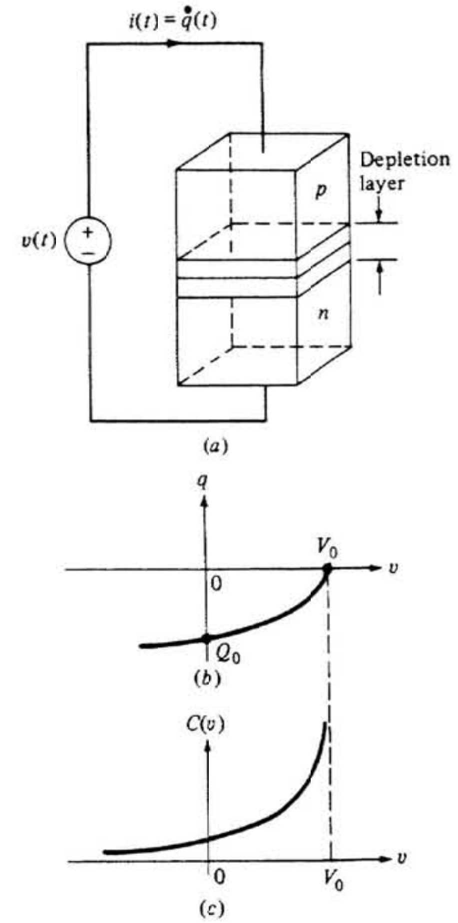


Figure 1.5 q-v characteristic of a varactor diode.

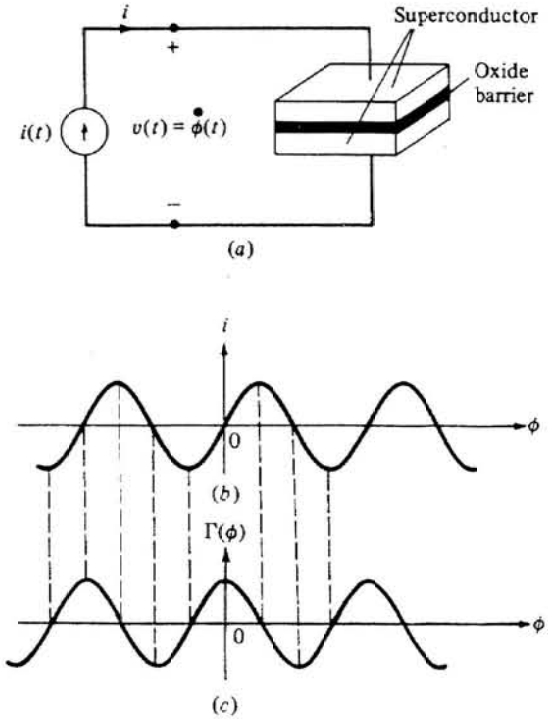


Figure 1.6 ϕ-i characteristic of a Josephson junction.

Example 3a (Varactor diode) The varactor diode[9] shown in Fig. 1.5*a* is a *pn*-junction diode designed specially to take advantage of the depletion layer when operating in reverse bias, i.e., when $v < V_0$ (typically, $0.2V < V_0 < 0.9V$). Semiconductor physics proves that the charge q accumulated on the top layer is equal to

Example 3b (Josephson junction) The Josephson junction[10] shown in Fig. 1.6*a* consists of two superconductors separated by an insulating layer (such as oxide). Superconductor physics proves that the current i varies sinusoidally with ϕ, namely,

[9] Varactor diodes are widely used in many communication circuits. For example, modern radio and TV sets are automatically tuned by applying a suitable dc bias voltage across such a diode.

[10] Josephson proposed this exotic device in 1961 and was awarded the Nobel prize in physics in 1969 for this discovery. The Josephson junction has been used in numerous applications.

$$q = -K(V_0 - v)^{1/2} \triangleq \hat{q}(v) \tag{1.9a}$$

provided $v < V_0$. Here, V_0 is the contact potential and

$$K = \frac{2\varepsilon N_a N_d}{N_a + N_d} A$$

where ε = permittivity of the material, N_a = number of acceptor atoms per cubic centimeter, N_d = number of donor atoms per cubic centimeter, and A = cross-sectional area in square centimeters.

Its small-signal capacitance (Fig. 1.5*c*) is obtained by differentiating Eq. (1.9*a*):

$$C(v) = \tfrac{1}{2} K(V_0 - v)^{-1/2} \qquad v < V_0 \tag{1.10a}$$

Note that unlike the previous examples, the q-v characteristic in Fig. 1.5*b* is *not* defined for $v > V_0$. Hence, this capacitor is *not* voltage-controlled for *all* values of $v > V_0$. (For $v > V_0$, the diode becomes forward biased and behaves like a nonlinear resistor.)

$$i = I_0 \sin k\phi \triangleq \hat{i}(\phi) \tag{1.9b}$$

where I_0 is a device parameter and

$$k = 4\pi \frac{e}{h}$$

where e = electron charge and h = Planck's constant.

Note that unlike the previous example, the ϕ-i characteristic in Fig. 1.6*b* is *not* current-controlled. Consequently, its small-signal inductance $L(i) \triangleq d\phi(i)/di$ is not uniquely defined.

However, the Josephson junction is *flux-controlled* and has a well-defined slope (Fig. 1.6*c*)

$$\Gamma(\phi) \triangleq \frac{d\hat{i}(\phi)}{d\phi} = kI_0 \cos k\phi \tag{1.10b}$$

We call $\Gamma(\phi)$ the *reciprocal small-signal inductance* since it has the unit of H^{-1}.

REMARK Note that Eqs. (1.9*a*) and (1.9*b*), as well as Eqs. (1.10*a*) and (1.10*b*), are not strictly dual equations because the corresponding variables are not duals of each other. However, if we solved for v in terms of q in Eq. (1.9*a*), we would obtain a dual function $v = \hat{v}(q)$. In this case, the derivative

$$S(q) \triangleq \frac{d\hat{v}(q)}{dq}$$

is called the *reciprocal small-signal capacitance*.

Exercise (*a*) Show that a one-port obtained by connecting port 2 of a *gyrator* (assume unity coefficient) across a k-H linear inductor is equivalent to that of a k-F linear capacitor in the sense that they have identical q-v characteristics. (*b*) Is it possible to give a simple *physical* interpretation of the *charge* associated with this element? (*c*) Generalize the property in (*a*) to the case where the inductor is nonlinear: $\phi = \hat{\phi}(i)$.

Exercise (*a*) Show that a one-port obtained by connecting port 2 of a *gyrator* (assume unity coefficient) across a k-F linear capacitor is equivalent to that of a k-H linear inductor in the sense that they have identical ϕ-i characteristics. (*b*) Is it possible to give a simple *physical* interpretation of the *flux* associated with this element? (*c*) Generalize the property in (*a*) to the case where the capacitor is nonlinear: $q = \hat{q}(v)$.

1.2 Time-Varying Capacitors and Inductors

The examples presented so far are *time-invariant* in the sense that the q-v and ϕ-i characteristics do not change with time.

If the q-v characteristic changes with time, the capacitor is said to be *time-varying*.

For example, suppose we vary the spacing between the parallel-plate capacitor in Fig. 1.1*a*, say by using a motor-driven cam mechanism, so that the capacitance C becomes some *prescribed function of time* $C(t)$. Then Eq. (1.8*a*) becomes

$$q(t) = C(t)v(t) \qquad (1.11a)$$

It follows from Eq. (1.2*a*) that

$$i(t) = C(t)\frac{dv(t)}{dt} + \frac{dC(t)}{dt}v(t) \qquad (1.12a)$$

Note that the current in a *time-varying linear capacitor* differs from Eq. (1.1*a*) *not only* in the replacement of C by $C(t)$, but also in the presence of an extra term.

If the ϕ-i characteristic changes with time, the inductor is said to be *time-varying*.

For example, suppose we vary the number of turns of the winding in Fig. 1.2*a*, say by using a motor-driven sliding contact, so that the inductance L becomes some *prescribed function of time* $L(t)$. Then Eq. (1.8*b*) becomes

$$\phi(t) = L(t)i(t) \qquad (1.11b)$$

It follows from Eq. (1.2*b*) that

$$v(t) = L(t)\frac{di(t)}{dt} + \frac{dL(t)}{dt}i(t) \qquad (1.12b)$$

Note that the voltage in a *time-varying linear inductor* differs from Eq. (1.1*b*) *not only* in the replacement of L by $L(t)$, but also in the presence of an extra term.

To be specific, assume

$$C(t) = 2 + \sin t \qquad (1.13a)$$

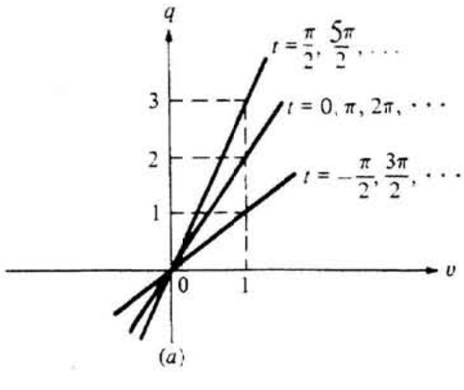
then

$$q(t) = (2 + \sin t)v(t) \qquad (1.14a)$$

and

$$i(t) = (2 + \sin t)\,\frac{dv(t)}{dt} + (\cos t)v(t) \qquad (1.15a)$$

The q-v characteristic of a time-varying linear capacitor consists of a *family* of straight lines, each line valid for a given instant of time. For example, the q-v characteristic of the above time-varying linear capacitor is shown in Fig. 1.7a. Its associated small-signal capacitance consists of a family of horizontal lines (Fig. 1.7b).



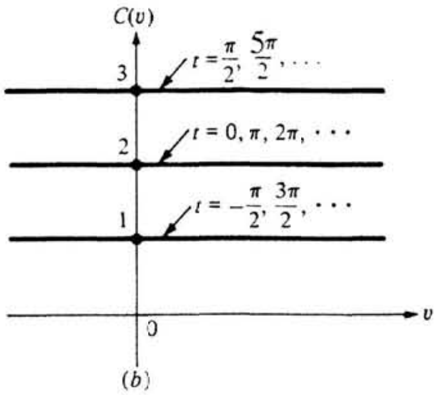


Figure 1.7 Time-varying q-v characteristic of Eq. (1.14a).

To be specific, assume

$$L(t) = 2 + \sin t \qquad (1.13b)$$

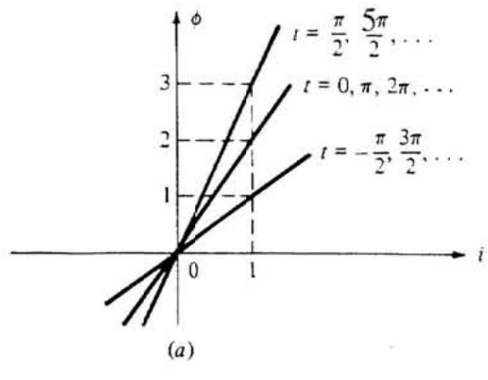
then

$$\phi(t) = (2 + \sin t)i(t) \qquad (1.14b)$$

and

$$v(t) = (2 + \sin t)\,\frac{di(t)}{dt} + (\cos t)i(t) \qquad (1.15b)$$

The ϕ-i characteristic of a time-varying linear inductor consists of a *family* of straight lines, each line valid for a given instant of time. For example, the ϕ-i characteristic of the above time-varying linear inductor is shown in Fig. 1.8b. Its associated small-signal inductance consists of a family of horizontal lines (Fig. 1.8b).



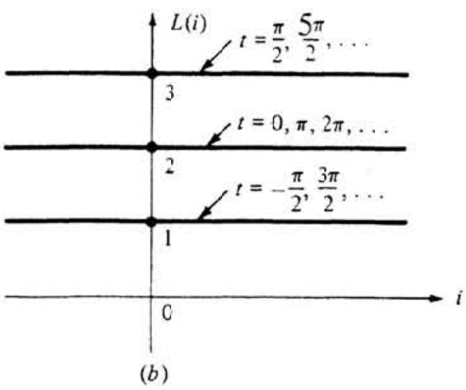


Figure 1.8 Time-varying ϕ-i characterisic of Eq. (1.14b).

Time-varying linear capacitors and inductors are useful in the modeling, analysis, and design of many communication circuits (e.g., modulators, demodulators, parametric amplifiers).

In the most general case, a *time-varying nonlinear capacitor* is defined by a family of time-dependent and nonlinear q-v characteristics, namely,

$$f_C(q, v, t) = 0 \qquad (1.16a)$$

In the most general case, a *time-varying nonlinear inductor* is defined by a family of time-dependent and nonlinear ϕ-i characteristics, namely,

$$f_L(\phi, i, t) = 0 \qquad (1.16b)$$

The two circuit variables used in defining a two-terminal resistor, inductor, or capacitor can be easily remembered with the help of the mnemonic diagram shown in Fig. 1.9. Note that out of the six exhaustive pairings of the four basic variables v, i, q, and ϕ, two are related *by definitions*, namely, $i = \dot{q}$ and $v = \dot{\phi}$. The remaining pairs are constrained by the *constitutive relation* of a two-terminal element, three of which give us the resistor, inductor, and capacitor.[11]

We will use the symbols shown in Fig. 1.9 to denote a nonlinear two-terminal resistor, inductor, or capacitor, respectively. Note that a dark band is included in each symbol in order to distinguish the two terminals. Just as in the case of a *nonbilateral* two-terminal resistor, such distinction is necessary if the q-v or ϕ-i characteristic is *not* odd symmetric. In the special case where the element is *linear*, the v-i, ϕ-i, and q-v characteristics are odd symmetric

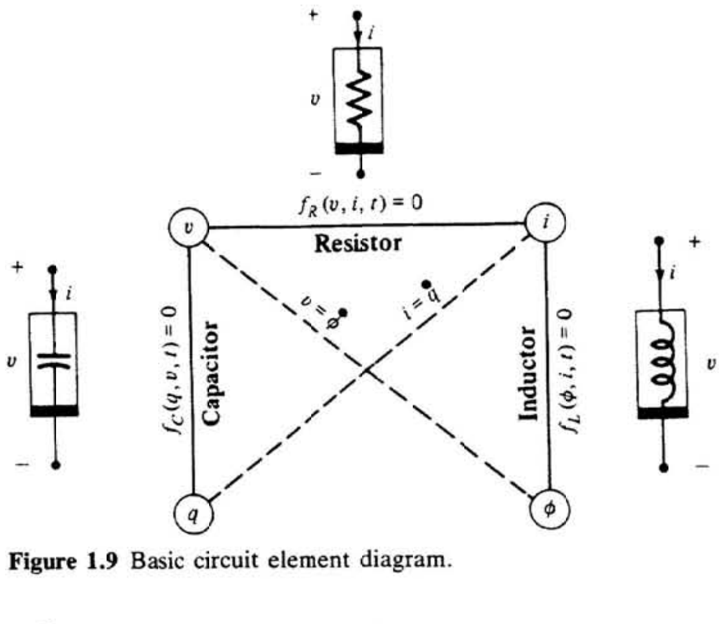


Figure 1.9 Basic circuit element diagram.

[11] A fourth nonlinear two-terminal element called the *memristor* is defined by the remaining relationship between q and ϕ. This circuit element is described in L. O. Chua, "Memristor—The Missing Circuit Element," *IEEE Trans. on Circuit Theory*, vol. 18, pp. 507–519, September 1971.

and hence remain unchanged after the two terminals are interchanged. In this case, we simply delete drawing the enclosing rectangle and revert to the standard symbol for a linear resistor, inductor, and capacitor.

2 BASIC PROPERTIES EXHIBITED BY TIME-INVARIANT CAPACITORS AND INDUCTORS

Capacitors and inductors behave differently from resistors in many ways. The following typical properties illustrate some fundamental differences.[12]

2.1 Memory Property

Suppose we drive the linear capacitor in Fig. 1.1*a* by a current source $i(t)$. The corresponding voltage at any time t is obtained by integrating both sides of Eq. (1.1*a*) from $\tau = -\infty$ to $\tau = t$. Assuming $v(-\infty) = 0$ (i.e., the capacitor has no initial charge when manufactured), we obtain

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau \tag{2.1}$$

Note that unlike the *resistor* voltage which depends on the resistor current only at one instant of time t, the above capacitor voltage depends on the *entire past history* (*i.e.*, $-\infty < \tau < t$) of $i(\tau)$. Hence, *capacitor has memory*.

Now suppose the voltage $v(t_0)$ at some time $t_0 < t$ is given, then Eq. (1.1*a*) integrated from $t = -\infty$ to t becomes

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d\tau \qquad t \geq t_0 \tag{2.2}$$

In other words, instead of specifying the entire past history, we need only specify $v(t)$ at some conveniently chosen initial time t_0. In effect, the *initial condition* $v(t_0)$ summarizes the effect of $i(\tau)$ from $\tau = -\infty$ to $\tau = t_0$ on the present value of $v(t)$.

By duality, it follows that *inductor has memory* and that the inductor current is given by

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)\, d\tau \qquad t \geq t_0 \tag{2.3}$$

The "memory" in a capacitor or inductor is best manifested by the "dual" equivalent circuits shown in Fig. 2.1*a* and *b* which asserts the following:

[12] Unless otherwise specified, all capacitors and inductors are assumed to be *time-invariant* in this book.

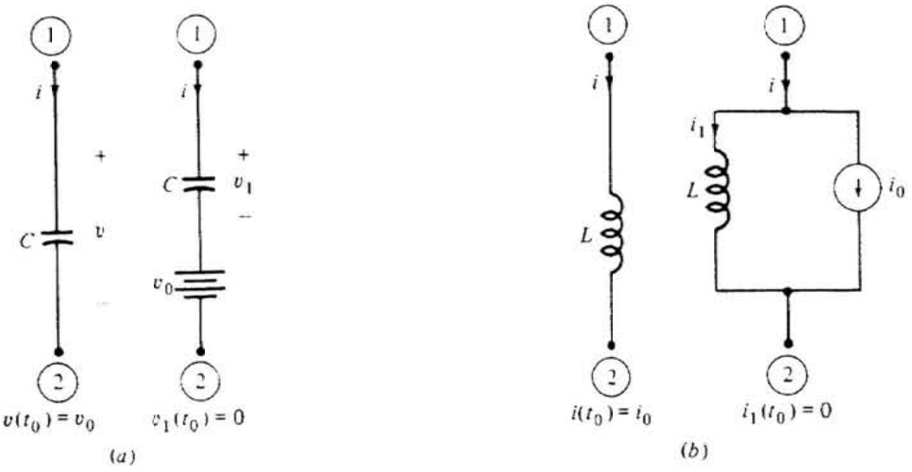


Figure 2.1 Initial condition transformation.

Initial capacitor voltage transformation A linear time-invariant capacitor with an *initial voltage* v_0 is indistinguishable externally from a one-port made of an *initially uncharged* capacitor (having the same capacitance) in series with a battery of v_0 volts.

The circuits shown in Fig. 2.1*a* are equivalent because they are characterized by the same equation, namely, Eq. (2.2).

Initial inductor current transformation A linear time-invariant inductor with an *initial current* i_0 is indistinguishable externally from a one-port made of an inductor (having the same inductance) with *zero initial current* in parallel with a current source of i_0 amperes.

The circuits shown in Fig. 2.1*b* are equivalent because they are characterized by the same equation, namely, Eq. (2.3).

The memory property of capacitors and inductors has been exploited in the design of many practical circuits. For example, consider the "peak detector" circuit shown in Fig. 2.2*a*. Since the ideal diode current vanishes whenever $v_{\text{in}}(t) \le v_o(t)$, it follows from Eq. (2.2) that at any time t_1, $v_o(t)$ is equal to the *maximum* value of $v_{\text{in}}(t)$ from $t=-\infty$ to $t=t_1$. A typical waveform of $v_o(t)$ and $v_{\text{in}}(t)$ is shown in Fig. 2.2*b*. In practice, this circuit is usually implemented as shown in Fig. 2.2*c*, where the op-amp circuit from Fig. 3.13 of Chap. 4 is used to simulate an ideal diode and where the op-amp buffer from Fig. 2.1 of Chap. 4 is used to avoid output loading effects.

Exercise The switch S in the "track-and-hold" circuit shown in Fig. 2.3 is periodically open and closed every Δt seconds. Sketch $v_o(t)$, and suggest a typical application.

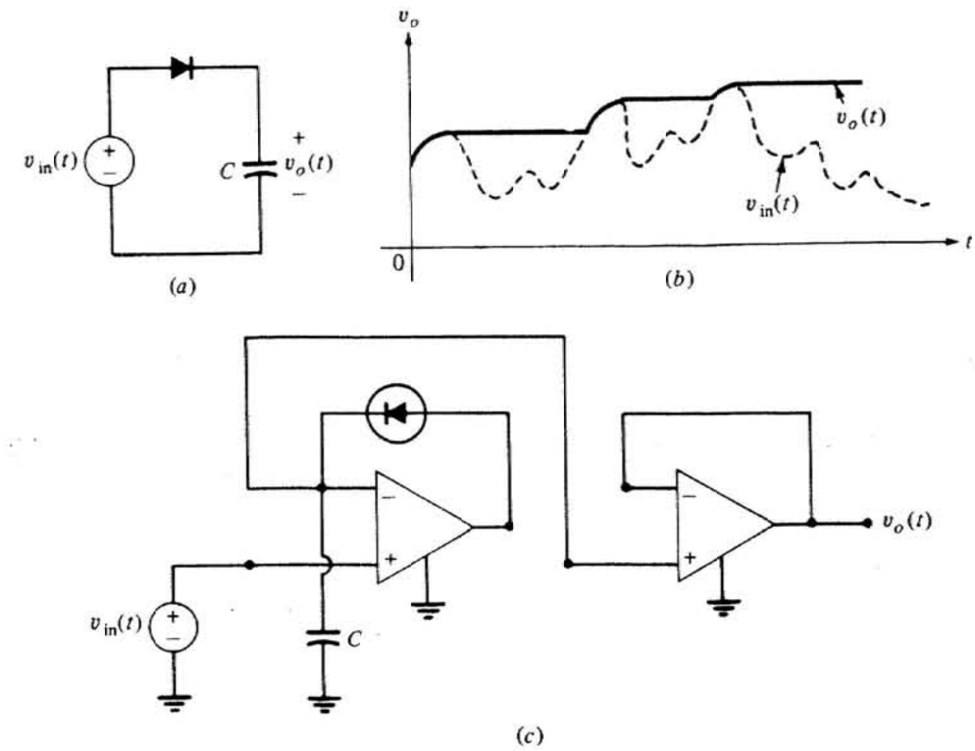


Figure 2.2 A peak detector circuit.

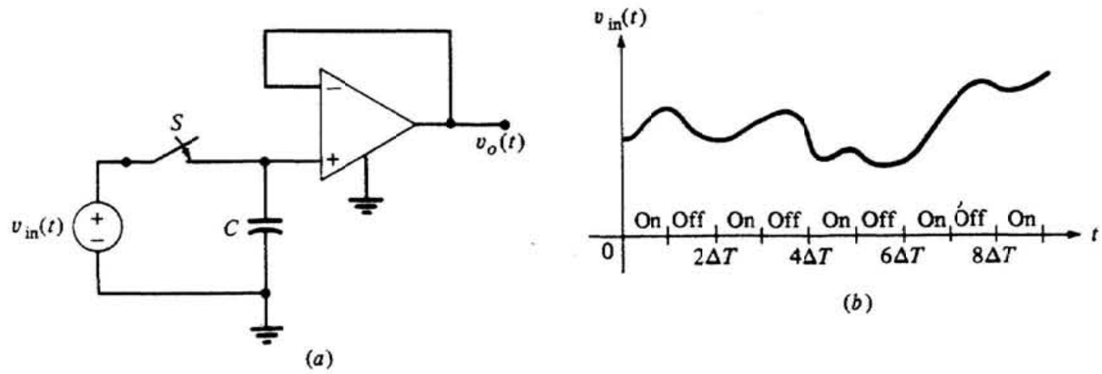


Figure 2.3 A track-and-hold circuit.

Equation (2.2) and Fig. 2.1*a* are valid only when the capacitor is linear. To show that *nonlinear* capacitors also exhibit memory, note that its voltage $v(t)$ depends on the charge and by Eq. (1.2*a*),

$$q(t) = q(t_0) + \int_{t_0}^{t} i(\tau)\, d\tau \qquad t > t_0 \tag{2.4a}$$

Equation (2.3) and Fig. 2.1*b* are valid only when the inductor is linear. To show that *nonlinear* inductors also exhibit memory, note that its current $i(t)$ depends on the flux and by Eq. (1.2*b*),

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} v(\tau)\, d\tau \qquad t > t_0 \tag{2.4b}$$

which in turn depends on the past history of $i(\tau)$ for $-\infty < \tau < t_0$.

which in turn depends on the past history of $v(\tau)$ for $-\infty < \tau < t_0$.

2.2 Continuity Property

Consider the circuit shown in Fig. 2.4*a*, where the current source is described by the "discontinuous" square wave shown in Fig. 2.4*b*. Assuming that $v_C(0) = 0$ and applying Eq. (2.2), we obtain the "continuous" capacitor voltage waveform shown in Fig. 2.4*c*. This "smoothing" phenomenon turns out to be a general property shared by both capacitor voltages and inductor currents. More precisely, we can state this important property as follows:

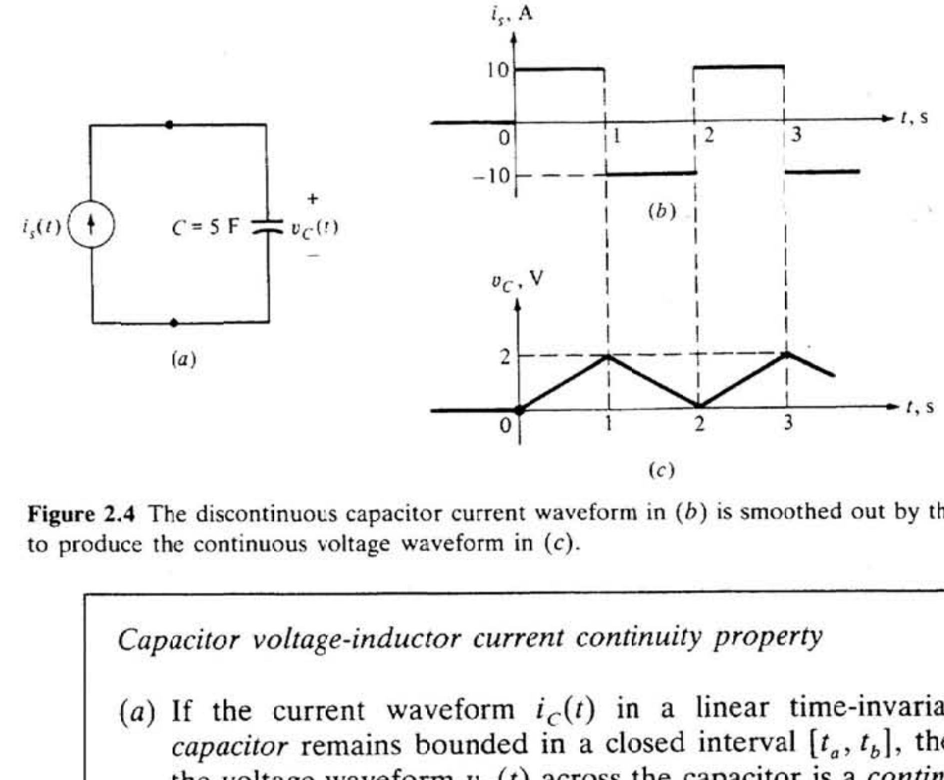


Figure 2.4 The discontinuous capacitor current waveform in (*b*) is smoothed out by the capacitor to produce the continuous voltage waveform in (*c*).

> *Capacitor voltage-inductor current continuity property*
>
> (*a*) If the current waveform $i_C(t)$ in a linear time-invariant *capacitor* remains bounded in a closed interval $[t_a, t_b]$, then the voltage waveform $v_C(t)$ across the capacitor is a *continuous* function in the open interval (t_a, t_b). In particular,[13] for any time T satisfying $t_a < T < t_b$, $v_C(T-) = v_C(T+)$.
>
> (*b*) If the voltage waveform $v_L(t)$ in a linear time-invariant *inductor* remains bounded in a closed interval $[t_a, t_b]$, then the current waveform $i_L(t)$ through the inductor is a *continuous* function in the open interval (t_a, t_b). In particular, for any time T satisfying $t_a < T < t_b$, $i_L(T-) = i_L(T+)$.

[13] We denote the left-hand limit and right-hand limit of a function $f(t)$ at $t = T$ by $f(T-)$ and $f(T+)$, respectively.

PROOF We will prove only (*a*) since (*b*) follows by duality. Substituting $t = T$ and $t = T + dt$ into Eq. (2.2), where $t_a < T < t_b$ and $t_a < T + dt \leq t_b$, and subtracting, we get

$$v_C(T + dt) - v_C(T) = \frac{1}{C} \int_T^{T+dt} i_C(t')\, dt' \tag{2.5}$$

Since $i_C(t)$ is *bounded* in $[t_a, t_b]$, there is a *finite* constant M such that $|i_C(t)| < M$ for all t in $[t_a, t_b]$. It follows that the area under the curve $i_C(t)$ from T to $T + dt$ is at most $M\, dt$ (in absolute value), which tends to zero as $dt \to 0$. Hence Eq. (2.5) implies $v_C(T + dt) \to v_C(T)$ as $dt \to 0$. This means that the waveform $v_C(\cdot)$ is continuous at $t = T$. ∎

REMARK The above continuity property does *not* hold if the capacitor current (respectively, inductor voltage) is *unbounded*. Before we illustrate this remark, let us first give an example showing how a capacitor current can become unbounded—at least in theory.

Suppose we apply a voltage source across a 1-F linear capacitor having the waveform shown in Fig. 2.5*b*. It follows from Eq. (1.1*a*) that the capacitor current waveform $i_C(t)$ is a rectangular pulse with height equal to $1/\Delta$ and width equal to Δ, as shown in Fig. 2.5*c*. Note that the pulse height increases as Δ decreases. It is important to note that the area of this pulse is equal to 1, *independent of* Δ. Now in the limit where $\Delta \to 0$, $v_s(t)$ tends to the discontinuous "unit step" function [henceforth denoted by $1(t)$] shown in Fig. 2.5*d*, i.e.,[14]

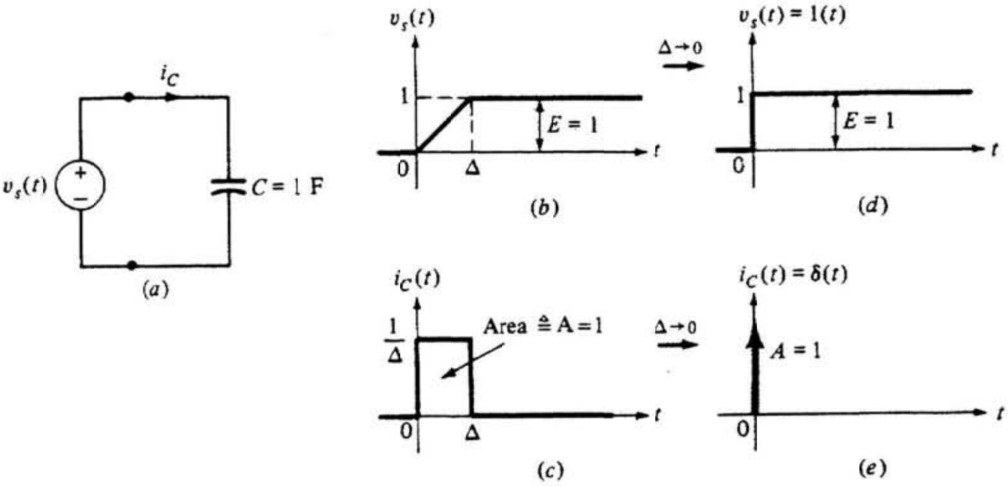


Figure 2.5 Circuit for generating a unit current impulse.

[14] The value of the unit step function $1(t)$ at $t = 0$ does not matter from the physical point of view. However, sometimes it is convenient to define it to be equal to $\frac{1}{2}$ in circuit theory.

$$\lim_{\Delta \to 0} v_s(t) = 1(t) \triangleq \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{2.6}$$

To show that this discontinuity in the capacitor voltage does *not* contradict the preceding continuity property, note that the corresponding capacitor *current* is *unbounded* in this case, namely,

$$\lim_{\Delta \to 0} i_C(t) = \infty \qquad \text{at } t = 0 \tag{2.7}$$

Since this waveform is of great importance in engineering analysis, let us pause to study its properties carefully.

Impulse (delta function) As $\Delta \to 0$, the height of the "rectangular" pulse in Fig. 2.5c tends to infinity at $t = 0$, and to zero elsewhere, while the area under the pulse remains unchanged, i.e., $A = 1$. This limiting waveform is called an *impulse* and will be denoted by $\delta(t)$.

More precisely, an unbounded signal $\delta(t)$ is called a *unit impulse*[15] iff it satisfies the following *two* properties:

1.	$\delta(t) \triangleq \begin{cases} \text{singular} & t = 0 \\ 0 & t \neq 0 \end{cases}$		(2.8a) (2.8b)
2.	$\int_{-\varepsilon_1}^{\varepsilon_2} \delta(t)\, dt = 1$	for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$	(2.8c)

Since the unit impulse is *unbounded*, we will denote it symbolically by a "bold" arrowhead as shown in Fig. 2.5e.

Now, if $E \neq 1$ and $C \neq 1$ in Fig. 2.5, the above discussion still holds provided the area A of the impulse is changed from $A = 1$ to $A = CE$. Note that this situation can be simulated by connecting an E-V battery across a C-F capacitor *at* $t = 0$. The resulting current waveform would then be an impulse with an area equal to $A = CE$.[16]

Now that we have demonstrated how a current impulse can be generated, let us drive the circuit in Fig. 2.4a with a current impulse of area $A = 10$ applied at $t = 5$ s, as shown in Fig. 2.6a. It follows from Eq. (2.8) that this impulse can be denoted by

$$i_s(t) = A\delta(t - 5) \tag{2.9}$$

[15] In physics, the unit impulse is called a *delta function*. Using the theory of distribution in advanced mathematics, the unit impulse can be rigorously defined as a "generalized" function imbued with most of the standard properties of a function. In particular, most of the time $\delta(t)$ can be manipulated like an ordinary function.

[16] In practice, only a very large (but finite) current pulse is actually observed because all *physical* batteries have a small but nonzero internal resistance (recall Fig. 2.4a of Chap. 2). Given the value of R, we will be able to calculate the exact current waveform $i_C(t)$ in Sec. 3.1.

Substituting $i_s(t)$ into Eq. (2.2) with $v(0) = 0$ we obtain

$$v_C(t) = \frac{1}{5}\int_0^t A\delta(\tau - 5)\,d\tau \qquad t \geq 0 \tag{2.10}$$

Defining a new dummy variable $x \triangleq \tau - 5$ and using Eq. (2.8), we obtain

$$\begin{aligned} v_C(t) &= \frac{A}{5}\int_{-5}^{t-5} \delta(x)\,dx \qquad t \geq 0 \\ &= \begin{cases} 0 & t < 5 \quad \text{[in view of Eq. (2.8}b\text{)]} \\ 2 & t > 5 \quad \text{[in view of Eq. (2.8}c\text{)]} \end{cases} \end{aligned} \tag{2.11}$$

The resulting capacitor voltage waveform is shown in Fig. 2.6*b*. Note that it is *discontinuous* at $t = 5$ s.

Exercise Prove that whenever the current waveform $i_C(t)$ entering a C-F linear time-invariant capacitor contains an *impulse* of area A at $t = t_0$, the associated capacitor voltage waveform $v_C(t)$ *will change abruptly* at t_0 by an amount equal to A/C.

Exercise Prove that whenever the voltage waveform $v_L(t)$ across an L-H linear time-invariant inductor contains an *impulse* of area A at $t = t_0$, the associated inductor current waveform $i_L(t)$ will change *abruptly* at t_0 by an amount equal to A/L.

2.3 Lossless Property

Since $p(t) = v(t)\,i(t)$ is the *instantaneous power* in *watts* entering a two-terminal element at any time t, the total *energy* $w(t_1, t_2)$ in joules entering the element during any time interval $[t_1, t_2]$ is given by

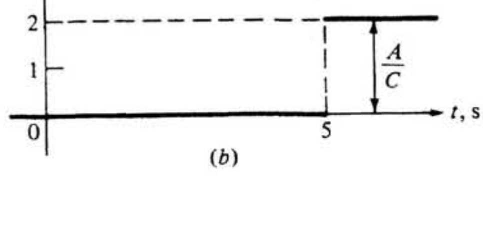


Figure 2.6 The voltage waveform $v_C(t)$ is discontinuous at $t = 5$ s.

$$w(t_1, t_2) = \int_{t_1}^{t_2} v(t)\, i(t)\, dt \qquad \text{joules} \tag{2.12}$$

For example, the total energy $w_R(t_1, t_2)$ entering a linear resistor with resistance $R > 0$ is given by

$$w_R(t_1, t_2) = \int_{t_1}^{t_2} [R\, i(t)]\, i(t)\, dt = R \int_{t_1}^{t_2} i^2(t)\, dt = \frac{1}{R} \int_{t_1}^{t_2} v^2(t)\, dt \tag{2.13}$$

This energy is dissipated in the form of heat and is lost as far as the circuit is concerned. Such an element is therefore said to be *lossy*.

In general, the energy $w(t_1, t_2)$ entering a two-terminal element during $[t_1, t_2]$ depends on the entire voltage waveform $v(t)$ or current waveform $i(t)$ over the *entire interval* $[t_1, t_2]$. For example, if we drive the 10-Ω resistor in Fig. 2.7*a* by the waveforms shown in Fig. 2.7*b* and *c*, respectively, the energy dissipated during the interval $[\frac{1}{4}, \frac{3}{4}]$ is given respectively by

$$w_1(\tfrac{1}{4}, \tfrac{3}{4}) = 10 \int_{1/4}^{3/4} (2 \sin 2\pi t)^2\, dt = 10.00 \text{ joules} \tag{2.14}$$

$$w_2(\tfrac{1}{4}, \tfrac{3}{4}) = 10 \int_{1/4}^{3/4} [-8(t - \tfrac{1}{2})]^2\, dt = 6.67 \text{ joules} \tag{2.15}$$

Note that $w_1(\frac{1}{4}, \frac{3}{4}) \neq w_2(\frac{1}{4}, \frac{3}{4})$ even though the resistor currents $i_1(t)$ and $i_2(t)$, and hence also their voltages $v_1(t)$ and $v_2(t)$, are *identical at the end points*, namely, $i_1(\frac{1}{4}) = i_2(\frac{1}{4}) = 2$ A and $i_1(\frac{3}{4}) = i_2(\frac{3}{4}) = -2$ A.

In sharp contrast to the above typical observations, the following calculation shows that

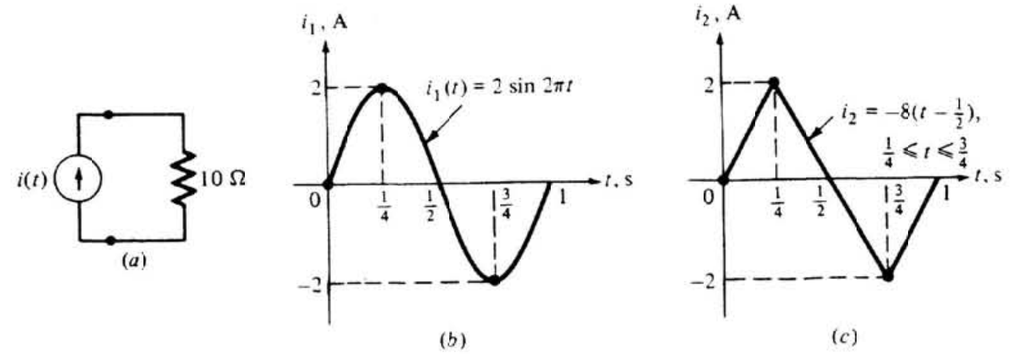


Figure 2.7 Resistor driven by two distinct current waveforms whose values coincide at $t_1 = \frac{1}{4}$ s and $t_2 = \frac{3}{4}$ s.

The energy $w_C(t_1, t_2)$ entering a charge-controlled capacitor during any time interval $[t_1, t_2]$ *is independent of the capacitor voltage or charge waveforms*: It is uniquely determined by the capacitor charge at the end points, namely by $q(t_1)$ and $q(t_2)$. Indeed,

$$w_C(t_1, t_2) = \int_{t_1}^{t_2} \hat{v}(q(t)) \frac{dq(t)}{dt} dt \tag{2.16a}$$

It follows from Eq. (2.16a) that

$$w_C(q_1, q_2) = \int_{q_1}^{q_2} \hat{v}(q)\, dq \tag{2.17a}$$

where we switched from t to q as the dummy variable, and $q_1 \triangleq q(t_1)$ and $q_2 \triangleq q(t_2)$.

Example For a C-F linear capacitor, we have $\hat{v}(q) = q/C$ and hence Eq. (2.17a) reduces to

$$\begin{aligned} w_C(q_1, q_2) &= \frac{1}{2C}[q_2^2 - q_1^2] \\ &= \tfrac{1}{2} C[V_2^2 - V_1^2] \end{aligned} \tag{2.18a}$$

where

$$V_1 \triangleq v(t_1) \quad \text{and} \quad V_2 \triangleq v(t_2).$$

The energy $w_L(t_1, t_2)$ entering a flux-controlled inductor during any time interval $[t_1, t_2]$ *is independent of the inductor current or flux waveforms*: It is uniquely determined by the inductor flux at the end points, namely, by $\phi(t_1)$ and $\phi(t_2)$. Indeed,

$$w_L(t_1, t_2) = \int_{t_1}^{t_2} \hat{i}(\phi(t)) \frac{d\phi(t)}{dt} dt \tag{2.16b}$$

It follows from Eq. (2.16b) that

$$w_L(\phi_1, \phi_2) = \int_{\phi_1}^{\phi_2} \hat{i}(\phi)\, d\phi \tag{2.17b}$$

where we switched from t to ϕ as the dummy varaible, and $\phi_1 \triangleq \phi(t_1)$ and $\phi_2 \triangleq \phi(t_2)$.

Example For an L-H linear inductor, we have $\hat{i}(\phi) = \phi/L$ and hence Eq. (2.17b) reduces to:

$$\begin{aligned} w_L(\phi_1, \phi_2) &= \frac{1}{2L}[\phi_2^2 - \phi_1^2] \\ &= \tfrac{1}{2} L[I_2^2 - I_1^2] \end{aligned} \tag{2.18b}$$

where

$$I_1 \triangleq i(t_1) \quad \text{and} \quad I_2 \triangleq i(t_2).$$

Exercises

1. Derive Eq. (2.18a) directly by substituting Eq. (1.1a) into Eq. (2.12).
2. Derive Eq. (2.18b) directly by substituting Eq. (1.1b) into Eq. (2.12).
3. Give an example showing that Eq. (2.17a) does not hold if the capacitor is *time-varying*.
4. Give an example showing that Eq. (2.17b) does not hold if the inductor is *time-varying*.

Equation (2.17a) shows that the energy $w_C(t_1, t_2)$ entering a charge-controlled capacitor is equal to the shaded area shown in Fig. 2.8a. Any waveform pair $[v(\cdot), q(\cdot)]$ taking the values $[v(t_1), q(t_1)]$ at t_1 and $[v(t_2), q(t_2)]$ at t_2 will give the same $w_C(t_1, t_2)$.

Now suppose $v(t)$ and $q(t)$ are *periodic* with period $T = t_2 - t_1$. Then $q(t_2) = q(t_1 + T) = q(t_1)$, and hence $w_C(t_1, t_2) = 0$. In this case, P_1 and P_2 in Fig. 2.8a coincide, thereby resulting in a zero area.

This observation can be summarized as follows: *Under periodic excitation, the total energy entering a charge-controlled capacitor is zero over any period.*

Equation (2.17b) shows that the energy $w_L(t_1, t_2)$ entering a flux-controlled inductor is equal to the shaded area shown in Fig. 2.8b. Any waveform pair $[i(\cdot), \phi(\cdot)]$ taking the values $[i(t_1), \phi(t_1)]$ at t_1 and $[i(t_2), \phi(t_2)]$ at t_2 will give the same $w_L(t_1, t_2)$.

Now suppose $i(t)$ and $\phi(t)$ are *periodic* with period $T = t_2 - t_1$. Then $\phi(t_2) = \phi(t_1 + T) = \phi(t_1)$, and hence $w_L(t_1, t_2) = 0$. In this case, P_1 and P_2 in Fig. 2.8b coincide, thereby resulting in a zero area.

This observation can be summarized as follows: *Under periodic excitation, the total energy entering a flux-controlled inductor is zero over any period.*

It follows from the above observation that the instantaneous power entering any charge-controlled capacitor or flux-controlled inductor is positive only during parts of each cycle, and must necessarily become negative elsewhere in order for the net area over each cycle to cancel out. Hence, unlike resistors, the power entering the capacitor or inductor is *not* dissipated. Rather, energy is stored during parts of each cycle and is "spit" out during the remaining part of the cycle. Such elements are therefore said to be *lossless*.

One immediate consequence of this lossless property is that in a periodic regime where $v(t) = v(t + T)$ and $i(t) = i(t + T)$ for all t, the voltage waveform $v(t)$ and current waveform $i(t)$ associated with any capacitor or inductor must necessarily *cross the time axis at different instants of time*. Otherwise, the integrand in Eq. (2.12) would always be positive, or negative, for all t, thereby implying $w(t_1, t_2) \neq 0$.

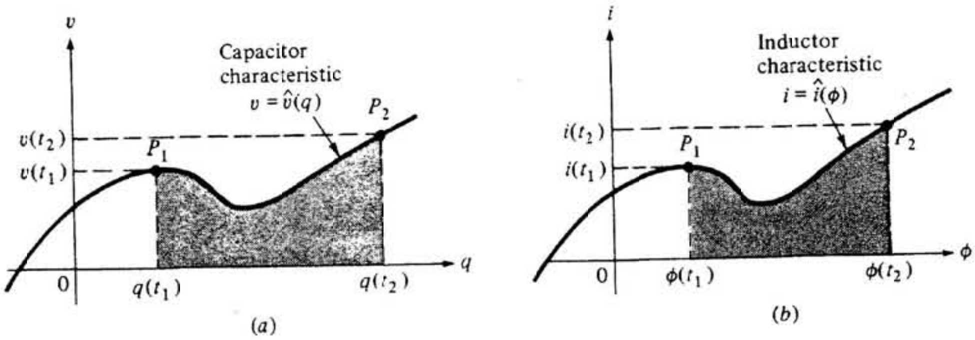


Figure 2.8 Geometric interpretation of $w_C(t_1, t_2)$ and $w_L(t_1, t_2)$.

For a linear capacitor or inductor operating in a sinusoidal steady state, this distinct "zero-crossing property" manifests itself as a 90° *phase shift* between the voltage and current, respectively. For example, if we drive the linear capacitor in Fig. 1.1*a* with a sinusoidal voltage $v(t) = E \sin \omega t$ as shown in Fig. 2.9*a*, then the corresponding current $i(t) = \omega CE \cos \omega t$ *leads* the voltage by 90° as shown in Fig. 2.9*b*. The locus $\mathcal{L}(v, i)$ in the v-i plane is therefore an ellipse as shown in Fig. 2.9*c*. Note that this locus is *frequency dependent*. Indeed, by adjusting ω from $\omega = 0$ to $\omega = +\infty$, the locus can be made to pass through *any* point lying within the vertical strip $-E < v < E$. Hence, it does not make sense to describe a capacitor, or an inductor, by a characteristic in the v-i plane.

2.4 Energy Stored in a Linear Time-Invariant Capacitor or Inductor

Consider a C-F linear capacitor having an initial voltage $v(t_1) = V$ and an initial charge $q(t_1) = Q = CV$ at $t = t_1$. Let the capacitor be connected to an external circuit, as shown in Fig 2.10*a*, at $t = t_1$. The energy entering the capacitor during $[t_1, t_2]$ is given by Eq. (2.17*a*):

$$w_C(t_1, t_2) = \frac{1}{2C}[q^2(t_2) - Q^2] = \tfrac{1}{2}C[v^2(t_2) - V^2] \tag{2.19}$$

Note that whenever $q(t_2) < Q$, or $v(t_2) < V$, then $w_C(t_1, t_2) < 0$. This can also be seen in Fig. 2.10*b* where $w_C(t_1, t_2)$ is negative because we are integrating from right (P_1) to left (P_2) in the first quadrant. Note that $w_C(t_1, t_2) < 0$ means energy is actually being spit out of the capacitor and returned to the external circuit $\mathcal{N}$. It follows from Eq. (2.19) and Fig. 2.10*b* that $w_C(t_1, t_2)$ is *most negative* when $q(t_2) = v(t_2) = 0$, whereupon $w_C(t_1, t_2) = -Q^2/2C = -\tfrac{1}{2}CV^2$.

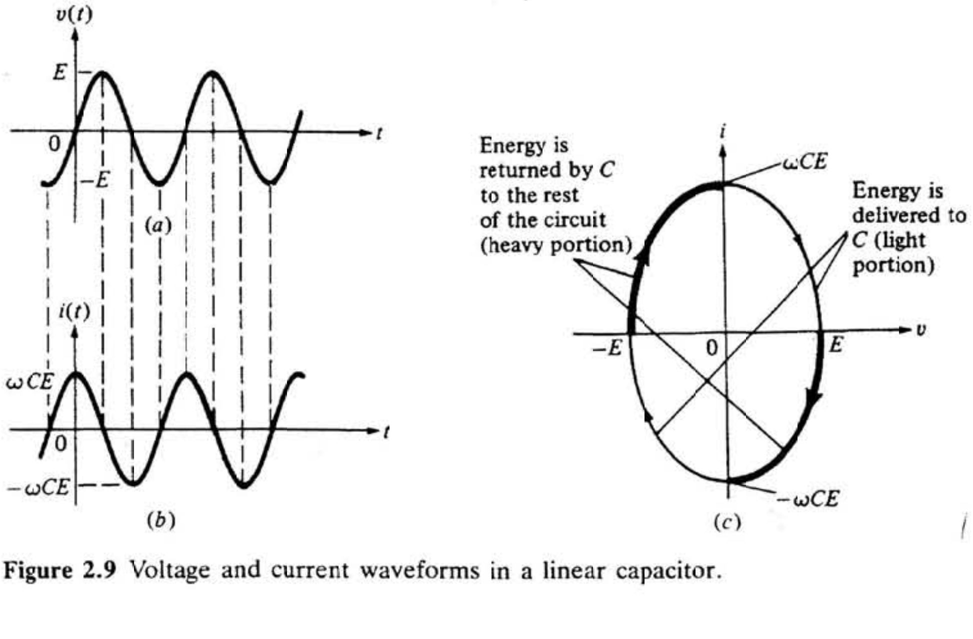
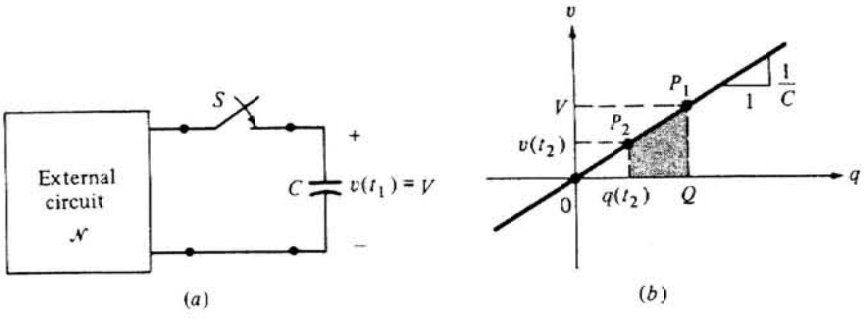


Figure 2.9 Voltage and current waveforms in a linear capacitor.



Since this represents the maximum amount of energy that could be *extracted* from the capacitor, it is natural to say that *an energy equal to*

$$\mathscr{E}_C(Q) = \frac{1}{2C} Q^2 = \tfrac{1}{2} CV^2 \tag{2.20}$$

is stored in a *linear capacitor* C having an initial voltage $v(t_1) = V$ or initial charge $q(t_1) = Q = CV$.

By duality, *an energy equal to*

$$\mathscr{E}_L(\phi) = \frac{1}{2L} \phi^2 = \tfrac{1}{2} LI^2 \tag{2.21}$$

is stored in a linear inductor L having an initial current $i(t_1) = I$ or initial flux $\phi(t_1) = \phi = LI$.

2.5 Energy Stored in a Nonlinear Time-Invariant Capacitor or Inductor[17]

Following the same reasoning for the linear case, we define the *energy stored in a nonlinear capacitor or inductor* to be equal *to the magnitude of* the *maximum energy* that can be extracted from the element at a given initial condition. Since the q-v (or ϕ-i) characteristic need not pass through the origin (Fig. 2.11*a*) and

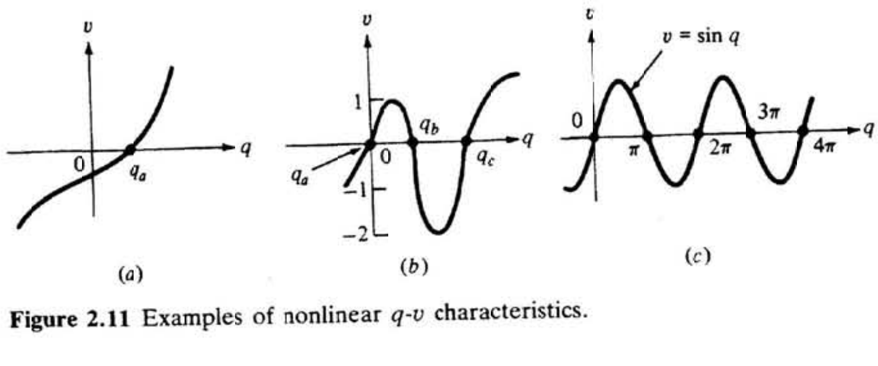


Figure 2.11 Examples of nonlinear q-v characteristics.

[17] May be omitted without loss of continuity.

can have several zero crossings (Fig. 2.11*b*) or infinitely many zero crossings (Fig. 2.11*c*), special care is needed to derive a formula for stored energy in the *nonlinear* case.

Suppose it is possible to find a point q_* on the q-v characteristic such that

$$\int_{q_*}^{q} \hat{v}(q')\,dq' \geq 0 \qquad \text{for all } -\infty < q < \infty \tag{2.22}$$

Now given any initial charge $q(t_1) = Q$, the energy entering a charge-controlled capacitor during $[t_1, t_2]$ is given by Eq. (2.17*a*):

$$\begin{aligned} w_C(t_1, t_2) &= \int_{Q}^{q(t_2)} \hat{v}(q)\,dq \\ &= \int_{Q}^{q_*} \hat{v}(q)\,dq + \int_{q_*}^{q(t_2)} \hat{v}(q)\,dq \end{aligned} \tag{2.23}$$

It follows from Eq. (2.22) that the *first term is negative* while the second term is positive if $q(t_2) \neq q_*$ in Eq. (2.23). Consequently, $w_C(t_1, t_2)$ is most negative when we choose $q(t_2) = q_*$, and the maximum energy that can be extracted is equal to $w_C(t_1, t_2) = \int_Q^{q_*} \hat{v}(q)\,dq < 0$. It follows from the lossless property that *the energy stored in a charge-controlled capacitor having an initial charge* $q(t_1) = Q$ *is equal* to

$$\boxed{\mathscr{E}_C(Q) = \int_{q_*}^{Q} \hat{v}(q)\,dq} \tag{2.24}$$

where q_* is *any* point satisfying Eq. (2.22). Note that $\mathscr{E}_C(Q) \geq 0$ for all Q in view of Eq. (2.22).

Since $\mathscr{E}_C(q_*) = 0$, it follows that no energy is stored when the initial charge is equal to q_* and the capacitor is therefore said to be initially relaxed. Consequently, any point q_* satisfying Eq. (2.22) is called a *relaxation point.*

For the q-v characteristic shown in Fig. 2.11, we find q_a to be the only relaxation point in Fig. 2.11*a* and q_c to be the only relaxation point in Fig. 2.11*b*. On the other hand, all points $q = \pm k2\pi$, $k = 0, 1, 2, \ldots$, qualify as relaxation points in Fig. 2.11*c* because each of these points satisfies Eq. (2.22).

By duality, the energy stored in a flux-controlled inductor having an initial flux $\phi(t_1) = \Phi$ is equal to

$$\boxed{\mathscr{E}_L(\Phi) = \int_{\phi_*}^{\Phi} \hat{i}(\phi)\,d\phi} \tag{2.25}$$

where ϕ_* is *any* relaxation point, namely,

$$\int_{\phi_*}^{\phi} \hat{i}(\phi)\, d\phi \geq 0 \qquad \text{for all } -\infty < \phi < \infty \tag{2.26}$$

Special case In the special case where the

q-v characteristic passes through the origin and satisfies

$$\int_0^q \hat{v}(q)\, dq \geq 0 \qquad -\infty < q < \infty \tag{2.27a}$$

the origin is a relaxation point and the stored energy is simply given by

$$\mathscr{E}_C(Q) = \int_0^Q \hat{v}(q)\, dq \tag{2.28a}$$

In this case, $\mathscr{E}_C(Q)$ is equal to the net area under the q-v characteristic from $q = 0$ to $q = Q$, as shown in Fig. 2.12a.

ϕ-i characteristic passes through the origin and satisfies

$$\int_0^\phi \hat{i}(\phi)\, d\phi > 0 \qquad -\infty < \phi < \infty \tag{2.27b}$$

the origin is a relaxation point and the stored energy is simply given by

$$\mathscr{E}_L(\Phi) = \int_0^\Phi \hat{i}(\phi)\, d\phi \tag{2.28b}$$

In this case, $\mathscr{E}_L(\Phi)$ is equal to the net area under the ϕ-i characteristic from $\phi = 0$ to $\phi = \Phi$, as shown in Fig. 2.12b.

REMARK The results presented throughout Sec. 2 are valid only if the capacitors and inductors are time-invariant. When the element is time-varying, additional energy is contributed by an external energy source which causes the time variation.

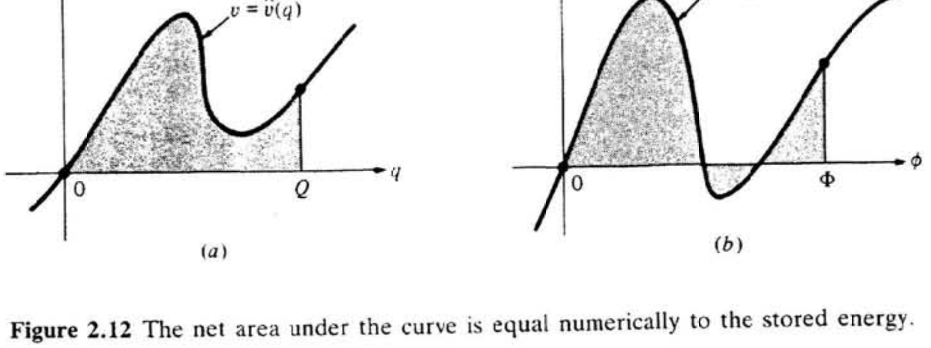


Figure 2.12 The net area under the curve is equal numerically to the stored energy.

Exercises

1. Verify that out of the three zero crossings in Fig. 2.11b, only q_C qualifies as a relaxation point.
2. Find all relaxation points associated with the Josephson junction defined earlier by Eq. (1.9b).
3. Prove that if a nonlinear capacitor or inductor has more than one relaxation point, then each point will give the *same* stored energy $\mathscr{E}_C(Q)$ or $\mathscr{E}_L(\Phi)$.

3 FIRST-ORDER LINEAR CIRCUITS

Circuits made of *one* capacitor (or one inductor), resistors, and independent sources are called *first-order circuits*. Note that "resistor" is understood in the broad sense: It includes controlled sources, gyrators, ideal transformers, etc.

In this section, we study first-order circuits made of linear time-invariant elements and independent sources. Any such circuit can be redrawn as shown in either Fig. 3.1a or b, where the one-port N is assumed to include all other elements (e.g., independent sources, resistors, controlled sources, gyrators, ideal transformers, etc.).[18]

Applying the *Thévenin-Norton equivalent one-port theorem* from Chap. 5, we can, in most instances, replace N by the equivalent circuit shown in Fig. 3.2a and b, respectively.

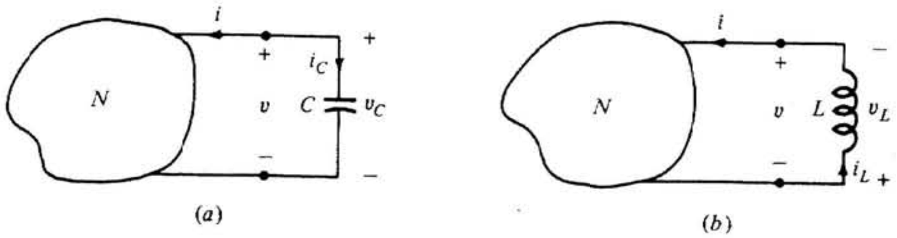


Figure 3.1 (a) First-order RC circuit. (b) First-order RL circuit.

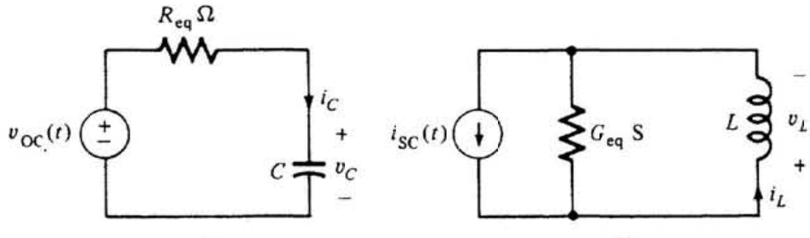


Figure 3.2 Equivalent first-order circuits.

[18] Without loss of generality, we draw v_L and i_L as shown in Fig. 3.1b so that $i_L = i$ (the dual of $v_C = v$ in Fig. 3.1a). This will guarantee the state equation (3.2b) will come out to be the dual of Eq. (3.2a).

Applying KVL we obtain

$$R_{eq} i_C + v_C = v_{OC}(t) \quad (3.1a)$$

Substituting $i_C = C\dot{v}_C$ and solving for $\dot{v}_C$, we obtain

$$\boxed{\dot{v}_C = -\frac{v_C}{R_{eq}C} + \frac{v_{OC}(t)}{R_{eq}C}} \quad (3.2a)$$

Applying KCL we obtain

$$G_{eq} v_L + i_L = i_{SC}(t) \quad (3.1b)$$

Substituting $v_L = L\dot{i}_L$ and solving for $\dot{i}_L$, we obtain

$$\boxed{\dot{i}_L = -\frac{i_L}{G_{eq}L} + \frac{i_{SC}(t)}{G_{eq}L}} \quad (3.2b)$$

When written in the above standard form, this *first*-order linear differential equation is called a *state equation* and the variable v_C (respectively, i_L) is called a *state variable*.

Given any *initial condition* $v_C(t_0)$ at any initial time t_0, our objective is to find the solution $v_C(t)$ for all $t \ge t_0$. We will show that $v_C(t)$ depends only on the initial condition $v_C(t_0)$ and the waveform $v_{OC}(\cdot)$ over $[t_0, t]$.

Once the solution $v_C(\cdot)$ is found, we can apply the *substitution theorem* from Chap. 5 and replace the capacitor in Fig. 3.1*a* by a voltage source $v_C(t)$.

Given any *initial condition* $i_L(t_0)$ at any initial time t_0, our objective is to find the solution $i_L(t)$ for all $t \ge t_0$. We will show that $i_L(t)$ depends only on the initial condition $i_L(t_0)$ and the waveform $i_{SC}(\cdot)$ over $[t_0, t]$.

Once the solution $i_L(\cdot)$ is found, we can apply the *substitution theorem* from Chap. 5 and replace the inductor in Fig. 3.1*b* by a current source $i_L(t)$.

The resulting equivalent circuit, being resistive, can then be solved using techniques developed in the preceding chapters.

In Sec. 3.1 we show that the solution of any first-order linear circuit can be found by inspection, provided N contains only *dc* sources. By repeated application of this "inspection method," Sec. 3.2 shows how the solution can be easily found if N contains only *piecewise-constant* sources. This method is then applied in Sec. 3.3 for finding the solution—called the *impulse response*—when the circuit is driven by an *impulse* $\delta(t)$. Finally, Sec. 3.4 gives an explicit integration formula for finding solutions under arbitrary excitations.

3.1 Circuits Driven by DC Sources

When N contains only *dc* sources, $v_{OC}(t) = v_{OC}$ and $i_{SC}(t) = i_{SC}$ are constants in Fig. 3.2 and in Eq. (3.2). Let us rewrite Eqs. (3.2*a*) and (3.2*b*) as follows:

State equation

$$\boxed{\dot{x} = -\frac{x}{\tau} + \frac{x(t_\infty)}{\tau}} \tag{3.3}$$

where

$$x \triangleq v_C \qquad x(t_\infty) \triangleq v_{OC} \qquad \tau \triangleq R_{eq}C \tag{3.4a}$$

for the RC circuit.

where

$$x \triangleq i_L \qquad x(t_\infty) \triangleq i_{SC} \qquad \tau \triangleq G_{eq}L \tag{3.4b}$$

for the RL circuit.

Given any initial condition $x = x(t_0)$ at $t = t_0$, Eq. (3.3) has a unique solution[19]

$$\boxed{x(t) - x(t_\infty) = [x(t_0) - x(t_\infty)] \exp \frac{-(t-t_0)}{\tau}} \tag{3.5}$$

which holds for *all* times t, i.e., $-\infty < t < \infty$. To verify that this is indeed the solution, simply substitute Eq. (3.5) into Eq. (3.3) and show that both sides are identical. Observe that at $t = t_0$, both sides of Eq. (3.5) reduce to $x(t_0) - x(t_\infty)$. Note also that the solution given by Eq. (3.5) is valid whether τ is positive or negative.

The solution (3.5) is determined by only three parameters: $x(t_0)$, $x(t_\infty)$, and τ. We call them *initial state*, *equilibrium state*, and *time constant*, respectively. To see why $x(t_\infty)$ is called the equilibrium state, note that if $x(t_0) = x(t_\infty)$, then Eq. (3.3) gives $\dot{x}(t_0) = 0$ and thus $x(t) = x(t_\infty)$ for all t. Hence the circuit remains "motionless," or in equilibrium.

Since the "inspection method" to be developed in this section depends crucially on the ability to sketch the *exponential* waveform quickly, the following properties are extremely useful.

A. Properties of exponential waveforms Depending on whether τ is positive or negative, the exponential waveform in Eq. (3.5) tends either to a constant or to infinity, as the time t tends to infinity. Hence, it is convenient to consider these two cases separately.

$\tau > 0$ (*Stable case*) When $\tau > 0$, Eq. (3.5) shows that $x(t) - x(t_\infty)$, i.e., the distance between the present state and the equilibrium state $x(t_\infty)$, *decreases* exponentially: For all initial states, the solution $x(t)$ is *sucked* into the equilibrium and $|x(t) - x(t_\infty)|$ decreases exponentially with a time constant τ.

[19] We write $x(t_\infty)$ on the left side to make it easier to remember this important formula.

The solution (3.5) for $\tau > 0$ is sketched in Fig. 3.3 for two different initial states $\tilde{x}(t_0)$ and $x(t_0)$ for $t \geq t_0$. Observe that because the time constant τ is positive,

$$x(t) \to x(t_\infty) \qquad \text{as } t \to \infty \tag{3.6}$$

Thus, when $\tau > 0$, we say the equilibrium state $x(t_\infty)$ is *stable* because any initial deviation $x(t_0) - x(t_\infty)$ decays exponentially and $x(t) \to x(t_\infty)$ as $t \to \infty$.

The *exponential* waveforms in Fig. 3.3 can be accurately sketched using the following observations:

1. The *tangent* at $t = t_0$ passes through the point $[t_0, x(t_0)]$ and the point $[t_0 + \tau, x(t_\infty)]$.
2. After one time constant τ, the distance between $x(t)$ and $x(t_\infty)$ decreases approximately by 63 percent of the initial distance $|x(t_0) - x(t_\infty)|$.
3. After five time constants, $x(t)$ practically attains the *steady-state* value $x(t_\infty)$. (Indeed, $e^{-5} \approx 0.007$.)

Example 1 (Op-amp voltage follower: Stable configuration) Consider the op-amp circuit shown in Fig. 3.4*a*. Using the ideal op-amp model, this circuit was analyzed earlier in Sec. 2.2 (Fig. 2.1) of Chap. 4. Assuming the switch is closed at $t = 0$, we found $v_0(t) = v_{in}(t) = 10$ V for $t \geq 0$.

In practice, the output is observed to reach the 10-V solution after a small but finite time. In order to predict the *transient* behavior before the

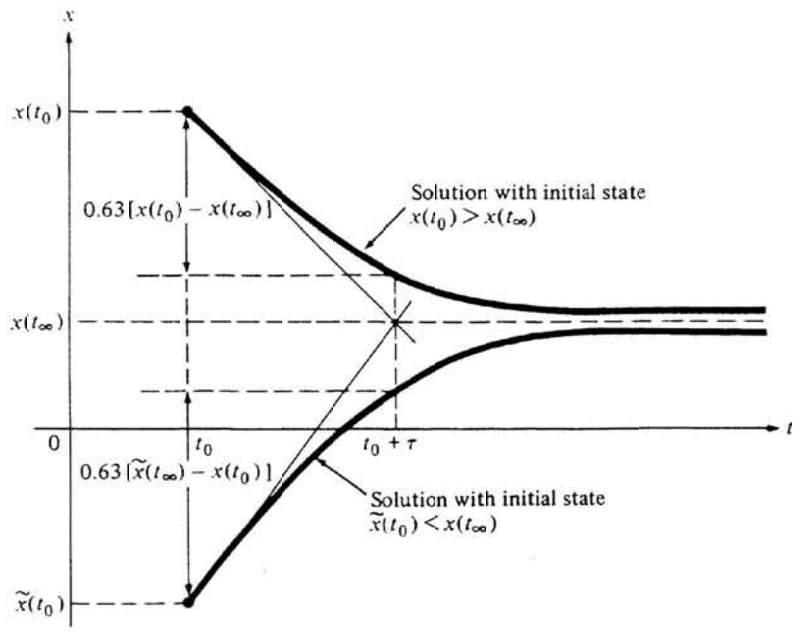


Figure 3.3 The solution tends to the equilibrium state $x(t_\infty)$ as $t \to \infty$ when the time constant τ is *positive*.

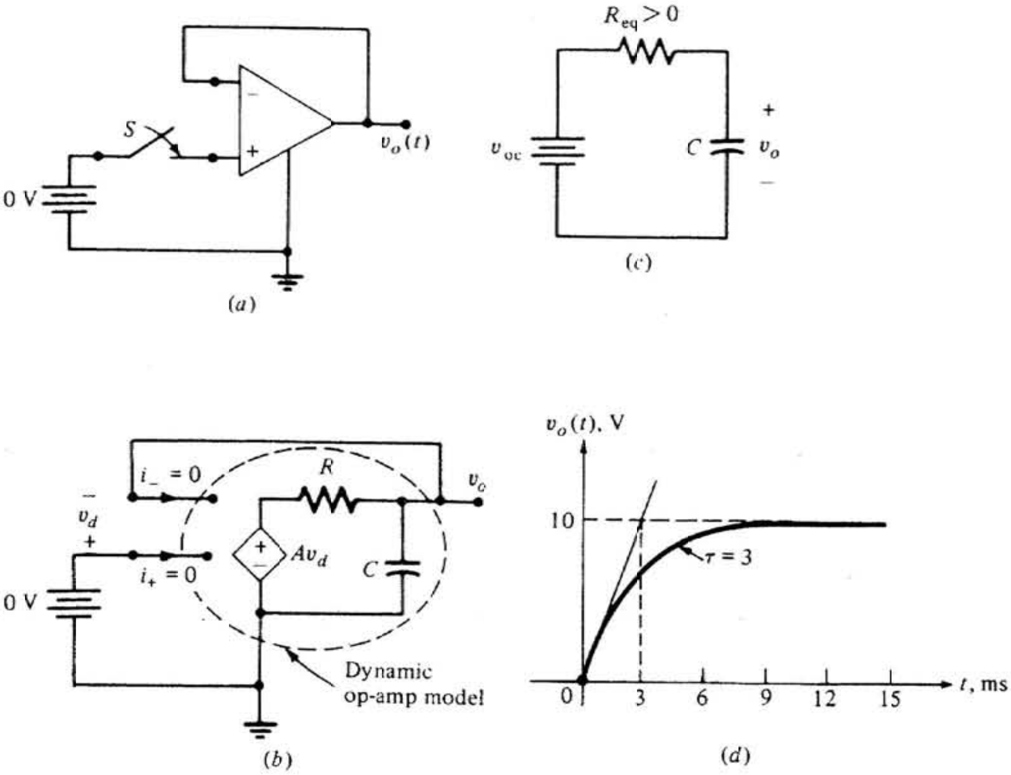


Figure 3.4 Transient behavior of op-amp voltage follower circuit.

equilibrium is reached, let us replace the op amp in Fig. 3.4*a* by the *dynamic* circuit model shown in Fig. 3.4*b*.[20] To analyze this first-order circuit, we extract the capacitor and replace the remaining circuit by its Thévenin equivalent as shown in Fig. 3.4*c*, where

$$R_{eq} = \frac{R}{A+1} \approx \frac{R}{A} \qquad \text{since } A \gg 1 \tag{3.7}$$

$$v_{OC} = \frac{10A}{A+1} \approx 10 \qquad \text{since } A \gg 1 \tag{3.8}$$

Assuming $A = 10^5$, $R = 100\,\Omega$, and $C = 3\,\text{F}$, we obtain $R_{eq} \approx 10^{-3}\,\Omega$ and $v_{OC} \approx 10\,\text{V}$. Consequently, the time constant and equilibrium state are given respectively by $\tau = R_{eq}C = 3\,\text{ms}$ and $v_0(t_\infty) = v_{OC} \approx 10\,\text{V}$. Assuming the capacitor is initially uncharged, i.e., $v_0(0) = 0$, the resulting output voltage can be easily sketched as shown in Fig. 3.4*d*. Note that after five time constants or 15 ms, the output is practically equal to 10 V.

[20] A more realistic *dynamic* op-amp circuit model for high-frequency applications would require several linear capacitors. The one-capacitor model chosen in Fig. 3.4, though not valid in general, does predict the transient behavior correctly for the voltage follower circuit.

$\tau < 0$ (*Unstable case*) When $\tau < 0$, Eq. (3.5) shows that the quantity $x(t) - x(t_\infty)$ *increases* exponentially for all initial states, i.e., the solution $x(t)$ *diverges* from the equilibrium, and $x(t) - x(t_\infty)$ *increases* exponentially with a time constant τ.

The solution (3.5) for $\tau < 0$ is sketched in Fig. 3.5 for two different initial states $x(t_0)$ and $\tilde{x}(t_0)$.

Observe that, since the time constant τ is negative, as $t \to \infty$, $x(t) \to \infty$ if $x(t_0) > x(t_\infty)$, and $x(t) \to -\infty$ if $x(t_0) < x(t_\infty)$.

Thus, when $\tau < 0$, we say the equilibrium state $x(t_\infty)$ is *unstable* because any initial deviation $x(t_0) - x(t_\infty)$ *grows* exponentially with time and $|x(t)| \to \infty$ as $t \to \infty$.

However, if we run time *backward*, then

$$x(t) \to x(t_\infty) \qquad \text{as } t \to -\infty \tag{3.9}$$

Consequently, $x(t_\infty)$ can be interpreted as a *virtual equilibrium state*.

The exponential waveform in Fig. 3.5 can be accurately sketched using the following observations:

1. The *tangent* at $t = t_0$ passes through the point $[t_0, x(t_0)]$ and the point $[t_0 - |\tau|, x(t_\infty)]$.
2. At $t = t_0 + |\tau|$, the distance $|x(t_0 + |\tau|) - x(t_\infty)|$ is approximately 1.72 times the initial distance $|x(t_0) - x(t_\infty)|$.

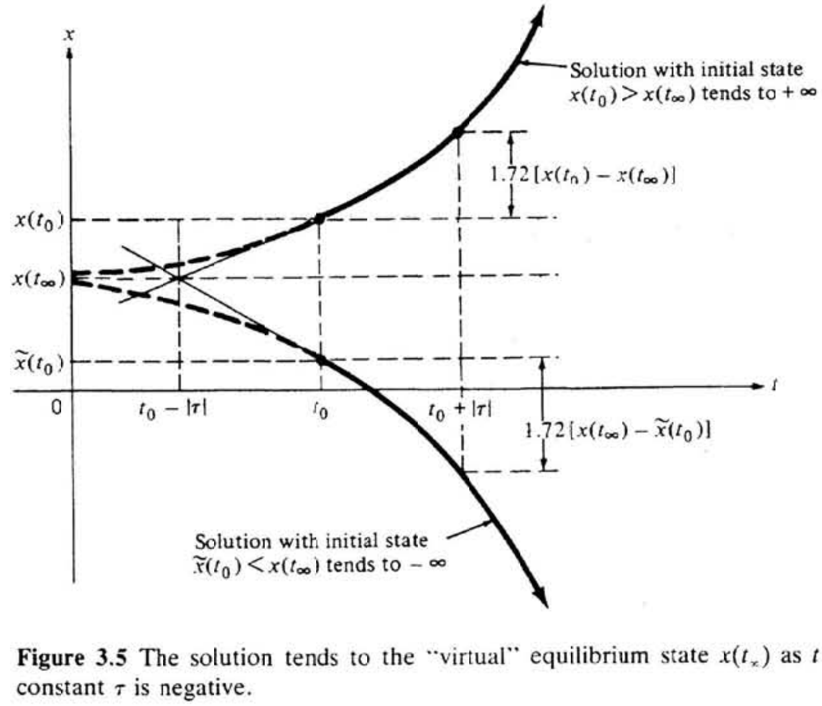


Figure 3.5 The solution tends to the "virtual" equilibrium state $x(t_\infty)$ as $t \to -\infty$ when the time constant τ is negative.

Example 2 (Op-amp voltage follower: Unstable configuration) The op-amp circuit in Fig. 3.6*a* is identical to that of Fig. 3.4*a* except for an interchange between the inverting (−) and the noninverting (+) terminals. Using the ideal op-amp model in the linear region, we would obtain exactly the same answer as before, namely, $v_0 = 10$ V for $t \geq 0$, provided $E_{sat} > 10$ V. Let us see what happens if the op amp is replaced by the dynamic model adopted earlier in Fig. 3.4*b*. The resulting circuit shown in Fig. 3.6*b* resembles that of Fig. 3.4*b* except for an important difference: The polarity of v_d is now reversed. The parameters in the Thévenin equivalent circuit now become

$$R_{eq} = -\frac{R}{A-1} \approx -\frac{R}{A} \qquad \text{since } A \gg 1 \tag{3.10}$$

$$v_{OC} = \frac{10A}{A-1} \approx 10 \qquad \text{since } A \gg 1 \tag{3.11}$$

Assuming the same parameter values as in Example 1, we obtain $R_{eq} \approx -10^{-3}\ \Omega$ and $v_{OC} \approx 10$ V. Consequently, the time constant and equilibrium state are given respectively by $\tau \approx -3$ ms and $v_0(t_\infty) \approx 10$ V. Assuming $v_0(0) = 0$ as in Example 1, the resulting output voltage can be easily sketched as shown in Fig. 3.6*d*.

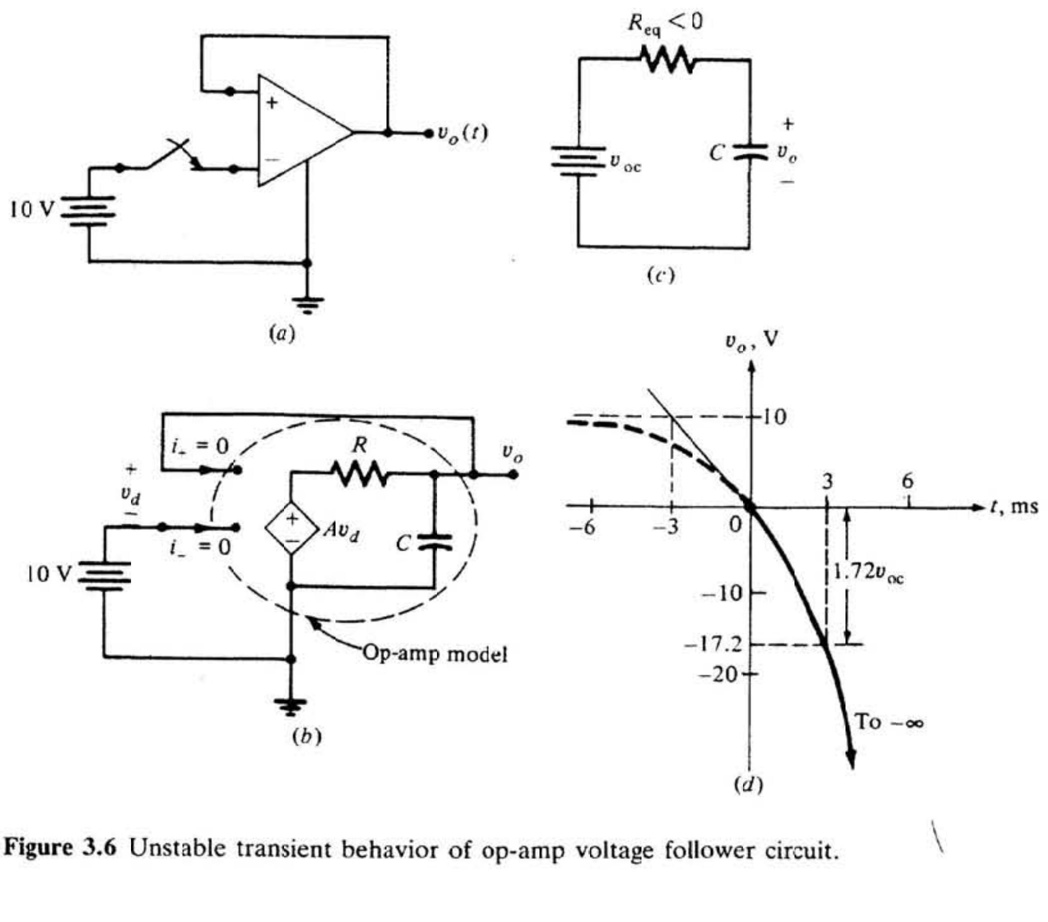


Figure 3.6 Unstable transient behavior of op-amp voltage follower circuit.

Note that the solution differs drastically from that of Fig. 3.4*d*: It tends to $-\infty$! Of course, in practice, when $v_0(t)$ decreases to $-E_{sat}$, the op-amp negative saturation voltage, the solution would remain constant at $-E_{sat}$. Clearly, this circuit would not function as a voltage follower in practice.

B. Elapsed time formula We will often need to calculate the time interval between two prescribed points on an exponential waveform. For example, to obtain the actual solution waveform for the circuit in Fig. 3.6, we need to calculate the time that elapsed when v_0 decreases from $v_0 = 0$ to $v_0 = -15$ V (assuming $E_{sat} = 15$ V) in Fig. 3.6*d*.

Given any two points $[(t_j, x(t_j))$ and $(t_k, x(t_k))]$ on an exponential waveform (see, e.g., Figs. 3.3 and 3.5), the time it takes to go from $x(t_j)$ to $x(t_k)$ is given by

Elapsed time formula

$$\boxed{t_k - t_j = \tau \ln \frac{x(t_j) - x(t_\infty)}{x(t_k) - x(t_\infty)}} \tag{3.12}$$

To derive Eq. (3.12), let $t = t_j$ and $t = t_k$ in Eq. (3.5), respectively:

$$x(t_j) - x(t_\infty) = [x(t_0) - x(t_\infty)] \exp \frac{-(t_j - t_0)}{\tau} \tag{3.13}$$

$$x(t_k) - x(t_\infty) = [x(t_0) - x(t_\infty)] \exp \frac{-(t_k - t_0)}{\tau} \tag{3.14}$$

Dividing Eq. (3.13) by Eq. (3.14) and taking the logarithm on both sides, we obtain Eq. (3.12).

REMARK The above derivation does not depend on whether τ is positive or negative.

C. Inspection method (First-order linear time-invariant circuits driven by dc sources) Consider first the first-order *RC* circuit in Fig. 3.1*a* where all independent sources inside N are dc sources. Equation (3.5) gives us the voltage waveform across the capacitor, namely,

$$v_C(t) = v_C(t_\infty) + [v_C(t_0) - v_C(t_\infty)] \exp \frac{-(t - t_0)}{\tau} \tag{3.15}$$

Suppose we replace the capacitor by a *voltage source* defined by Eq. (3.15). Assuming the resulting resistive circuit is uniquely solvable, we can apply the *substitution theorem* to conclude that the solution inside N of the resistive circuit is identical to that of the first-order *RC* circuit.

Let v_{jk} denote the voltage across any pair of nodes, say ⓙ and ⓚ and assume that N contains α independent dc voltage sources $V_{s1}, V_{s2}, \ldots, V_{s\alpha}$ and

β independent dc current sources $I_{s1}, I_{s2}, \ldots, I_{s\beta}$. Applying the *superposition theorem* from Chap. 5, we know the solution $v_{jk}(t)$ is given by an expression of the form

$$v_{jk}(t) = H_0 v_C(t) + \sum_{j=1}^{\alpha} H_j V_{sj} + \sum_{j=1}^{\beta} K_j I_{sj} \tag{3.16}$$

where H_0, H_j, and K_j are *constants* (which depend on element values and circuit configuration). Substituting Eq. (3.15) for $v_C(t)$ in Eq. (3.16) and rearranging terms, we obtain

$$v_{jk}(t) - v_{jk}(t_\infty) = [v_{jk}(t_0) - v_{jk}(t_\infty)] \exp \frac{-(t - t_0)}{\tau} \tag{3.17}$$

where

$$v_{jk}(t_\infty) \triangleq H_0 v_C(t_\infty) + \sum_{j=1}^{\alpha} H_j V_{sj} + \sum_{j=1}^{\beta} K_j I_{sj} \tag{3.18}$$

and

$$v_{jk}(t_0) \triangleq H_0 v_C(t_0) + \sum_{j=1}^{\alpha} H_j V_{sj} + \sum_{j=1}^{\beta} K_j I_{sj} \tag{3.19}$$

Since Eq. (3.17) has exactly the same form as Eq. (3.5), and since nodes ⓙ and ⓚ are arbitrary, we conclude that:

The voltage $v_{jk}(t)$ across any pair of nodes in a first-order RC circuit driven by dc sources is an exponential waveform having the same time constant τ as that of $v_C(t)$.

By the same reasoning, we conclude that:

The current $i_j(t)$ in any branch j of a first-order RC circuit driven by dc sources is an exponential waveform having the same time constant τ as that of $v_C(t)$.

It follows from duality that the voltage $v_{jk}(t)$ across any pair of nodes, or the current $i_j(t)$ in any branch j of a first-order RL circuit driven by dc sources is an exponential waveform having the same time constant τ as that of $i_L(t)$.

The above "exponential solution waveform" property, of course, assumes that the first-order circuit is *not* degenerate, i.e., that it is uniquely solvable and that $0 < |\tau| < \infty$.

It is important to remember that all voltage and current waveforms in a given first-order circuit have the *same time constant* τ as defined in Eq. (3.4).

Moreover, as we approach the *equilibrium*, i.e., when $t \to +\infty$ (if $\tau > 0$) or $t \to -\infty$ (if $\tau < 0$), *the capacitor current and the inductor voltage both tend to zero*. This follows from Figs. 3.3 and 3.5, $i_C = C\dot{v}_C$, and $v_L = L\dot{i}_L$.

Since an exponential waveform is uniquely determined by only three parameters [initial state $x(t_0)$, equilibrium state $x(t_\infty)$, and time constant τ], the following "inspection method" can be used to find the voltage solution $v_{jk}(t)$

across any pair of nodes j and k or the current solution $i_j(t)$ in any branch j, in any uniquely solvable linear first-order circuit driven by dc sources:

RC circuit: given $v_C(t_0)$.	*RL circuit*: given $i_L(t_0)$.
1. Replace the *capacitor* by a dc *voltage source* with a terminal voltage equal to $v_C(t_0)$. Label the voltage across node-pair j, k as $v_{jk}(t_0)$ and the current i_j as $i_j(t_0)$. Solve the resulting resistive circuit for $v_{jk}(t_0)$ or $i_j(t_0)$.	1. Replace the *inductor* by a dc *current source* with a terminal current equal to $i_L(t_0)$. Label the voltage across node-pair j, k as $v_{jk}(t_0)$ and the current i_j as $i_j(t_0)$. Solve the resulting resistive circuit for $v_{jk}(t_0)$ or $i_j(t_0)$.
2. Replace the *capacitor* by any *open circuit*. Label the voltage across node-pair j, k as $v_{jk}(t_\infty)$ and the current i_j as $i_j(t_\infty)$. Solve for $v_{jk}(t_\infty)$ or $i_j(t_\infty)$.	2. Replace the *inductor* by a *short circuit*. Label the voltage across node-pair j, k as $v_{jk}(t_\infty)$ and the current i_j as $i_j(t_\infty)$. Solve for $v_{jk}(t_\infty)$ or $i_j(t_\infty)$.
3. Find the Thévenin equivalent circuit of N. Calculate the time constant $\tau = R_{eq}C$.	3. Find the Norton equivalent circuit of N. Calculate the time constant $\tau = G_{eq}L$.
4. If $0<\lvert\tau\rvert<\infty$, use the above three parameters to sketch the *exponential* solution waveform.	4. If $0<\lvert\tau\rvert<\infty$, use the above three parameters to sketch the *exponential* solution waveform.

REMARKS

1. The above inspection method eliminates the usual step of writing the differential equation: It reduces each step to resistive circuit calculations.
2. The above method is valid only if the circuit is uniquely solvable. For example, if the one-port N in Fig. 3.1 does not have a Thévenin *and* Norton equivalent circuit, it is not uniquely solvable.
3. The above method assumes the circuit is *not* degenerate in the sense that $0<\lvert\tau\rvert<\infty$. This means that $R_{eq} \neq 0$ and is finite in Fig. 3.2*a*, and that $G_{eq} \neq 0$ and is finite in Fig. 3.2*b*.

3.2 Circuits Driven by Piecewise-Constant Signals

Consider next the case where the independent sources in N of Fig. 3.1 are piecewise-constant for $t > t_0$. This means that the semi-infinite time interval $t_0 \le t < \infty$ can be partitioned into *subintervals* $[t_j, t_{j+1})$, $j = 1, 2, \ldots$, such that all sources assume a *constant* value during each subinterval. Hence, we can analyze the circuit as a sequence of first-order circuits driven by dc sources, each one analyzed separately by the inspection method. Since the circuit remains unchanged except for the sources, the *time constant* τ remains unchanged throughout the analysis.

The *initial state* $x(t_0)$ and *equilibrium state* $x(t_\infty)$ will of course vary from one subinterval to another. Although the same procedure holds in the determination of $x(t_\infty)$, one must be careful in calculating the *initial value* at the beginning of each subinterval t_j because at least one source changes its value *discontinuously* at each boundary time t_j between two consecutive subintervals. In general, $x(t_j-) \neq x(t_j+)$, where the $-$ and $+$ denote the *limit* of $x(t)$ as $t \to t_j$ *from the left* and *from the right*, respectively. The *initial value* to be used in the calculation during the subinterval $[t_j, t_{j+1})$ is $x(t_j+)$.

Although in general both $v_{jk}(t)$ and $i_j(t)$ can jump, the "continuity property" in Sec. 2.2 guarantees that in the usual case where the capacitor current (respectively, inductor voltage) waveform is bounded, the capacitor voltage (respectively, inductor current) waveform is a *continuous* function of time and therefore cannot jump. This property is the key to finding the solution by inspection, as illustrated in the following examples.

Example 1 Consider the *RC* circuit shown in Fig. 3.7*a*: $v_s(\cdot)$ is given by Fig. 3.7*a* and $v_C(0) = 0$. Our objective is to find $i_C(t)$, $v_C(t)$, and $v_R(t)$ for

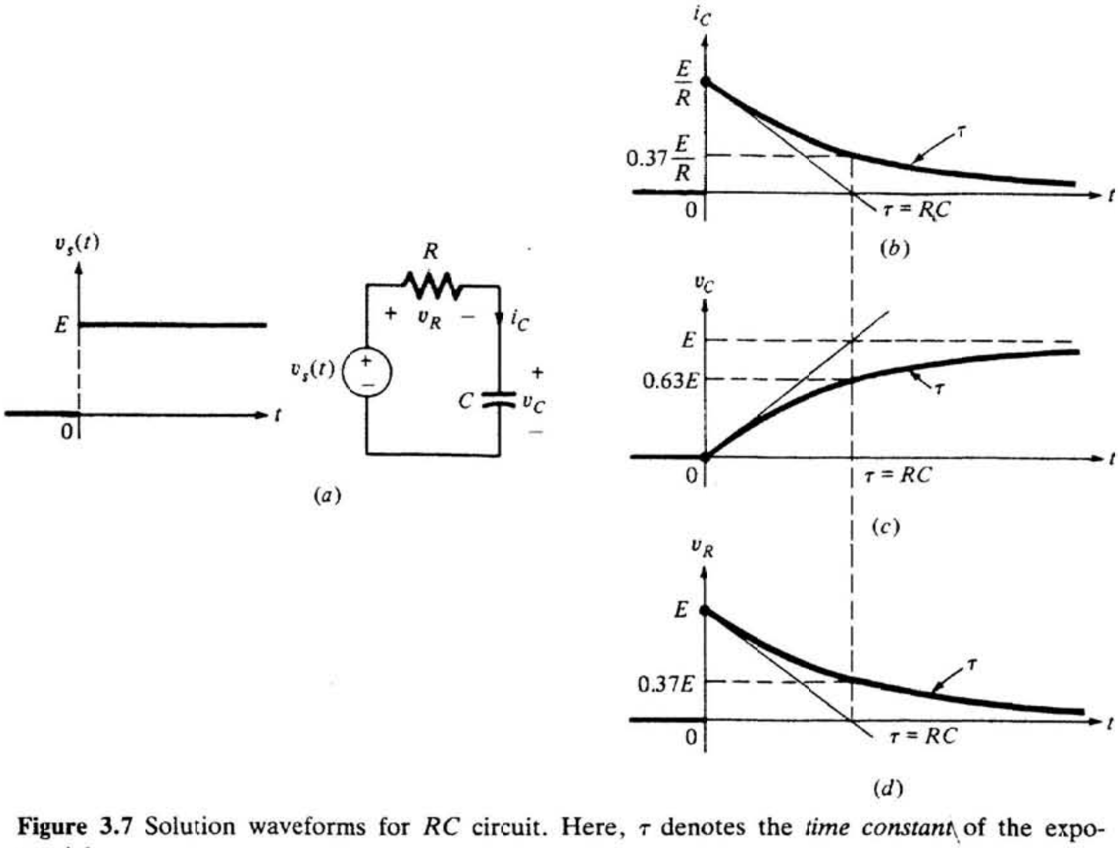


Figure 3.7 Solution waveforms for *RC* circuit. Here, τ denotes the *time constant* of the exponential.

$t \geq 0$ by *inspection*. Since $v_C(0) = 0$ and $v_s(t) = 0$ for $t \leq 0$, it follows that $i_C(t) = v_C(t) = V_R(t) = 0$ for $t \leq 0$.

The solution waveforms for $t > 0$ consists of *exponentials* with a time constant $\tau = RC$. At $t = 0+$, using the continuity property, we have $v_C(0+) = v_C(0-) = 0$. Therefore, $v_R(0+) = v_s(0+) - v_C(0+) = E$ and $i_C(0+) = v_R(0+)/R = E/R$. To find the equilibrium state, we *open* the capacitor and find $i_C(t_\infty) = 0$, $v_C(t_\infty) = E$, and $v_R(t_\infty) = 0$.

These three pieces of information allow us to sketch $i_C(t)$, $v_C(t)$, and $v_R(t)$ for $t \geq 0$ as shown in Fig. 3.7*b*, *c*, and *d*, respectively. Note that $i_C(t) = C\,dv_C(t)/dt$ and $v_R(t) + v_C(t) = E$ for $t \geq 0$, as they should. Observe also that whereas $v_R(t)$ is *discontinuous* at $t = 0$, $v_C(t)$ is *continuous* for all t, as expected.

REMARKS

1. The circuit in Fig. 3.7 is often used to model the situation where a dc voltage source is suddenly connected across a resistive circuit which normally draws a zero-input current. The linear capacitor in this case is used to model the small *parasitic* capacitance between the connecting wires. Without this capacitor, the input voltage would be identical to $v_s(t)$. However, in practice, a "transient" is always observed and the circuit in Fig. 3.7*a* represents a more realistic situation. In this case, the time constant τ gives a measure of how "fast" the circuit can respond to a step input. Such a measure is of crucial importance in the design of high-speed circuits, say in computers, measuring equipment, etc.
2. Since the term *time constant* is meaningful only for *first-order* circuits, a more general measure of "response speed" called the *rise time* is used in specifying practical equipments.

The *rise time* t_r is defined as the time it takes the output waveform to rise from *10 percent* to *90 percent* of the steady-state value after application of a step input.

For first-order circuits, the following simple relationship between t_r and τ follows directly from Eq. (3.12):

Rise time

$$\boxed{t_r = \tau \ln \frac{0.1E - E}{0.9E - E} = \tau \ln 9 \approx 2.2\tau} \tag{3.20}$$

Example 2 Consider the *RL* circuit shown in Fig. 3.8*a*, driven by a periodic square-wave current source in Fig. 3.8*b*. Our objective is to find $i_o(t)$ through the resistor when (*a*) $R = 10\,\text{k}\Omega$, $L = 1$ mH and (*b*) $R = 1\,\text{k}\Omega$, $L = 10$ mH.

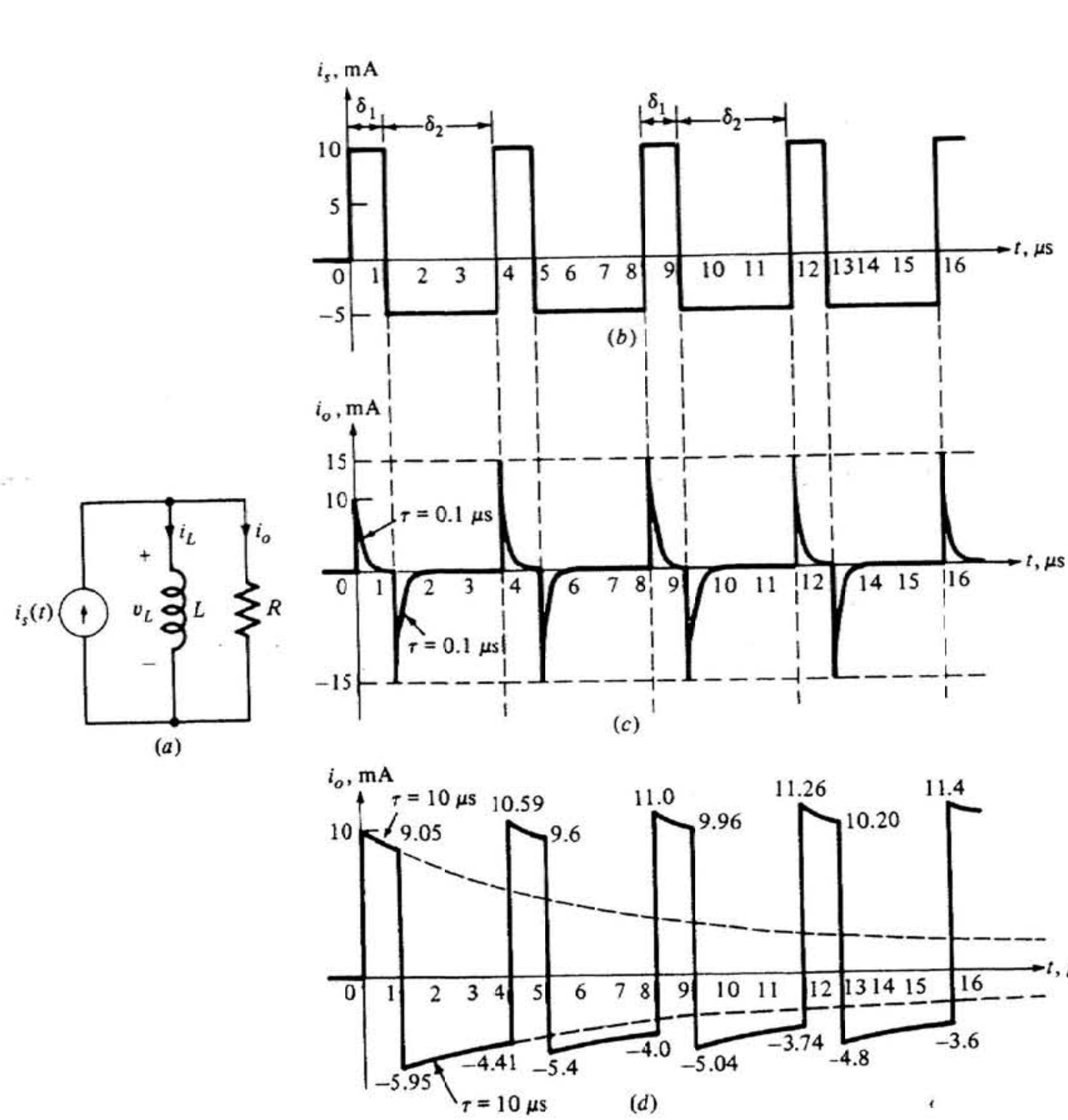


Figure 3.8 (*a*) *RL* circuit. (*b*) Input current waveform with $\delta_1 = 1\ \mu$s and $\delta_2 = 3\ \mu$s. (*c*) Output current waveform when $\tau \ll \delta_1$. (*d*) Output current waveform when $\tau \gg \delta_1$.

(*a*) *Small time constant case*: $\tau = GL = L/R = 0.1\ \mu$s. Since $\tau \ll \delta_1 = 10\tau$, the exponential waveform solution in each subinterval of width δ_1 or δ_2 will have essentially reached its steady state and we only need to calculate $i_o(t)$ over one period. In other words, the solution is periodic for all practical purposes.

Since $i_s(t) = 0$ for $t \le 0$, the inductor is in equilibrium and can be replaced by a *short circuit* at $t = 0-$ so that $i_L(0+) = i_L(0-) = 0$. Hence $i_o(0+) = i_s(0+) - i_L(0+) = 10 - 0 = 10$ mA.

To find $i_o(t_\infty)$ for the circuit during the subinterval $[0, \delta_1)$, we replace the inductor by a short circuit and obtain $i_L(t_\infty) = 10$ mA and $i_o(t_\infty) = 0$.

At $t = \delta_1 = 1\ \mu s$, $i_L(\delta_1+) = i_L(\delta_1-) = 10$ mA. Hence $i_o(\delta_1+) = i_s(\delta_1+) - i_L(\delta_1+) = -5 - 10 = -15$ mA. Hence i_o jumps at $t = \delta_1$ from 0 to -15 mA.

To find $i_o(t_\infty)$ for the circuit during the subinterval $[\delta_1, \delta_1 + \delta_2)$, we replace the inductor by a short circuit again and obtain $i_o(t_\infty) = 0$.

At $t = \delta_1 + \delta_2 = 4\ \mu s$, $i_o(t)$ jumps again from 0 to 15 mA, and the solution repeats itself thereafter, as shown in Fig. 3.8*c*.

(*b*) *Large time constant case*: $\tau = 10\ \mu s$. Since $\tau \gg \delta_1 = 0.1\tau$, the exponential waveform does not have enough time to reach a steady state during each subinterval. Consequently, the solution $i_o(t)$ is *not* periodic and we will have to partition $0 \le t < \infty$ into *infinitely* many subintervals $[0, \delta_1)$, $[\delta_1, \delta_1 + \delta_2)$, $[\delta_1 + \delta_2, 2\delta_1 + \delta_2)$, ... We will see, however, that $i_o(t)$ will tend to a periodic waveform after a few periods.

Starting at $t = 0$ as in (*a*), we find $i_o(0+) = 10$ mA and $i_o(t_\infty) = 0$. The exponential solution is drawn in a solid line during $0 \le t < \delta_1$ and in a dotted line thereafter in Fig. 3.8*d* to emphasize the relative magnitudes of τ and δ_1.

To determine $i_o(\delta_1+) = i_o(1+)$, it is necessary to write the solution $i_o(t) = 10\exp(-t/10)$ in order to calculate $i_o(1-) = 9.05$ mA. This gives us $i_L(1-) = i_s(1-) - i_o(1-) = 10 - 9.05 = 0.95$ mA. Since $i_L(1+) = i_L(1-) = 0.95$ mA, $i_o(1+) = i_s(1+) - i_L(1+) = -5 - 0.95 = -5.95$ mA. Hence $i_o(t)$ jumps from 9.05 to -5.95 mA at $t = 1\ \mu s$, as shown in Fig. 3.8*d*.

Again, the exponential solution during $[1, 4)$ has not reached steady state when $i_s(t)$ changes from -5 to 10 mA at $t = 4\ \mu s$. To calculate $i_o(t)$ at $t = 4+$, it is necessary to write the solution $i_o(t) = -5.95\exp\{-[(t-1)/10]\}$ and obtain $i_o(4-) = -4.41$ mA. This gives $i_L(4+) = i_L(4-) = i_s(4-) - i_o(4-) = -5 - (-4.41) = -0.59$ mA and $i_o(4+) = i_s(4+) - i_L(4+) = 10 - (-0.59) = 10.59$ mA. Hence $i_o(t)$ jumps from -4.41 to 10.59 mA at $t = 4\ \mu s$, as shown in Fig. 3.8*d*.

Repeating the above procedure, we find $i_o(t)$ jumps from 9.6 to -5.4 mA at $t = 5\ \mu s$, from -4.0 to 11.0 mA at $t = 8\ \mu s$, from 9.96 to -5.04 mA at $t = 9\ \mu s$, from -3.74 to 11.26 mA at $t = 12\ \mu s$, from 10.20 to -4.8 mA at $t = 13\ \mu s$, and from -3.6 to 11.4 mA at $t = 16\ \mu s$, etc., as shown in Fig. 3.8*d*.

It is clear from Fig. 3.8*d* that $i_o(t)$ is tending toward a periodic waveform. To determine this periodic waveform, note that if we let I_o denote the "peak" value of each "falling" exponential segment in Fig. 3.8*d* (e.g., $I_o = 10$, 10.59, 11, 11.26, and 11.4 mA at $t = 0$, 4, 8, 12, 16 μs, etc.) then this periodic waveform must satisfy the following *periodicity condition*:

$$I_o \exp\frac{-\delta_1}{\tau} - 15\exp\frac{-\delta_2}{\tau} + 15 = I_o$$

where $\delta_1 = 1\ \mu s$, $\delta_2 = 3\ \mu s$, and $\tau = 10\ \mu s$. The solution of this equation gives one point on the periodic solution, namely, the *peak value*.

Exercise
(*a*) Calculate the peak value I_o from the periodicity condition.
(*b*) Specify the initial inductor current $i_L(0)$ in Fig. 3.8*a* so that the solution $i_o(t)$ is periodic for $t \geq 0$.
(*c*) Sketch this periodic solution.

3.3 Linear Time-Invariant Circuits Driven by an Impulse

Consider the *RC* circuit shown in Fig. 3.9*a* and the *RL* circuit shown in Fig. 3.9*b*. Let the input voltage source $v_s(t)$ and input current source $i_s(t)$ be a square pulse $p_\Delta(t)$ of width Δ and height $1/\Delta$, as shown in Fig. 3.9*c*. Assuming *zero initial* state [i.e., $v_C(0-) = 0$, $i_L(0-) = 0$], the response voltage $v_C(t)$ and current $i_L(t)$ are given by the same waveform shown in Fig. 3.9*d*, where $\tau = RC$ for the *RC* circuit and $\tau = GL$ for the *RL* circuit, and

$$h_\Delta(\Delta) \triangleq \frac{1-\exp(-\Delta/\tau)}{\Delta} \triangleq \frac{f(\Delta)}{g(\Delta)} \tag{3.21}$$

The *input* and *response* corresponding to $\Delta = 1, \frac{1}{2}$, and $\frac{1}{3}$ s are shown in Fig. 3.9*e* and *f*, respectively. Note that as $\Delta \to 0$, $p_\Delta(t)$ tends to the unit *impulse* shown in Fig. 3.9*g* [recall Eq. (2.8)], namely,

$$\lim_{\Delta \to 0} p_\Delta(t) = \delta(t) \tag{3.22}$$

Note also that the "peak" value $h_\Delta(\Delta)$ of the response waveform in Fig. 3.9*d* increases as Δ decreases. To obtain the limiting value of $h_\Delta(\Delta)$ as $\Delta \to 0$, we apply L'Hospital's rule:

$$\lim_{\Delta \to 0} h_\Delta(\Delta) = \lim_{\Delta \to 0} \frac{f'(\Delta)}{g'(\Delta)} = \lim_{\Delta \to 0} \frac{(1/\tau)\exp(-\Delta/\tau)}{1} = \frac{1}{\tau} \tag{3.23}$$

Hence, the response waveform in Fig. 3.9*f* tends to the exponential waveform

$$h(t) = \begin{cases} \dfrac{1}{\tau}\exp\left(-\dfrac{t}{\tau}\right) & t > 0 \\ 0 & t < 0 \end{cases} \tag{3.24}$$

shown in Fig. 3.9*h*. Using the *unit step function* $1(t)$ defined earlier in Eq. (2.6), we can rewrite Eq. (3.24) as follows:

$$h(t) = \frac{1}{\tau}\exp\left(\frac{-t}{\tau}\right)1(t) \tag{3.24$'$}$$

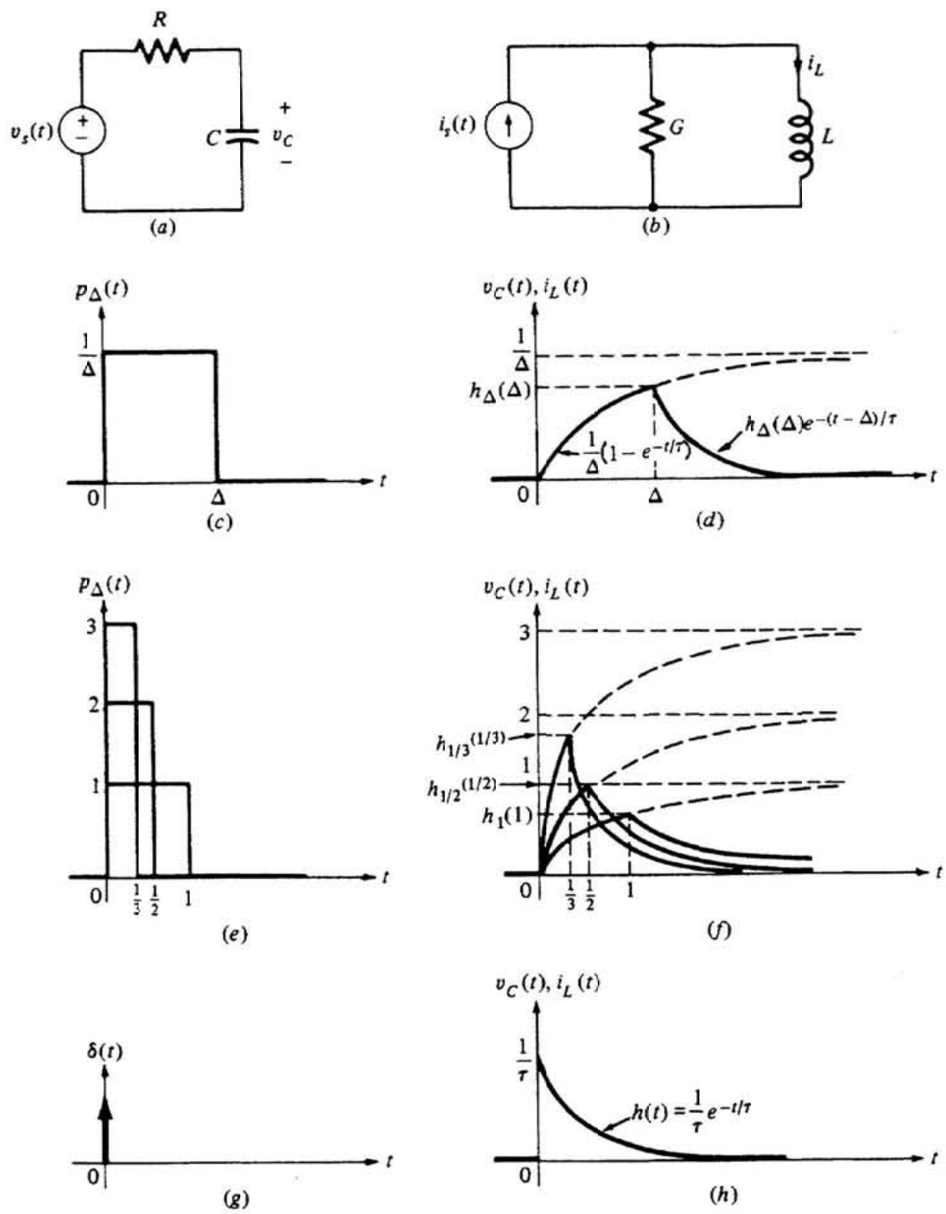


Figure 3.9 As $\Delta \rightarrow 0$, the square pulse in (c) tends to the *unit* impulse $\delta(\cdot)$ in (g). The corresponding response tends to the *impulse* response $h(t)$ in (h).

Because $h(t)$ is the response of the circuit when driven by a unit impulse under *zero initial condition*, it is called an *impulse response*. Note that $h(t) = 0$ for $t < 0$.

In Chap. 10, we will show that given the impulse response of any linear time-invariant circuit, we can use it to calculate the response when the circuit is driven by any other input waveform.

3.4 Circuits Driven by Arbitrary Signals

Let us consider now the general case where the one-port N in Fig. 3.1 contains arbitrary independent sources. This means that the Thévenin equivalent voltage source $v_{OC}(t)$, or the Norton equivalent current source $i_{SC}(t)$, in Fig. 3.2 can be *any* function of time, say, in practice, a piecewise-continuous function of time: square wave, triangular wave, synchronization signal of a TV set, etc. Our objective is to derive an explicit solution and draw the consequences.

Consider first the RC circuit in Fig. 3.2a whose state equation is

$$\dot{v}_C(t) = -\frac{v_C(t)}{\tau} + \frac{v_{OC}(t)}{\tau} \tag{3.25}$$

where $\tau \triangleq R_{eq}C$.

Explicit solution for first-order linear time-invariant *RC* circuits Given *any* prescribed waveform $v_{OC}(t)$, the solution of Eq. (3.25) corresponding to *any* initial state $v_C(t_0)$ at $t = t_0$ is given by

$$\boxed{v_C(t) = \underbrace{v_C(t_0)\exp\frac{-(t-t_0)}{\tau}}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t}\frac{1}{\tau}\exp\frac{-(t-t')}{\tau}\,v_{OC}(t')dt'}_{\text{zero-state response}}} \tag{3.26}$$

for all $t \geq t_0$. Here, $\tau = R_{eq}C$.

PROOF

(a) At $t = t_0$, Eq. (3.26) reduces to

$$v_C(t)\big|_{t=t_0} = v_C(t_0) \tag{3.27}$$

Hence Eq. (3.26) has the correct initial condition.

(b) To prove that Eq. (3.26) is a solution of Eq. (3.25), let us differentiate both sides of Eq. (3.26) with respect to t: First we rewrite Eq. (3.26) as

$$v_C(t) = v_C(t_0)\exp\frac{-(t-t_0)}{\tau} + \left(\frac{1}{\tau}\exp\frac{-t}{\tau}\right)\int_{t_0}^{t}\exp\frac{t'}{\tau}\,v_{OC}(t')\,dt' \tag{3.28}$$

Then upon differentiating we obtain for $t > 0$,

$$\dot{v}_C(t) = -\frac{1}{\tau}\,v_C(t_0)\exp\frac{-(t-t_0)}{\tau} + \left(-\frac{1}{\tau^2}\exp\frac{-t}{\tau}\right) \times \int_{t_0}^{t}\exp\frac{t'}{\tau}\,v_{OC}(t')\,dt' + \left(\frac{1}{\tau}\exp\frac{-t}{\tau}\right)\left[\exp\frac{t}{\tau}\,v_{OC}(t)\right] \tag{3.29}$$

where we used the fundamental theorem of calculus:

$$\frac{d}{dt}\int_0^t f(t')\,dt' = f(t) \qquad \text{if } f(\cdot) \text{ is continuous at time } t$$

Simplifying Eq. (3.29), we obtain

$$\begin{aligned}\dot{v}_C(t) &= -\frac{1}{\tau}\, v_C(t_0)\exp\frac{-(t-t_0)}{\tau} \\ &\qquad -\frac{1}{\tau}\left[\int_{t_0}^{t}\frac{1}{\tau}\exp\frac{-(t-t')}{\tau}\, v_{OC}(t')\,dt'\right] + \frac{1}{\tau}\, v_{OC}(t) \\ &= -\frac{v_C(t)}{\tau} + \frac{v_{OC}(t)}{\tau} \qquad (3.30)\end{aligned}$$

Hence Eq. (3.26) is a solution of Eq. (3.25).

(*c*) From mathematics we learned that the differential equation (3.25) has a unique solution. Hence Eq. (3.26) is indeed the solution. ∎

Zero-input response and zero-state response The solution (3.26) consists of two terms. The first term is called the *zero-input response* because when all independent sources in N are set to zero, we have $v_{OC}(t) = 0$ for all times, and $v_C(t)$ reduces to the first term only. The second term is called the *zero-state response* because when the initial state $v_C(t_0) = 0$, $v_C(t)$ reduces to the second term only.

Example Let us find the solution $v_C(t)$ of Fig. 3.7*a* using the above general formula. In this case, we have

$$v_C(t_0) = 0 \qquad t_0 = 0 \qquad \text{and} \qquad v_{OC}(t) = E \qquad t \geq 0$$

Substituting these parameters into Eq. (3.26), we obtain

$$\begin{aligned}v_C(t) &= 0 \times \exp\left[-\frac{(t-0)}{\tau}\right] + \int_0^t \frac{1}{\tau}\exp\left[-\frac{(t-t')}{\tau}\right]\cdot E\,dt' \\ &= \frac{E}{\tau}\exp\left(-\frac{t}{\tau}\right)\int_0^t \exp\frac{t'}{\tau}\,dt' = \frac{E}{\tau}\exp\left(-\frac{t}{\tau}\right)\left(\exp\frac{t}{\tau} - 1\right)\tau \\ &= E\left[1 - \exp\left(-\frac{t}{\tau}\right)\right] \qquad t \geq 0 \qquad (3.31)\end{aligned}$$

which coincides with that shown in Fig. 3.7*c*, as it should.

By duality, we have the following:

Explicit solution for first-order linear time-invariant *RL* circuit Given any prescribed waveform $i_{SC}(t)$, the solution of Eq. (3.26) corresponding to any initial state $i_L(t_0)$ at $t = t_0$ is given by

$$\boxed{i_L(t) = \underbrace{i_L(t_0)\exp\frac{-(t-t_0)}{\tau}}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t}\frac{1}{\tau}\exp\frac{-(t-t')}{\tau}\, i_{SC}(t')\,dt'}_{\text{zero-state response}}} \tag{3.32}$$

for all $t \geq t_0$. Here, $\tau = G_{eq}L$.

REMARKS

1. In both Eqs. (3.26) and (3.32), the *zero-input* response does *not* depend on the inputs and the *zero-state response* does *not* depend on the initial condition. In both cases, the total response can be interpreted as the superposition of two terms, one due to the initial condition acting alone (with all independent sources set to zero) and the other due to the input acting alone (with the initial condition set to zero).
2. Formulas (3.26) and (3.32) are valid for both $\tau > 0$ and $\tau < 0$. Consider the stable case $\tau > 0$. For values of t' such that $t - t' \gg \tau$, the factor $\exp[-(t-t')/\tau]$ is very small; consequently the values of $v_{OC}(t)$ [respectively, $i_{SC}(t)$] for such times contribute almost nothing to the integral in Eq. (3.26) [respectively, Eq. (3.32)]. In other words, the stable *RC* circuit (respectively, the stable *RL* circuit) has a *fading memory*: Inputs that have occurred many time constants ago have practically no effect at the present time.

   Thus we may say that the time constant τ is a measure of the memory time of the circuit.
3. Using the impulse response $h(t)$ for the *RC* circuit derived earlier in Eq. (3.24), we can rewrite the zero-state response in Eq. (3.26) as follows:

$$\int_{t_0}^{t} h(t-t')\, v_{OC}(t')\,dt' \tag{3.33}$$

Equation (3.33) is an example of a *convolution integral* to be developed in Chap. 10.

4. Once $v_C(t)$ is found using Eq. (3.26), we can replace the capacitor in Fig. 3.2*a* by an independent voltage source described by $v_C(t)$. We can then apply the *substitution theorem* to find the corresponding solution inside N by solving the resulting linear resistive circuit using the methods from the preceding chapters.
5. The *zero-state response* due to a *unit step* input $1(t)$ is called the *step response*, and will be denoted in this book by $s(t)$. The step response for a first-order *RC* (respectively, *RL*) circuit can be found by the *inspection method* in Sec. 3.1C, upon choosing $v_C(0) = 0$ (respectively, $i_L(0) = 0$).

The significance of the *step response* is that for *any* linear time-invariant circuit, the *impulse response* $h(t)$ needed in the convolution integral (6.5) of Chap. 10 can be derived from $s(t)$ (which is usually much easier to derive) via the formula

$$h(t) = \frac{ds(t)}{dt} \tag{3.34}$$

This important relationship is the subject of Exercise 1 in Chap. 10, page 615 [Eq. (4.64)].

The dual remark of course applies to the RL circuit in Fig. 3.2b.

4 FIRST-ORDER LINEAR SWITCHING CIRCUITS

Suppose now that the one-port N in Fig. 3.1 contains one or more *switches*, where the *state* (open or closed) of each switch is specified for all $t \geq t_0$. Typically, a switch may be open over several disjoint time intervals, and closed during the remaining times. Although a switch is a *time-varying* linear resistor, such a linear switching circuit may be analyzed as a sequence of first-order linear time-invariant circuits, each one valid over a time interval where all switches remain in a given state. This class of circuits can therefore be analyzed by the same procedure used in the preceding section. The only difference here is that unlike Sec. 3, the time constant τ will generally vary whenever a switch changes state, as demonstrated in the following example.

Example Consider the RC circuit shown in Fig. 4.1a, where the switch S is assumed to have been open for a long time prior to $t = 0$.

Given that the switch is *closed* at $t = 1$ s and then *reopened* at $t = 2$ s, our objective is to find $v_C(t)$ and $v_o(t)$ for all $t \geq 0$.

Since we are only interested in $v_C(t)$ and $v_o(t)$, let us replace the remaining part of the circuit by its Thévenin equivalent circuit. The result is shown in Fig. 4.1b and c corresponding to the case where S is "open" or "closed," respectively. The corresponding *time constant* is $\tau_2 = 1$ s and $\tau_1 = 0.9$ s, respectively.

Since the switch is initially open and the capacitor is initially in equilibrium, it follows from Fig. 4.1b that $v_C(t) = 6$ V and $v_o(t) = 0$ for $t \leq 1$ s. At $t = 1+$ we change to the equivalent circuit in Fig. 4.1c. Since, by continuity, $v_C(1+) = v_C(1-) = 6$ V, we have $i_C(1+) = (10 - 6)\text{V}/(2 + 1.6)\,\text{k}\Omega \approx 1.11$ mA and hence $v_o(1+) = (1.6\,\text{k}\Omega)(1.11\,\text{mA}) \approx 1.78$ V.

To determine $v_C(t_\infty)$ and $v_o(t_\infty)$ for the equivalent circuit in Fig. 4.1c, we open the capacitor and obtain $v_C(t_\infty) = 0$. The waveforms of $v_C(t)$ and $v_o(t)$ during [1, 2) are drawn as solid lines in Figs. 4.1d and e, respectively. The dotted portion shows the respective waveform if S had been left closed for all $t \geq 1$ s.

Since S is closed at $t = 2$ s, we must write the equation of these two waveforms to calculate $v_C(2-) = 8.68$ V and $v_o(2-) = 0.59$ V.

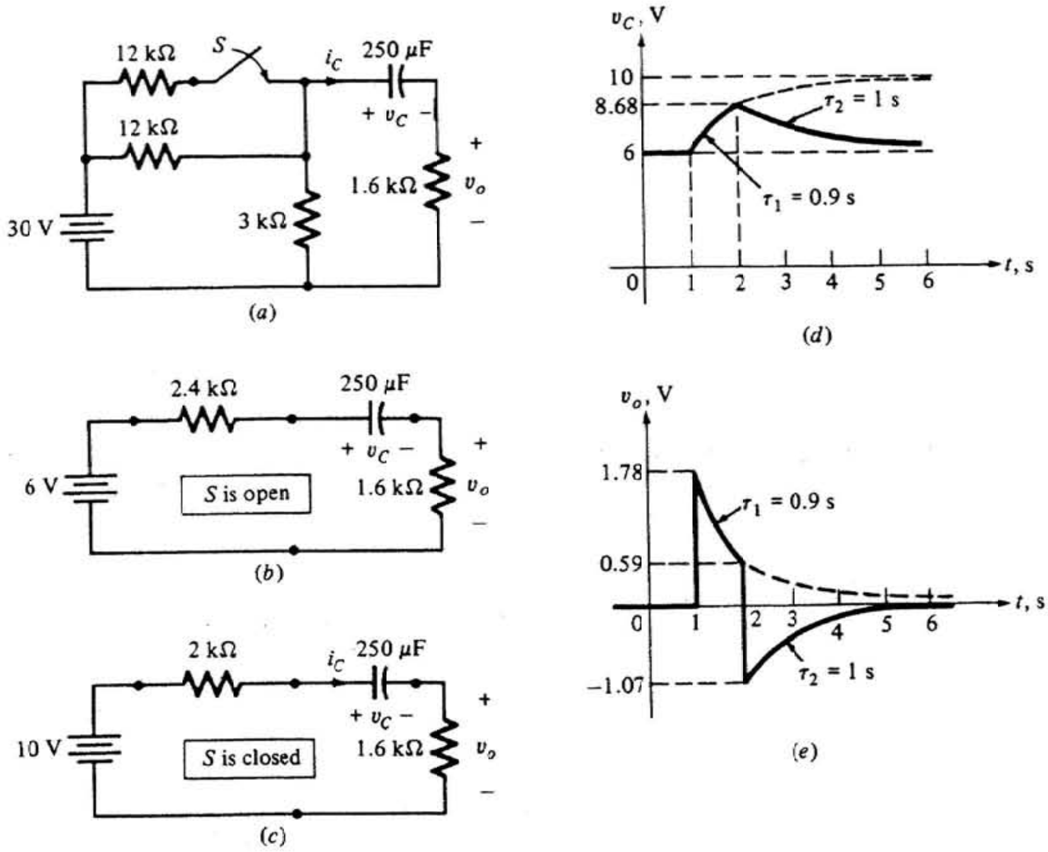


Figure 4.1 An RC switching circuit and the solution waveforms corresponding to the case where S is open during $t < 1$ s and $t \geq 2$ s, and closed during $1 \leq t < 2$.

At $t = 2+$, we return to the equivalent circuit in Fig. 4.1b. Since $v_C(2+) = v_C(2-) = 8.68$ V, we have $i_C(2+) = (6 - 8.68)\text{V}/(2.4 + 1.6)\ \text{k}\Omega \approx -0.67$ mA and $v_o(2+) = (1.6\ \text{k}\Omega)(-0.67\ \text{mA}) \approx -1.07$ V.

To determine $v_C(t_\infty)$ and $v_o(t_\infty)$ for the circuit in Fig. 4.1b, we open the capacitor and obtain $v_C(t_\infty) = 6$ V and $v_o(t_\infty) = 0$. The remaining solution waveforms are therefore as shown in Figs. 4.1d and e, respectively.

5 FIRST-ORDER PIECEWISE-LINEAR CIRCUITS

Consider the first-order circuit shown in Fig. 5.1 where the resistive one-port N may now contain *nonlinear* resistors in addition to linear resistors and dc sources. As before, all resistors and the capacitor are time-invariant. This class of circuits includes many important nonlinear electronic circuits such as multivibrators, relaxation oscillators, time-base generators, etc. In this section, we assume that all nonlinear elements inside N are *piecewise-linear* so that the one-port N is described by a *piecewise-linear driving-point characteristic.*

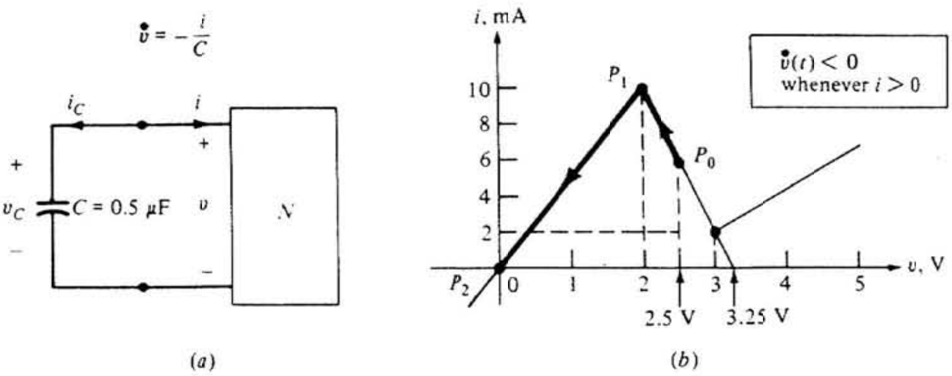


Figure 5.1 (*a*) A piecewise-linear *RC* circuit. (*b*) Driving-point characteristic of *N*.

Our main problem is to find the solution $v_C(t)$ for the *RC* circuit, or $i_L(t)$ for the *RL* circuit, subject to any given initial state. Since the corresponding port variables of *N*, namely, $[v(t), i(t)]$, must fall on the driving-point characteristic of *N*, the evolution of $[v(t), i(t)]$ can be visualized as the motion of a point on the characteristic starting from a given initial point.

5.1 The Dynamic Route

Since the driving-point characteristic is *piecewise-linear*, the solution $[v(t), i(t)]$ can be found by determining first the specific "route" and "direction," henceforth called the *dynamic route*, along the characteristic where the motion actually takes place. Once this route is identified, we can apply the "inspection method" developed in Sec. 3.1 to obtain the solution traversing along *each segment* separately, as illustrated in the following examples.

Example 1 Consider the *RC* circuit shown in Fig. 5.1*a*, where the one-port *N* is described by the voltage-controlled *piecewise-linear* characteristic shown in Fig. 5.1*b*.

Given the initial capacitor voltage $v_C(0) = 2.5$ V, our objective is to find $v_C(t)$ for all $t \geq 0$.

Step 1. Identify the initial point. Since $v(t) = v_C(t)$, for all t, initially $v(0) = v_C(0) = 2.5$ V. Hence the initial point on the driving-point characteristic of *N* is P_0, as shown on Fig. 5.1*b*.

Step 2. Determine the dynamic route. The dynamic route starting from P_0 contains two pieces of information: (*a*) the route traversed and (*b*) the direction of motion. They are determined from the following information:

Key to dynamic route for *RC* circuit

(*a*) The driving-point characteristic of *N*

(*b*) $\dot{v}(t) = -\frac{i(t)}{C}$

Since $\dot{v}(t) = -i(t)/C < 0$ whenever $i(t) > 0$, the voltage $v(t)$ *decreases* so long as the associated current $i(t)$ is positive. Hence, for $i(t) > 0$, the dynamic route starting at P_0 must always move *along* the v-i curve *toward the left*, as indicated by the *bold directed* line segments $P_0 \rightarrow P_1$ and $P_1 \rightarrow P_2$ in Fig. 5.1*b*. The dynamic route for this circuit ends at P_2 because at P_2, $i = 0$, so $\dot{v} = 0$. Hence the capacitor is in equilibrium.

Step 3. Obtain the solution for each straight line segment. Replace N by a sequence of *Thévenin equivalent circuits* corresponding to *each* line segment in the dynamic route. Using the method from Sec. 3.1, find a sequence of solutions $v_C(t)$. For this example, the dynamic route $P_0 \rightarrow P_1 \rightarrow P_2$ consists of only two segments. The corresponding equivalent circuits are shown in Fig. 5.2*a* and *b*, respectively.

To obtain $v_C(t)$ for segment $P_0 \rightarrow P_1$, we calculate $\tau = -62.5\ \mu$s, $v_C(0) = 2.5$ V, and $v_C(t_\infty) = 3.25$ V. Since the time constant in this case is *negative*, $v_C(t)$ consists of an "unstable" exponential passing through $v_C(0) = 2.5$ V and tending asymptotically to the "unstable" equilibrium value $v_C(t_\infty) = 3.25$ V as $t \rightarrow -\infty$. This solution is shown in Fig. 5.2*c* from P_0 to P_1. To calculate the time t_1 when $v_C(t) = 2$ V, we apply Eq. (3.12) and obtain

$$t_1 - 0 = 62.5\ \mu\text{s} \times \ln\left[\frac{2.5\ \text{V} - 3.25\ \text{V}}{2\ \text{V} - 3.25\ \text{V}}\right] = 31.9\ \mu\text{s} \tag{5.1}$$

Applying Eq. (3.5), we can write the solution from P_0 to P_1 analytically as follows (all voltages are in volts):

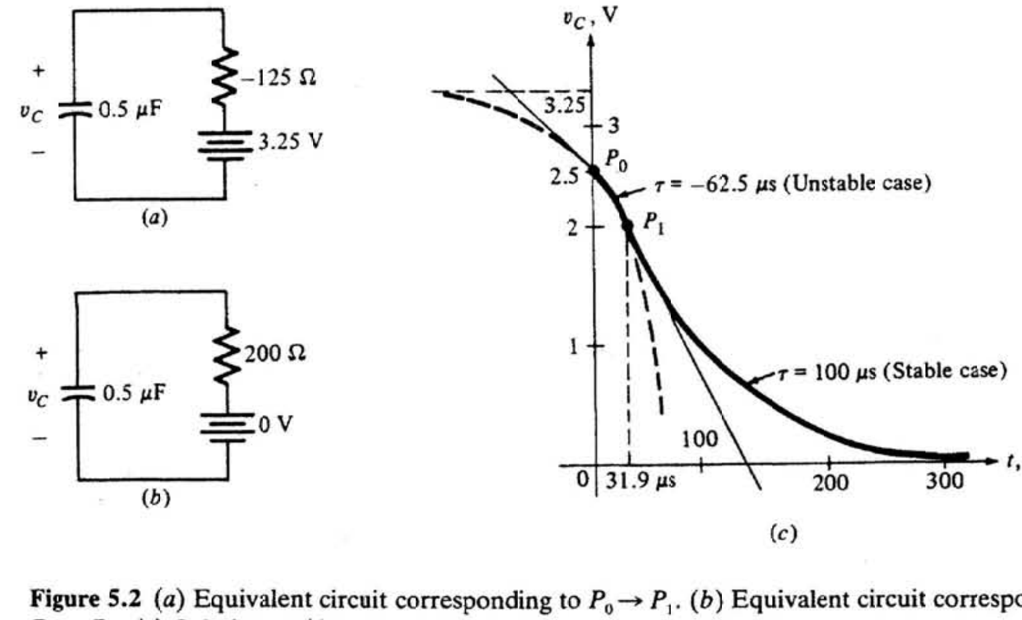


Figure 5.2 (*a*) Equivalent circuit corresponding to $P_0 \rightarrow P_1$. (*b*) Equivalent circuit corresponding to $P_1 \rightarrow P_2$. (*c*) Solution $v_C(t)$.

$$v_C(t) = 3.25 + [2.5 - 3.25] \exp \frac{-t}{62.5}\ \mu s$$

$$= 3.25 - 0.75 \exp \frac{-t}{62.5}\ \mu s \qquad 0 \le t \le 31.9\ \mu s \tag{5.2}$$

To obtain $v_C(t)$ for segment $P_1 \to P_2$, we calculate $\tau_2 = 100\ \mu s$, $v_C(t_0) = 2$ V, $t_0 = 31.9\ \mu s$, and $v_C(t_\infty) = 0$ V. The resulting exponential solution is shown in Fig. 5.2*c*. Applying Eq. (3.5), we can write the solution from P_1 to P_2 analytically as follows:

$$v_C(t) = 2 \exp \frac{-t - 31.9\ \mu s}{100\ \mu s} \qquad t \ge 31.9\ \mu s \tag{5.3}$$

Example 2 Consider the *RL* circuit shown in Fig. 5.3*a*, where *N* is described by the piecewise-linear characteristic shown in Fig. 5.3*b*.

Given the *initial inductor current* $i_L(t_0) = -I_0$, our objective is to find $i_L(t)$ for all $t \ge t_0$. (Note I_0 is the initial current *into* the one-port).

Step 1. Identify initial point. Since $i(t_0) = I_0$, we identify the initial point at P_0 on Fig. 5.3*b*.

Step 2. Determine the dynamic route. The dynamic route starting from P_0 is determined from the following information:

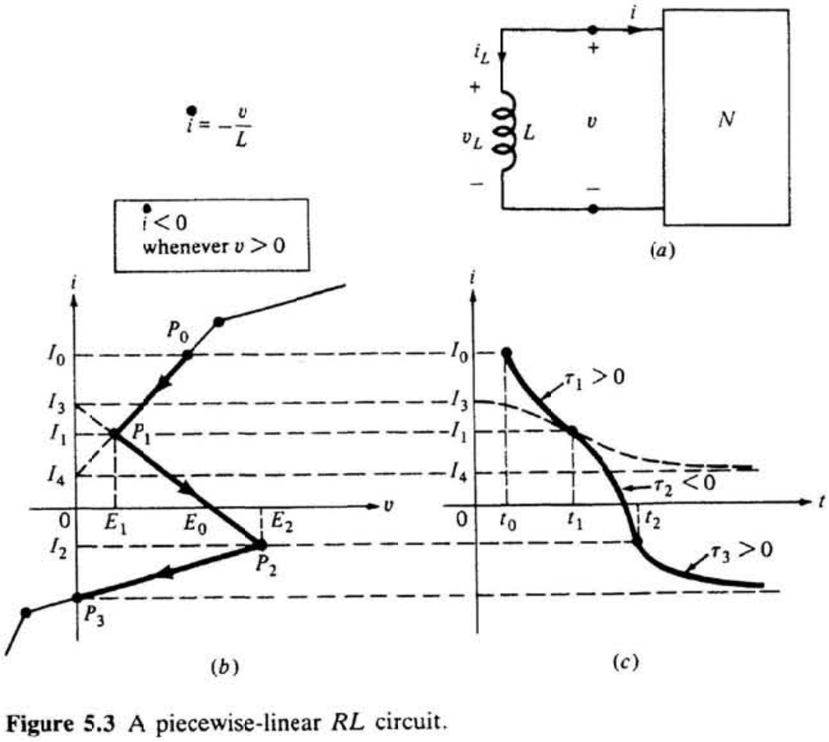


Figure 5.3 A piecewise-linear *RL* circuit.

Key to dynamic route for RL circuit

(a) The driving-point characteristic of N

(b) $\dot{i}(t) = -\dfrac{v(t)}{L}$

Since $\dot{i}(t) = -v(t)/L < 0$ whenever $v(t) > 0$, it follows that the *current* solution $i(t)$ must decrease so long as the associated $v(t)$ is positive.[21] Hence the dynamic route from P_0 must always move downward and consists of three segments $P_0 \to P_1$, $P_1 \to P_2$, and $P_2 \to P_3$ as shown in Fig. 5.3b. The dynamic route ends at P_3 because at P_3, $v = 0$ so $\dot{i} = 0$. Hence the inductor is in equilibrium.

Step 3. Replacing N by a sequence of Norton equivalent circuits corresponding to *each* line segment in the dynamic route, we obtain the solution in Fig. 5.3c by inspection.

REMARKS

1. After some practice, one can obtain the solution in Figs. 5.2c and 5.3c *directly* from the dynamic route, i.e., without drawing the Thévenin or Norton equivalent circuits.
2. In the RC case, since $\dot{v}(t) = -i(t)/C$, when $\tau > 0$, the dynamic route always terminates upon intersecting the v axis ($i = 0$).
3. In the RL case, since $\dot{i}(t) = -v(t)/L$, when $\tau > 0$, the dynamic route always terminates upon intersecting the i axis ($v = 0$).

Exercise

(a) Calculate the time constants τ_1, τ_2, and τ_3 in Fig. 5.3c.
(b) Calculate t_1 and t_2.
(c) Write the solution $i_L(t)$ analytically for $t \geq t_0$.
(d) Write the solution $v_L(t)$ analytically for $t \geq t_0$.

5.2 Jump Phenomenon and Relaxation Oscillation

Consider the RC op-amp circuit shown in Fig. 5.4a. The driving-point characteristic of the resistive one-port N was derived earlier in Fig. 3.8b of Chap. 4 and is reproduced in Fig. 5.4b for convenience.[22] Consider the four different initial points Q_1, Q_2, Q_3, and Q_4 (corresponding to four different initial capacitor voltages at $t = 0$) on this characteristic. Since $\dot{v}(t) = \dot{v}_C(t) = -i(t)/C$ and $C > 0$, we have

$$\dot{v}(t) > 0 \qquad \text{for all } t \text{ such that } i(t) < 0 \tag{5.4a}$$

[21] In order to use the v-i curve directly, we will find $i(t)$ first. The desired solution is then simply $i_L(t) = -i(t)$.

[22] Note that we have relabeled the two resistors R_1 and R_2 in Fig. 3.8b of Chap. 4 as R_A and R_B, respectively, in Fig. 5.4a. The symbols R_1, R_2, and R_3 in Fig. 5.4 denote the reciprocal slope of segments 1, 2, and 3, respectively, in Fig. 5.4b.

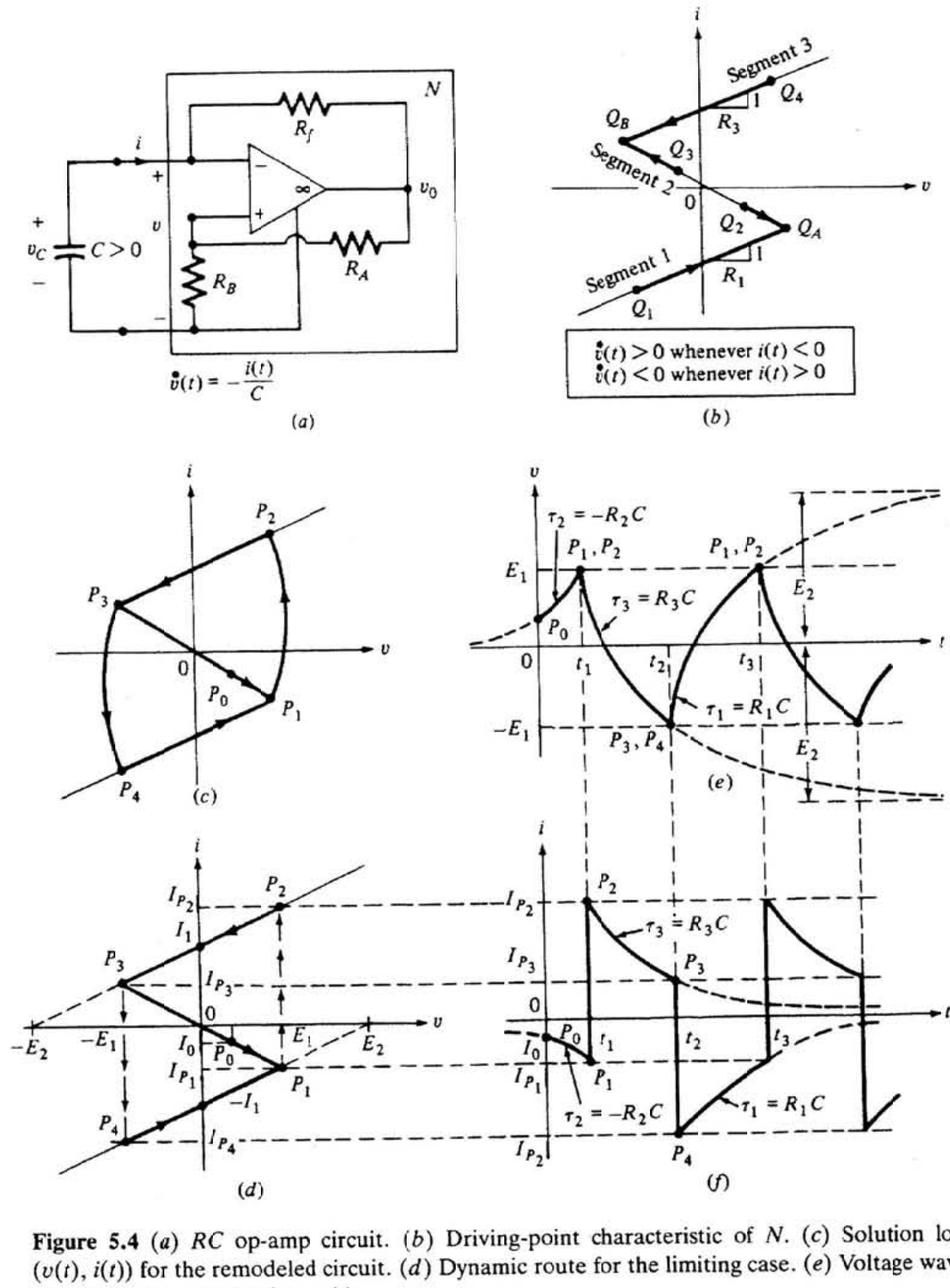


Figure 5.4 (*a*) *RC* op-amp circuit. (*b*) Driving-point characteristic of *N*. (*c*) Solution locus of $(v(t), i(t))$ for the remodeled circuit. (*d*) Dynamic route for the limiting case. (*e*) Voltage waveform $v(t)$. (*f*) Current waveform $i(t)$.

and $$\mathring{v}(t) < 0 \qquad \text{for all } t \text{ such that } i(t) > 0 \tag{5.4b}$$

Hence the *dynamic route* from any initial point must move *toward the left in the upper half plane*, and *toward the right in the lower half plane*, as indicated by the arrow heads in Fig. 5.4*b*.

Since $i \neq 0$ at the two *breakpoints* Q_A and Q_B, they are *not* equilibrium points of the circuit. It follows from Eq. (3.12) that the amount of time T it takes to go from any initial point to Q_A or Q_B is finite [because $x(t_k) \neq x(t_\infty)$].

Since the arrowheads from Q_1 and Q_2 (or from Q_3 and Q_4) *are oppositely directed*, it is impossible to continue drawing the dynamic route (from any initial point P_0) beyond Q_A or Q_B. In other words, an *impasse* is reached whenever the solution reaches Q_A or Q_B.

Any circuit which exhibits an impasse is the result of poor modeling. For the circuit of Fig. 5.4*a*, the impasse can be resolved by inserting a *small* linear *inductor* in series with the capacitor; this inductor models the inductance L of the connecting wires.

As will be shown in Chap. 7, the remodeled circuit has a well-defined solution for all $t \geq 0$ so long as $L > 0$. A typical solution locus of $(v(t), i(t))$ corresponding to the initial condition at P_0 is shown in Fig. 5.4*c*. Our analysis in Chap. 7 will show that the *transition time* from P_1 to P_2, or from P_3 to P_4, decreases with L. In the limit $L \to 0$, the solution locus tends to the limiting case shown in Fig. 5.4*d* with a *zero* transition time. In other words in the limit where L decreases to zero, the solution *jumps* from the impasse point P_1 to P_2, and from the impasse point P_3 to P_4. We use *dotted* arrows to emphasize the *instantaneous* transition.

Both analytical and experimental studies support the existence of a *jump phenomenon*, such as the one depicted in Fig. 5.4*d*, whenever a solution reaches an *impasse point* such as P_1 or P_3. This observation allows us to state the following rule which greatly simplifies the solution procedure.

> *Jump rule*
>
> Let Q be an *impasse point* of any first-order RC circuit (respectively, RL circuit). Upon reaching Q at $t = T$, the dynamic route can be continued by jumping (instantaneously) to another point Q' on the driving-point characteristic of N such that $v_C(T+) = v_C(T-)$ [respectively, $i_L(T+) = i_L(T-)$] provided Q' is the only point having this property.

Note that the jump rule is also consistent with the continuity property of v_C, or i_L.

OBSERVATIONS

1. The concepts of an *impasse point* and the *jump rule* are applicable regardless of whether the driving-point characteristic of N is piecewise-linear or not.
2. A first-order RC circuit has at least one impasse point if N is described by a continuous *nonmonotonic* current-controlled driving-point characteristic. The instantaneous transition in this case consists of a *vertical jump* in the v-i plane, assuming i is the vertical axis.

3. A first-order RL circuit has at least one impasse point if N is described by a continuous *nonmonotonic* voltage-controlled driving-point characteristic. The instantaneous transition in this case consists of a *horizontal jump* in the v-i plane, assuming i is the vertical axis.
4. Once the dynamic route is determined, with the help of the jump rule, *for all* $t > t_0$, the solution waveforms of $v(t)$ and $i(t)$ can be determined *by inspection*, as illustrated below.

Example The solution waveforms $v(t)$ and $i(t)$ corresponding to the initial point P_0 in Fig. 5.4c can be found as follows:

Applying the *jump rule* at the two impasse points P_1 and P_3, we obtain the closed dynamic route shown in Fig. 5.4d. This means that the solution waveforms become *periodic* after the short transient time interval from P_0 to P_1. Since the two *vertical* routes occur instantaneously, the *period of oscillation* is equal to the sum of the time it takes to go from P_2 to P_3 and from P_4 to P_1.

Following the same procedure as in the preceding examples, we obtain the voltage waveform $v(\cdot)$ shown in Fig. 5.4e and the current waveform $i(\cdot)$ shown in Fig. 5.4f. As expected, these solution waveforms are periodic and the op-amp circuit functions as an *oscillator*.

Observe that the oscillation waveforms of $v(t)$ and $i(t)$ are far from being sinusoidal. Such oscillators are usually called *relaxation oscillators*.[23]

Exercise

(a) Find the time constants τ_1, τ_2, τ_3, and the time instants t_1, t_2, and t_3 indicated in Fig. 5.4e and f in terms of the element values in Fig. 5.4a. (Assume the ideal op-amp model.)

(b) Use the v_o-vs.-v_i transfer characteristic derived earlier in Fig. 3.8c of Chap. 4 to show that the op-amp output voltage waveform $v_0(\cdot)$ is a *square wave* of period T. Calculate T in terms of the element parameters.

5.3 Triggering a Bistable Circuit (Flip-Flop)

Suppose we replace the capacitor in Fig. 5.4a by the inductor-voltage source combination as shown in Fig. 5.5a. Consider first the case where $v_s(t) \equiv 0$ so that the inductor is directly connected across N. Since $\dot{i}(t) = -v(t)/L$ and $L > 0$, it follows that $di/dt > 0$ whenever $v < 0$ and $di/dt < 0$ whenever $v > 0$. Hence the current i *decreases* in the *right half* v-i plane and *increases* in the *left half* v-i plane, as depicted by the typical dynamic routes in Fig. 5.5b.

Since the equilibrium state of a first-order RL circuit is determined by replacing the inductor by a short circuit, i.e., $v = v_L = 0$, it follows that this

[23] Historically, relaxation oscillators are designed using only two vacuum tubes, or two transistors, such that one device is operating in a "cut-off" or *relaxing* mode, while the other device is operating in an "active" or "saturation" mode.

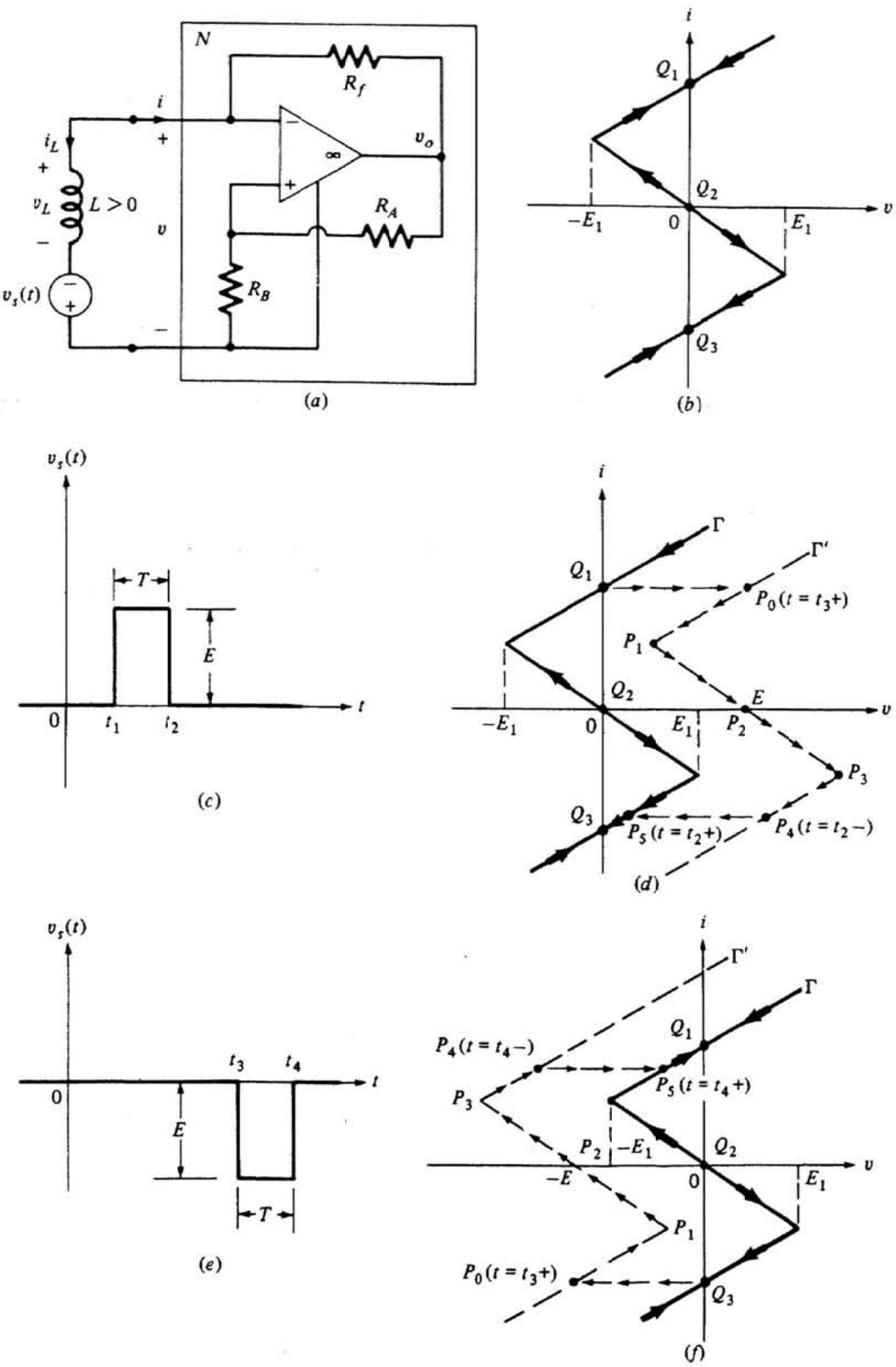


Figure 5.5 A bistable op-amp circuit and the dynamic routes corresponding to two typical triggering signals.

circuit has three *equilibrium points*; namely, Q_1, Q_2, and Q_3. These equilibrium points are the operating points of the associated resistive circuit obtained by short-circuiting the inductor L.

Since the dynamic route in Fig. 5.5*b* either tends to Q_1 or Q_3, but always diverges from Q_2, we say that the equilibrium point Q_2 is *unstable*. Hence even though the associated resistive circuit has three operating points, Q_2 can never be observed in practice—the slightest noise voltage will cause the dynamic route to diverge from Q_2, even if the circuit is operating initially at Q_2.

Whether Q_1 or Q_3 is actually observed depends on the initial condition. Such a circuit is said to be *bistable*.

Bistable circuits (flip-flops) are used extensively in digital computers, where the two stable equilibrium points correspond to the two binary states; say Q_1 denotes "1" and Q_3 denotes "0." In order to perform logic operations, it is essential to switch from Q_1 to Q_3, and vice versa. This is done by using a small *triggering signal*. We will now show how the voltage source in Fig. 5.5*a* can serve as a triggering signal.

Suppose initially the circuit is operating at Q_1. Let us at $t = t_1$ apply a *square pulse* of width $T = t_2 - t_1$ as shown in Fig. 5.5*c*. During the time interval $t_1 < t < t_2$, $v_s(t)$ can be replaced by an E-V battery, so that the inductor sees a translated driving-point characteristic as shown in Fig. 5.5*d* in broken line segments. Let us denote the original and the translated piecewise-linear driving-point characteristics by Γ and Γ' respectively. Then Γ holds over the time intervals $t < t_1$ and $t > t_2$, whereas Γ' holds over the time interval $t_1 < t < t_2$.

Since the inductor current cannot change instantaneously $[i_L(t_1-) = i_L(t_1+)]$, the dynamic route must jump horizontally from Q_1 to P_0 at time $t = t_1$. From P_0, the current i must subsequently decrease so long as $v > 0$. Hence, the dynamic route will be as indicated ($Q_1 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4$) in Fig. 5.5*d*. Here, we assume that at time $t = t_2-$, the dynamic route arrives at some point P_4 in the lower half plane. At time $t = t_2+$, Γ' switches back to Γ, and the dynamic route must jump horizontally from P_4 to P_5 at $t = t_2+$. After approximately five time constants, the dynamic route has essentially reached Q_3, and we have succeeded in triggering the circuit from equilibrium point Q_1 to equilibrium point Q_2.

To trigger from Q_3 back to Q_1, simply apply a similar triggering pulse of *opposite* polarity, as shown in Fig. 5.5*e*. The resulting dynamic route is shown in Fig. 5.5*f*.

Triggering criteria The following two conditions must be satisfied by the triggering signal in order to trigger from Q_1 to Q_3, or vice versa.

Minimum pulse width condition If t_2 occurs before the dynamic route in Fig. 5.5*d* (respectively, *f*) crosses the v axis at P_2, the route will jump (horizontally) to a point on Γ in the upper *left half* plane (respectively, lower *right half* plane) and return to Q_1 (respectively, Q_3). Hence, for successful triggering, we must require $T > T_{\min}$, where $T_{\min}$ is the time it takes to go from P_0 to P_2 in Fig. 5.5*d* or *f*.

Minimum pulse height condition If E is too small such that the breakpoint P_1 on Γ' is located in the left half plane, (respectively, the right half plane), then the dynamic route will also return to Q_1 (respectively, Q_3). Hence, for successful triggering, we must require $E > E_{\min}$, where $E_{\min} = E_1$.

Exercise
(*a*) Express $T_{\min}$ and $E_{\min}$ in terms of the circuit parameters.
(*b*) Sketch the solution waveforms of $i(t)$ and $v_o(t)$ for the case when $T = 1.5T_{\min}$ and $E = 1.5E_{\min}$.
(*c*) Repeat (*b*) for the case where $T = 0.5T_{\min}$ and $E = 0.5E_{\min}$.

SUMMARY

- A two-terminal element described by a q-v characteristic $f_C(q, v) = 0$ is called a *time-invariant capacitor*.
- In the special case where $q = Cv$, where C is a constant called the *capacitance*, the capacitor is *linear* and *time-invariant*. In this case, it can be described by

$$i = C \frac{dv}{dt}$$

or

$$v(t) = v(t_0) + \frac{1}{C} \int_{t_0}^{t} i(\tau)\, d\tau$$

- A *linear time-varying capacitor* is described by

$$q = C(t)v$$

This implies that

$$i(t) = C(t) \frac{dv(t)}{dt} + \frac{dC(t)}{dt} v(t)$$

requires an additional term compared to the time-invariant case.

- A two-terminal element described by a ϕ-i characteristic $f_L(\phi, i) = 0$ is called a *time-invariant inductor*.
- In the special case where $\phi = Li$, where L is a constant called the *inductance*, the inductor is *linear* and *time-invariant*. In this case, it can be described by

$$v = L \frac{di}{dt}$$

or

$$i(t) = i(t_0) + \frac{1}{L} \int_{t_0}^{t} v(\tau)\, d\tau$$

- A *linear time-varying inductor* is described by

$$\phi = L(t)i$$

This implies that

$$v(t) = L(t) \frac{di(t)}{dt} + \frac{dL(t)}{dt} i(t)$$

requires an additional term compared to the time-invariant case.

- Memory property: The capacitor voltage at any time T depends on the *entire* capacitor current waveform for all $t < T$.
- *Initial capacitor voltage transformation*: A C-F capacitor with an initial voltage $v_C(0)$ is equivalent to a C-F capacitor with *zero* initial voltage in series with a $v_C(0)$-V voltage source.
- *Capacitor voltage continuity property*: For any $t \in (t_1, t_2)$,

$$v_C(t+) = v_C(t-)$$

provided that, for some M,

$$|i_C(t)| \leq M < \infty \qquad \text{for all } t \in [t_1, t_2]$$

- Memory property: The inductor current at any time T depends on the *entire* inductor voltage waveform for all $t < T$.
- *Initial inductor current transformation*: An L-H inductor with an initial current $i_L(0)$ is equivalent to an L-H inductor with *zero* initial current in parallel with an $i_L(0)$-A current source.
- *Inductor current continuity property*: For any $t \in (t_1, t_2)$,

$$i_L(t+) = i_L(t-)$$

provided that, for some M,

$$|v_L(t)| \leq M < \infty \qquad \text{for all } t \in [t_1, t_2]$$

- A *unit step* function $1(t)$ is defined by

$$1(t) \triangleq \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

- A *unit impulse* (or delta function) $\delta(t)$ is defined by the following two properties:

1. $$\delta(t) \triangleq \begin{cases} \text{singular} & t=0 \\ 0 & t \neq 0 \end{cases}$$

2. $$\int_{\varepsilon_1}^{\varepsilon_2} \delta(t)\, dt = 1 \qquad \text{for any } \varepsilon_1 < 0 \text{ and } \varepsilon_2 > 0$$

- The zero-state response $h(t)$ to a *unit impulse* $\delta(t)$ is called the *impulse response*.
- The zero-state response $s(t)$ to a *unit step* $1(t)$ is called the *step response*.
- For any linear time-invariant circuit, the impulse response $h(t)$ and the step response $s(t)$ are related by

$$h(t) = \frac{ds(t)}{dt}$$

- *Lossless property*: A time-invariant charge-controlled capacitor cannot dissipate energy. Rather, energy is stored and can be recovered subsequently.
- *Lossless property*: A time-invariant flux-controlled inductor cannot dissipate energy. Rather, energy is stored and can be recovered subsequently.

- The energy w_C entering a time-invariant *charge-controlled* capacitor during $[t_1, t_2]$ depends on the charge at the end points, namely, $q_C(t_1)$ and $q_C(t_2)$. In particular,

$$w_C(q_1, q_2) = \int_{q_1}^{q_2} \hat{v}(q)\, dq$$

where $q_1 \triangleq q_C(t_1)$ and $q_2 \triangleq q_C(t_2)$.

- The energy $\mathscr{E}_C$ stored in a C-F capacitor with initial voltage $v_C(0) = V$ is equal to

$$\mathscr{E}_C = \tfrac{1}{2} CV^2$$

- q_* is called a *relaxation point* for a time-invariant charge-controlled capacitor iff

$$\int_{q_*}^{q} \hat{v}(q')\, dq' \geq 0 \qquad \text{for all } -\infty < q < \infty$$

The energy $\mathscr{E}_C$ stored in a time-invariant charge-controlled capacitor with initial charge $q(0) = Q$ is equal to

$$\mathscr{E}_C(Q) = \int_{q_*}^{Q} \hat{v}(q)\, dq$$

where q_* is any relaxation point of the capacitor.

- A first-order linear *parallel RC* circuit is described by a state equation

$$\dot{v}_C = -\frac{v_C}{R_{\text{eq}}C} + \frac{v_{\text{OC}}(t)}{R_{\text{eq}}C}$$

where R_{eq} is the Thévenin equivalent resistance and $v_{\text{OC}}(t)$ is the open-circuit voltage of the resistive one-port seen by the capacitor.

- The energy w_L entering a time-invariant *flux-controlled* inductor during $[t_1, t_2]$ depends *only* on the flux at the end points, namely, $\phi_L(t_1)$ and $\phi_L(t_2)$. In particular,

$$w_L(\phi_1, \phi_2) = \int_{\phi_1}^{\phi_2} \hat{i}(\phi)\, d\phi$$

where $\phi_1 \triangleq \phi_L(t_1)$ and $\phi_2 \triangleq \phi_L(t_2)$.

- The energy $\mathscr{E}_L$ stored in an L-H inductor with initial current $i_L(0) = I$ is equal to

$$\mathscr{E}_L = \tfrac{1}{2} LI^2$$

- ϕ_* is called a *relaxation point* for a time-invariant flux-controlled inductor iff

$$\int_{\phi_*}^{\phi} \hat{i}(\phi')\, d\phi' \geq 0 \qquad \text{for all } -\infty < \phi < \infty$$

The energy $\mathscr{E}_L$ stored in a time-invariant flux-controlled inductor with initial flux $\phi(0) = \Phi$ is equal to

$$\mathscr{E}_L(\Phi) = \int_{\phi_*}^{\Phi} \hat{i}(\phi)\, d\phi$$

where ϕ_* is any relaxation point of the inductor.

- A first-order linear *series RL* circuit is described by a state equation

$$\dot{i}_L = -\frac{i_L}{G_{\text{eq}}L} + \frac{i_{\text{SC}}(t)}{G_{\text{eq}}L}$$

where G_{eq} is the Norton equivalent conductance and $i_{\text{SC}}(t)$ is the short-circuit current of the resistive one-port seen by the inductor.

- Any first-order linear time-invariant circuit driven by dc sources is described by a state equation of the form

$$\dot{x} = -\frac{x}{\tau} + \frac{x(t_\infty)}{\tau}$$

where τ is called the *time constant*, and $x(t_\infty)$ is called the *equilibrium state*.

- $\tau = R_{eq}C$ for an *RC* circuit
- $\tau = G_{eq}L$ for an *RL* circuit

- The solution of the above state equation is always given explicitly by an *exponential* waveform:

$$x(t) - x(t_\infty) = [x(t_0) - x(t_\infty)] \exp \frac{-(t - t_0)}{\tau}$$

for *all time t*.

This solution is uniquely specified by three pieces of information: the *initial state* $x(t_0)$, the *equilibrium state* $x(t_\infty)$, and the *time constant* τ.

- Let $x(t_j)$ and $x(t_k)$ denote any two points on the above *exponential waveform*. The *elapsed time* between t_j and t_k can be calculated explicitly as follows:

$$t_k - t_j = \tau \ln \frac{x(t_j) - x(t_\infty)}{x(t_k) - x(t_\infty)}$$

- The solution of any first-order linear time-invariant circuit driven by dc sources, or by piecewise-constant signals, or circuits containing switches can be obtained *by* inspection (i.e., without writing the state equation): Simply determine the three relevant pieces of information over appropriate time intervals.
- The solution of any first-order *piecewise-linear* circuit can be determined by inspection by drawing the associated *dynamic route*.
- When the dynamic route contains *impasse points*, the capacitor voltage waveform and the inductor current waveform must exhibit one or more instantaneous jumps.

PROBLEMS

Nonlinear capacitors and inductors

1 A varactor diode behaves like a capacitor when $v < V_0$ ($V_0 = 0.5$ V in this case). Its *q-v* characteristic is given by

$$q = -10^{-15}(0.5 - v)^{1/2} \qquad v < 0.5 \text{ V}$$

(*a*) Calculate its small-signal capacitance $C(v)$ for $v < 0.5$ V.

(*b*) A voltage $v(t) = -1 + 0.3 \cos(2\pi 10^8 t)$ is applied. Obtain an explicit expression for the current through the capacitor.

2 A time-invariant nonlinear inductor has the characteristic shown in Fig. P6.2.

(*a*) Using the graph, estimate the small-signal inductance $L(i)$ when $i = 3$ mA.

(*b*) Repeat (*a*) for $i = 2$ mA.