Thus a sinusoidal voltage with angular frequency ω_1 applied to a linear time-varying resistor generates, in addition to a sinusoidal current with the same angular frequency ω_1, two sinusoids at angular frequencies $\omega + \omega_1$ and $\omega - \omega_1$. This property is the basis of several modulation schemes in communication systems. With linear *time-invariant* resistors, a sinusoidal input can only generate a sinusoidal response at the *same* frequency.

2 SERIES AND PARALLEL CONNECTIONS

In Chap. 1 we considered general circuits with arbitrary circuit elements. The primary objective was to learn Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL). KCL and KVL do not depend on the nature of the circuit elements. They lead to two sets of *linear* algebraic equations in terms of two sets of pertinent circuit variables: the *branch currents* and the *branch voltages*. These equations depend on the topology of the circuit, i.e., how the circuit elements are connected to one another. The branch currents and branch voltages are in turn related according to the characteristics of the circuit elements. As seen in the previous section, these characteristics for two-terminal resistors may be linear or nonlinear, time-invariant or time-varying. The equations describing the v-i characteristics are called *element equations* or *branch equations*. Together with the equations from KCL and KVL, they give a complete specification of the circuit. The purpose of circuit analysis is to write down the complete specification of any circuit and to obtain pertinent solutions of interest.

In this section we will consider a special but very important class of circuits: circuits formed by series and parallel connections of two-terminal resistors. First, we wish to generalize the concept of the v-i characteristic of a resistor to that of a two-terminal circuit made of two-terminal resistors, or more succinctly a *resistive one-port*. We will demonstrate that the series and parallel connections of two-terminal resistors will yield a one-port whose v-i characteristic is again that of a resistor. We say that two resistive one-ports are *equivalent* iff their v-i characteristics are the same.

When we talk about resistive one-ports, we naturally use *port voltage* and *port current* as the pertinent variables. The v-i characteristic of a one-port in terms of its port voltage and port current is often referred to as the *driving-point characteristic* of the one-port. The reason we call it the driving-point characteristic is that we may consider the one-port as being driven by an independent voltage source v_s or an independent current source i_s as shown in Fig. 2.1. In the former, the input is $v_s = v$, the port voltage; and the response is the port current i. In the latter, the input is $i_s = i$, the port current; and the response is the port voltage v. In the following subsections we will discuss the driving-point characteristics of one-ports made of two-terminal resistors connected in series, connected in parallel, and connected in series-parallel.

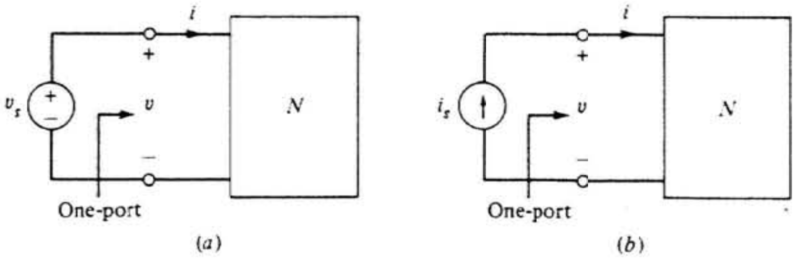

Figure 2.1 A one-port N driven (*a*) by an independent voltage source and (*b*) by an independent current source.

2.1 Series Connection of Resistors

From physics we know that the series connection of linear resistors yields a linear resistor whose resistance is the sum of the resistances of each linear resistor. Let us extend this simple result to the series connections of resistors in general.

Consider the circuit shown in Fig. 2.2 where two *nonlinear* resistors $\mathcal{R}_1$ and $\mathcal{R}_2$ are connected at node ②. Nodes ① and ③ are connected to the rest of the circuit, which is designated by $\mathcal{N}$. Looking toward the right from nodes ① and ③, we have a one-port which is formed by the *series connection of two resistors* $\mathcal{R}_1$ and $\mathcal{R}_2$. For our present purposes the nature of $\mathcal{N}$ is irrelevant. We are interested in obtaining the driving-point characteristic of the one-port with port voltage v and port current i.

Let us assume that both resistors are current-controlled, i.e.,

$$v_1 = \hat{v}_1(i_1) \qquad \text{and} \qquad v_2 = \hat{v}_2(i_2) \tag{2.1}$$

These are the two branch equations. Next, we consider the circuit topology and write the equations using KCL and KVL. KCL applied to nodes ① and ② gives

$$i = i_1 = i_2 \tag{2.2}$$

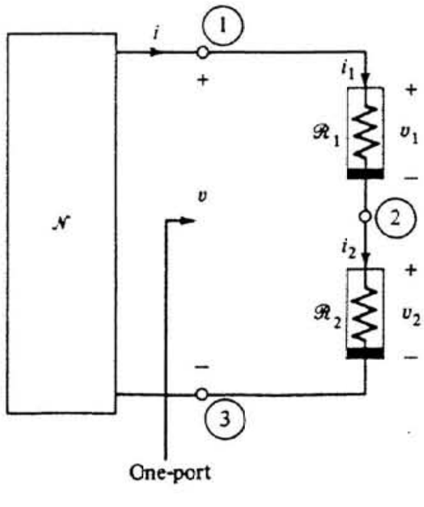

Figure 2.2 Two nonlinear resistors connected in series together with the rest of the circuit $\mathcal{N}$.

The KVL equation for the node sequence ①-②-③-① leads to

$$v = v_1 + v_2 \tag{2.3}$$

Combining Eqs. (2.1), (2.2), and (2.3), we obtain

$$v = \hat{v}_1(i) + \hat{v}_2(i) \tag{2.4}$$

which is the v-i characteristic of the one-port. It states that the driving-point characteristic of the one-port is again a current-controlled resistor

$$v = \hat{v}(i) \tag{2.5a}$$

where
$$\hat{v}(i) \triangleq \hat{v}_1(i) + \hat{v}_2(i) \qquad \text{for all } i \tag{2.5b}$$

Exercise If the two terminals of the nonlinear resistor $\mathscr{R}_1$ in Fig. 2.2 are turned around as shown in Fig. 2.3, show that the series connection gives a one-port which has a driving-point characteristic:

$$\hat{v}(i) = -\hat{v}_1(-i) + \hat{v}_2(i)$$

Example 1 (a battery model) A battery is a physical device which can be modeled by the series connection of a linear resistor and a dc voltage source, as shown in Fig. 2.4. Since both the independent voltage source and the linear resistor are current-controlled resistors, this is a special case of the circuit in Fig. 2.2. The branch equations are

$$v_1 = Ri_1 \qquad \text{and} \qquad v_2 = E \tag{2.6}$$

Adding v_1 and v_2 and setting $i_1 = i$, we obtain

$$v = Ri + E \tag{2.7}$$

Or, we can add the characteristics graphically to obtain the driving-point characteristic of the one-port shown in the i-v plane in the figure.[3] The

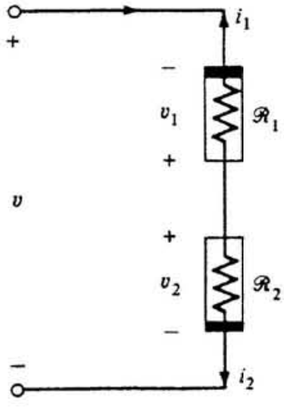

Figure 2.3 Series connection of $\mathscr{R}_1$ and $\mathscr{R}_2$ with the terminals of $\mathscr{R}_1$ turned around.

[3] We use the i-v plane to facilitate the addition of voltages.

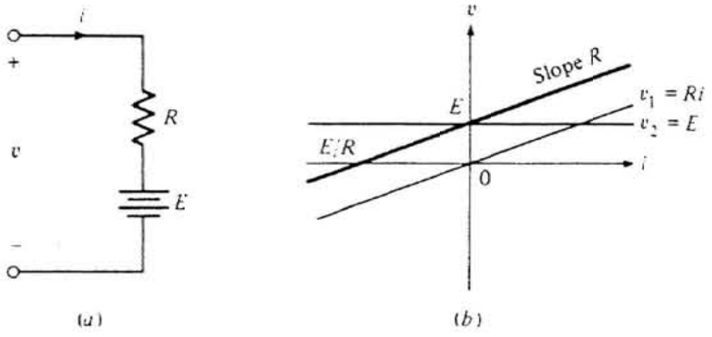

Figure 2.4 (*a*) Series connection of a linear resistor and a dc voltage source, and (*b*) its driving-point characteristic.

heavy line in Fig. 2.4*b* gives the characteristic of a battery with an internal resistance R. Usually R is small, thus the characteristic in the *i*-*v* plane is reasonably flat. However, it should be clear that a real battery does not behave like an independent voltage source because the port voltage v depends on the current i. If we connect the real battery to an external load, e.g., a linear resistor with resistance R, the actual voltage across the load will be $E/2$.

Since a battery is used to deliver power to an external circuit, we usually prefer to use the opposite of the associated reference direction when the battery is connected to an external circuit. The characteristic plotted on the i'-v plane where $i' = -i$ is shown in Fig. 2.5 together with the external circuit.

Example 2 (ideal diode circuit) Consider the series connection of a real battery and an ideal diode as shown in Fig. 2.6. Since the ideal diode is not a current-controlled resistor, we cannot use Eq. (2.5) to add the voltages directly. Instead, we consider each segment of the ideal diode characteristic independently. Recall the definition of the ideal diode, for $i_2 > 0$, $v_2 = 0$. Since by KCL, $i_2 = i_1 = i$, we have

$$v = v_1 = Ri + E \qquad \text{for } i > 0 \tag{2.8a}$$

In other words, in the right half of the *i*-*v* plane ($i > 0$), the one-port

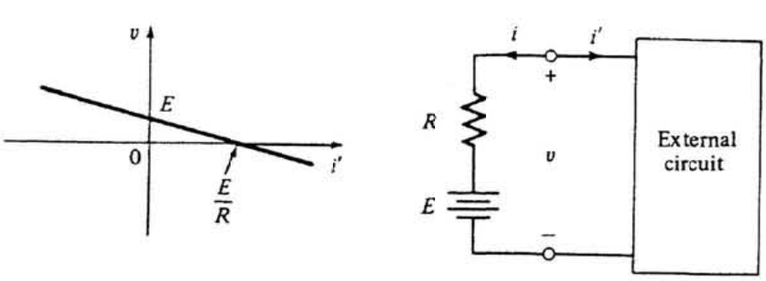

Figure 2.5 Characteristic of a real battery plotted on the i'-v plane and the external circuit connection.

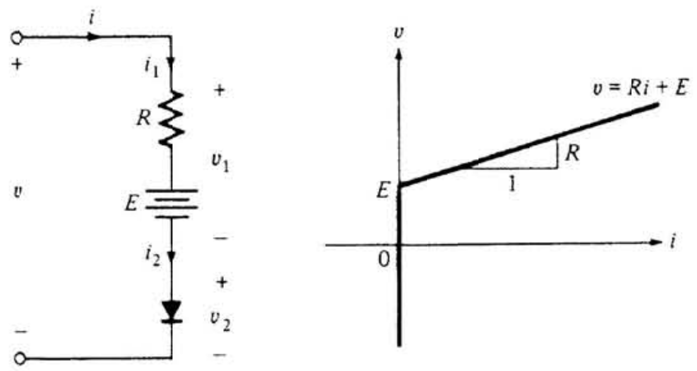

Figure 2.6 Series connection of a battery with an ideal diode and the driving-point characteristic of the resulting one-port.

characteristic is identical with that of Fig. 2.4, i.e., a straight line starting at the point $(0, E)$ with a slope R. Next, for $v_2 < 0$, $i_2 = 0$, i.e., the ideal diode is reversed biased, and its characteristic requires that $i = i_2 = 0$. Therefore there is no current flow. Hence $v_1 = E$, and thus

$$v = E + v_2 \qquad \text{and} \qquad i = 0 \qquad \text{for } v_2 \leq 0 \tag{2.8b}$$

In other words, when the diode is reversed biased, the one-port characteristic consists of the vertical half line lying on the v axis below the point $(0, E)$. (See Fig. 2.6.)

Exercise Determine the v-i characteristic of the series connection of the same three elements as in Fig. 2.6 except that the diode is turned around.

Example 3 (voltage sources in series) Consider m independent voltage sources in series as shown in Fig. 2.7. Since voltage sources are current-controlled, we only need to add the voltages of each independent source to obtain an equivalent one-port, which is an independent voltage source whose voltage is given by the sum of the m voltages.

Example 4 (current sources in series) Next, consider independent current sources in series as shown in Fig. 2.8. Applying KCL at nodes ①, ②, etc.,

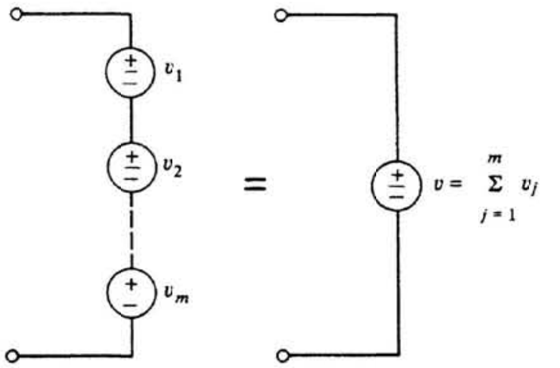

Figure 2.7 m Independent voltage sources in series.

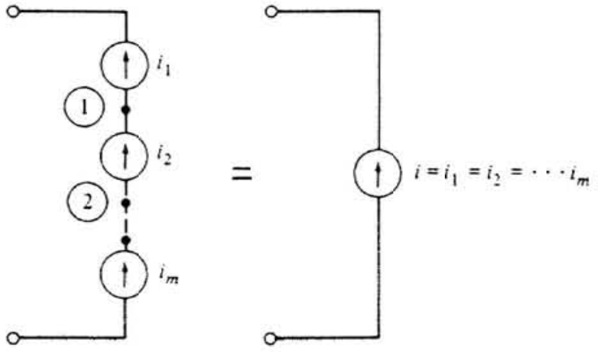

Figure 2.8 m Independent current sources in series.

we note that the m independent current sources must have the same current, i.e., $i_1 = i_2 = \cdots = i_m$. Otherwise, we reach a contradiction because (a) by definition, an independent current source has a current i_s irrespective of the external connection and (b) KCL cannot be violated at any node of a circuit. Therefore we conclude that (a) the only possibility for the connection to make sense is that all currents are the same and (b) the resulting one-port is a current source with the same current.

Example 5 (graphic method) Graphic methods are extremely useful in analyzing simple nonlinear circuits. Consider the series connection of a linear resistor and a *voltage-controlled* nonlinear resistor as shown in Fig. 2.9. The branch equations for the two resistors are

$$v_1 = R_1 i_1 \qquad \text{and} \qquad i_2 = \hat{i}_2(v_2) \tag{2.9}$$

First, we wish to find out whether the nonlinear resistor $\mathcal{R}_2$ is also current-controlled. Since, for $I_1 \leq i \leq I_2$, the voltage is multivalued, we know $\mathcal{R}_2$ is not current-controlled. Therefore we cannot solve the problem analytically as before [see Eqs. (2.1) and (2.5)]. However, it is possible to add the voltages v_1 and v_2 graphically point by point on the two curves.

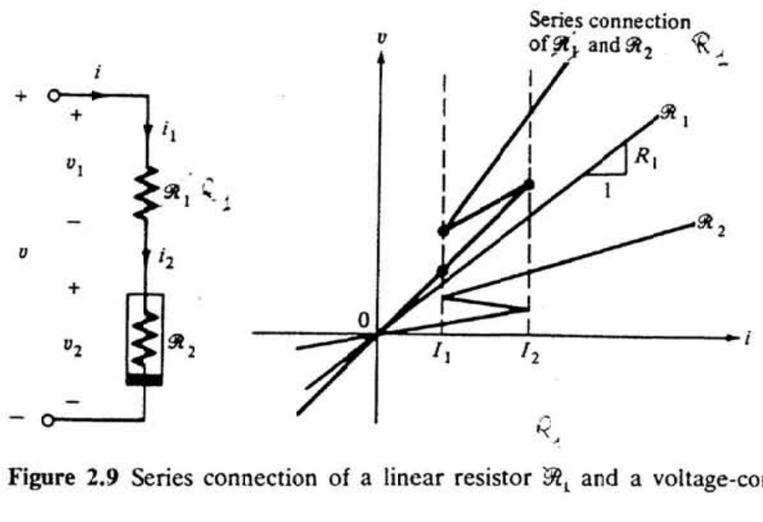

Figure 2.9 Series connection of a linear resistor $\mathcal{R}_1$ and a voltage-controlled resistor $\mathcal{R}_2$.

Thus for $I_1 \leq i \leq I_2$, the sum of v_1 and v_2 is multivalued. What is of interest is to note that the resulting characteristic is neither voltage-controlled nor current-controlled.

REMARK Figure 2.9 shows that $\mathscr{R}_2$ is voltage-controlled, i.e., $i_2 = \hat{i}_2(v_2)$ and $\mathscr{R}_1$ is both voltage-controlled and current-controlled. The characteristic of the series connection is neither voltage-controlled nor current-controlled but may be described by the *parametric representation*:

$$v = R_1\hat{i}_2(v_2) + v_2$$
$$i = \hat{i}_2(v_2)$$

where v_2 plays the role of a parameter.

Summary of series connection The key concepts used in obtaining the driving-point characteristic of a one-port formed by the series connection of two-terminal resistors are

1. KCL forces all branch currents to be equal to the port current.
2. KVL requires the port voltage v to be equal to the sum of the branch voltages of the resistors.
3. If each resistor is current-controlled, the resulting driving-point characteristic of the one-port is also a current-controlled resistor.

2.2 Parallel Connection of Resistors

Consider the circuit shown in Fig. 2.10 where two resistors $\mathscr{R}_1$ and $\mathscr{R}_2$ are connected in parallel at nodes ① and ② to the rest of the circuit designated $\mathscr{N}$. The nature of $\mathscr{N}$ is immaterial for the present discussion. We wish to determine the driving-point characteristic of the one-port defined by the two nodes ① and ② looking to the right, i.e., the parallel connection of $\mathscr{R}_1$ and $\mathscr{R}_2$. We assume that the resistors are both voltage-controlled, i.e.,

$$i_1 = \hat{i}_1(v_1) \qquad \text{and} \qquad i_2 = \hat{i}_2(v_2) \tag{2.10}$$

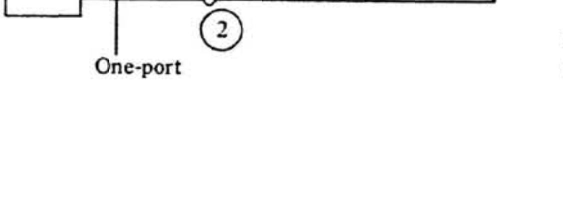

Figure 2.10 Two nonlinear resistors in parallel together with the rest of the circuit $\mathscr{N}$.

Picking node ② as the datum node, we have, from KVL

$$v = v_1 = v_2 \tag{2.11}$$

Applying KCL at node ①, we have

$$i = i_1 + i_2 \tag{2.12}$$

Combining the above equations, we obtain

$$i = \hat{i}_1(v) + \hat{i}_2(v) \tag{2.13}$$

Equation (2.13) states that the driving-point characteristic of the one-port is a voltage-controlled resistor defined by

$$i = \hat{i}(v) \tag{2.14a}$$

where $$\hat{i}(v) \triangleq \hat{i}_1(v) + \hat{i}_2(v) \qquad \text{for all } v \tag{2.14b}$$

Duality It is interesting to compare the two sets of equations: (2.1) to (2.5) for the series connection and (2.10) to (2.14) for the parallel connection. If we make the substitutions for all the v's with i's and for all the i's with v's in one set of equations, we obtain precisely the other set. For this reason, we can extend and generalize the concept of duality introduced earlier for resistors to *circuits*.

Let us redraw the two circuits of the series connection and of the parallel connection of two nonlinear resistors as shown in Fig. 2.11. Let us denote the series-connection circuit by $\mathcal{N}$ and the parallel-connection circuit by $\mathcal{N}^*$. Comparing Eqs. (2.1) to (2.5) with Eqs. (2.10) to (2.14) one by one together with the two circuits in Fig. 2.11, we can learn how to generalize the concept of duality. First, for the branch equations (2.1) and (2.10), note that substituting all the v's with i's and all the i's with v's in one equation, we obtain the other. This, however, requires that the function $\hat{v}_1(\cdot)$ be the same as $\hat{i}_1(\cdot)$ and the function $\hat{v}_2(\cdot)$ be the same as $\hat{i}_2(\cdot)$. In other words, the nonlinear resistor $\mathcal{R}_1^*$ in $\mathcal{N}^*$ must be the dual of the nonlinear resistor $\mathcal{R}_1$ in $\mathcal{N}$, and similarly $\mathcal{R}_2^*$ in $\mathcal{N}^*$ must be the dual of $\mathcal{R}_2$ in $\mathcal{N}$. Next, compare Eqs. (2.2) and (2.3) with Eqs.

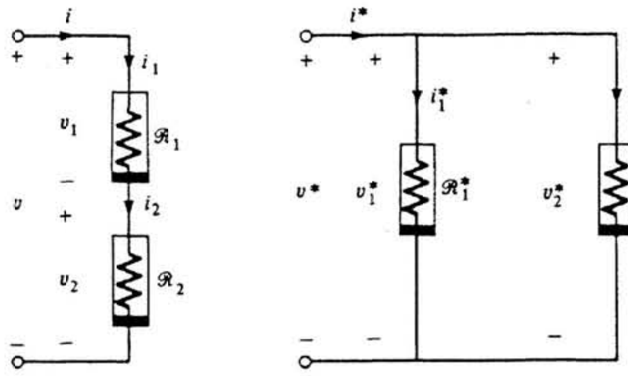

Figure 2.11 A circuit $\mathcal{N}$ and its dual $\mathcal{N}^*$.

(2.11) and (2.12). While Eq. (2.2) states KCL imposed by the series connection of $\mathscr{R}_1$ and $\mathscr{R}_2$ in $\mathcal{N}$, Eq. (2.11) states KVL imposed by the parallel connection of $\mathscr{R}_1^*$ and $\mathscr{R}_2^*$ in $\mathcal{N}^*$. Similarly, Eq. (2.3) is KVL which sums the two voltages v_1 and v_2 in $\mathcal{N}$, while Eq. (2.12) is KCL which sums the two currents i_1^* and i_2^* in $\mathcal{N}^*$. Finally, Eqs. (2.5a) and (2.5b) specify the driving-point characteristic of the one-port obtained by the series connection of two current-controlled resistors $\mathscr{R}_1$ and $\mathscr{R}_2$ as a current-controlled resistor. Equations (2.14a) and (2.14b) specify the driving-point characteristic of the one-port obtained by the parallel connection of two voltage-controlled resistors $\mathscr{R}_1^*$ and $\mathscr{R}_2^*$ as a voltage-controlled resistor.

In Table 2.1 we list two sets of terms S and S^* which we have encountered and which are said to be dual to one another. With these terms, we can generalize the concept of duality to define a dual circuit. Two circuits $\mathcal{N}$ and $\mathcal{N}^*$ are said to be *dual* to one another if the equations describing circuit $\mathcal{N}$ are identical to those describing circuit $\mathcal{N}^*$ after substituting every term in S for $\mathcal{N}$ by the corresponding dual term in S^*.

In this section we can take advantage of the duality concept in discussing the parallel connections of resistors. In later chapters, we shall enlarge the set of dual terms as we learn more about circuit theory.

Exercise Show that the parallel connection of m linear resistors gives a linear resistor whose conductance G is equal to the sum of the conductances of the m linear resistors.

Example 1 (dual one-port and equivalent one-port) Consider the parallel connection of a linear resistor with conductance G and an independent current source i_s as shown in Fig. 2.12. The branch equations are

$$i_1 = Gv_1 \qquad \text{and} \qquad i_2 = i_s \tag{2.15}$$

Adding i_1 and i_2, and setting $v_1 = v$, we obtain

$$i = Gv + i_s \tag{2.16}$$

Table 2.1 Dual terms

S	S^*
Branch voltage	Branch current
Current-controlled resistor	Voltage-controlled resistor
Resistance	Conductance
Open circuit	Short circuit
Independent voltage source	Independent current source
Series connection	Parallel connection
KVL	KCL
Port voltage	Port current

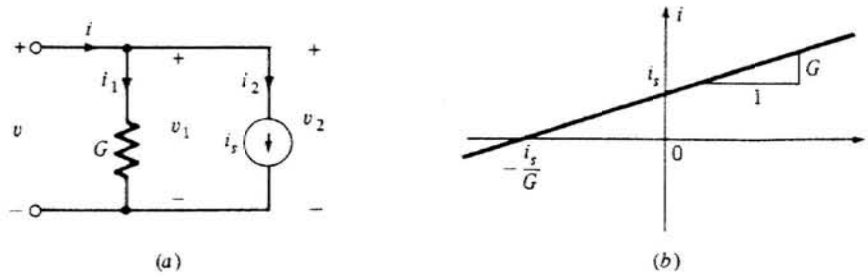

Figure 2.12 (*a*) Parallel connection of a linear resistor and an independent current source. (*b*) The driving-point characteristic of the resulting one-port.

The characteristic is shown in Fig. 2.12. Comparing Eq. (2.7) with Eq. (2.16) and Fig. 2.4 with Fig. 2.12, we know that the two circuits are dual to one another provided $i_s = E$ and $G = R$.

We also wish to use this example to illustrate the concept of *equivalent one-ports*. Recall two resistive one-ports are said to be *equivalent* if they have the same driving-point characteristics. Let $R' = 1/G$. Multiplying both sides of Eq. (2.16) by R' we obtain

$$R'i = v + R'i_s \tag{2.17}$$

Let $v_s' = R'i_s$; then the above equation can be written as

$$v = R'i - v_s' \tag{2.18}$$

This equation can be represented by a series connection of a linear resistor with resistance R' and an independent voltage source v_s' as shown in Fig. 2.13*a*. The driving-point characteristic is plotted in the *i-v* plane, which is shown in Fig. 2.13*b*. In order to compare it with the circuit in Fig. 2.12, we wish to plot the characteristic also in the *v-i* plane. This can be done by first drawing a straight line (dashed) passing through the origin with an angle 45° from the axis as shown in Fig. 2.13*c*. Next, taking the mirror image of the characteristic in Fig. 2.13*b* with respect to the 45° line, we obtain the characteristic in the *v-i* plane, which is exactly the same driving-point characteristic as that of Fig. 2.12. Therefore the two one-ports in Figs. 2.12 and 2.13 are equivalent. This particular equivalence turns out to be extremely important in circuit analysis. It allows us the flexibility of changing an independent voltage source in a circuit to an independent current source yet preserving the property of the circuit. We shall see later that the one-port in Fig. 2.13 is the Thévenin equivalent circuit and the one-port in Fig. 2.12 is the Norton equivalent circuit.

Example 2 (more on the ideal diode) Consider the parallel connection of a linear resistor, an independent current source, and an ideal diode as shown in Fig. 2.14. We wish to determine the driving-point characteristic of the one-port. Figure 2.15*a*, *b*, and *c* shows the branch characteristics. Note

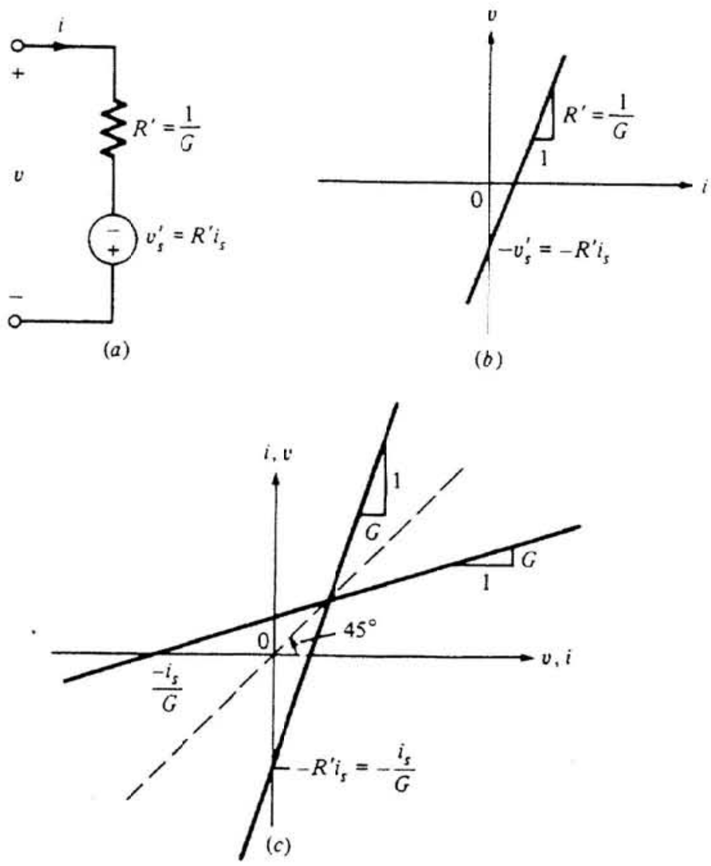

Figure 2.13 The one-port in (a) is equivalent to that of Fig. 2.12a since they have the same driving-point characteristics.

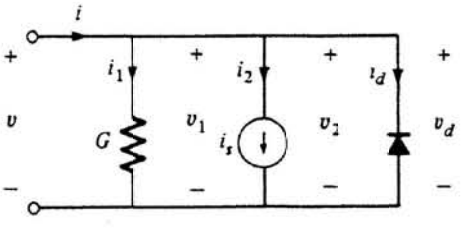

Figure 2.14 Parallel connection of a linear resistor, a current source, and an ideal diode.

that the characteristic of the ideal diode in Fig. 2.15c with the diode turned around is the mirror image with respect to the origin of that given in Fig. 1.8. In order to add the three branch currents we again consider the two individual segments of the ideal diode separately. Note that for $v > 0$, the summation of the current yields a half line with slope G, starting at the point $(0, i_s)$ as shown in Fig. 2.15d. For $v < 0$, we use a limiting process. First, consider that the ideal diode has a finite but very large slope G_d for the purpose of adding the three currents and observe that i_d dominates the other two currents. Then, we let $G_d \to \infty$, and the resulting characteristic becomes a half line on the i axis, which meets the other portion of the characteristic at $(0, i_s)$, as shown in Fig. 2.15d.

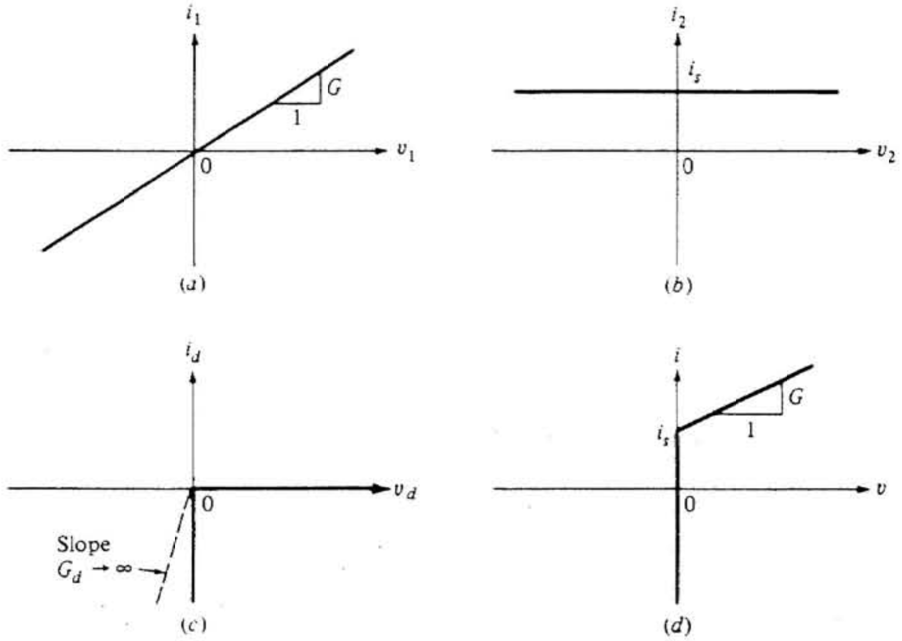

Figure 2.15 Branch v-i characteristics of (a) a linear resistor, (b) a current source, and (c) an ideal diode. (d) The resulting one-port characteristic.

Comparing the characteristic of this one-port with that shown in Fig. 2.6, we recognize that the two one-ports are dual to one another if $E = i_s$ and $R = G$.

Exercises

1. What is the dual of an ideal diode?
2. Determine the driving-point characteristic of the one-port in Fig. 2.14 with the terminals of the ideal diode turned around.
3. Repeat the above with the terminals of the independent current source turned around.

Example 3 (parallel connection of current sources) In Fig. 2.16 there are m independent current sources connected in parallel. This is the dual of m independent voltage sources connected in series. By KCL, it is clear that the parallel connection gives an equivalent one-port which is an independent current source whose current is $i_s = \Sigma_{j=1}^{m} i_j$.

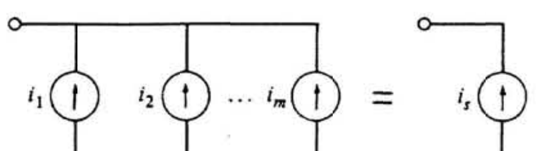

Figure 2.16 m Independent current sources in parallel.

Example 4 (parallel connection of voltage sources) The parallel connection of independent voltage sources violates KVL with the exception of the trivial case where all voltage sources are equal.

Summary of parallel connection The key concepts used in obtaining the driving-point characteristic of a one-port formed by the parallel connection of two-terminal resistors are

1. KVL forces all branch voltages to be equal.
2. KCL requires the port current i to be equal to the sum of the branch currents of the resistors.
3. If each resistor is voltage-controlled, the resulting driving-point characteristic of the one-port is also a voltage-controlled resistor.

2.3 Series-Parallel Connection of Resistors

We now extend the ideas introduced in the last two sections to series-parallel connections of two-terminal resistors. Let us give three examples to illustrate the method of analysis.

Example 1 (series-parallel connection of linear resistors) Consider a *ladder circuit*, i.e., a circuit formed by alternate series and parallel connection of two-terminal circuit elements, such as the one shown in Fig. 2.17. This ladder is made of four linear resistors with resistances, R_1, R_2, R_3, and R_4, respectively. The branch currents and branch voltages are indicated on the figure, and we wish to determine the driving-point characteristic of the one-port defined at terminals ① and ①'. We shall proceed to solve this problem from the back end of the ladder. Since the series connection forces $i_3 = i_4$, we can express the voltage v_2 by adding the voltages:

$$v_2 = v_3 + v_4 = (R_3 + R_4)i_3 \tag{2.19}$$

This equation specifies the characteristic of an equivalent linear resistor with resistance $R_3 + R_4$. We next consider the parallel connection of the resistor with resistance R_2 and the equivalent resistor just obtained. For a parallel connection, we add the currents; using Eq. (2.19), we obtain

$$i_1 = i_2 + i_3 = G_2 v_2 + \frac{1}{R_3 + R_4}\, v_2 = \left(G_2 + \frac{1}{R_3 + R_4}\right) v_2 \tag{2.20}$$

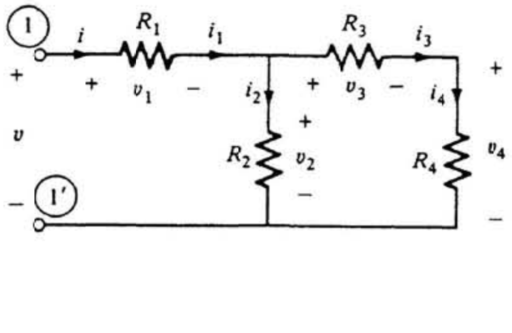

Figure 2.17 A ladder circuit with linear resistors.

where $G_2 = 1/R_2$. We next consider the series connection of the resistor with resistance R_1 and the equivalent resistor specified by Eq. (2.20). Adding the voltages v_1 and v_2, and using Eq. (2.20), we obtain

$$v = v_1 + v_2 = R_1 i_1 + \frac{1}{G_2 + 1/(R_3 + R_4)} i_1 \tag{2.21}$$

Since $i_1 = i$, we conclude that the driving-point characteristic of the one-port at ① and ①' is given by

$$v = Ri$$

where
$$R \triangleq R_1 + \frac{1}{G_2 + 1/(R_3 + R_4)} \tag{2.22}$$

is the resistance of the equivalent linear resistor.

Exercise Determine the resistance R of the one-port shown in Fig. 2.18.

Example 2 (series-parallel connection of nonlinear resistors) Consider the circuit in Fig. 2.19, where $\mathcal{R}_1$ is connected in series with the parallel connection of $\mathcal{R}_2$ and $\mathcal{R}_3$. All three resistors are nonlinear, and the problem is to determine the equivalent one-port resistor, $\mathcal{R}$. We will use the method of successive reduction from the back of the ladder as illustrated in the figure. Thus $\mathcal{R}^*$ is equivalent to the parallel connection of $\mathcal{R}_2$ and $\mathcal{R}_3$, and $\mathcal{R}$ is equivalent to the series connection of $\mathcal{R}_1$ and $\mathcal{R}^*$.

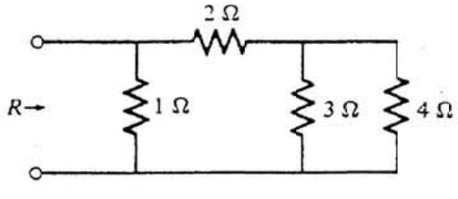

Figure 2.18 A ladder with four linear resistors.

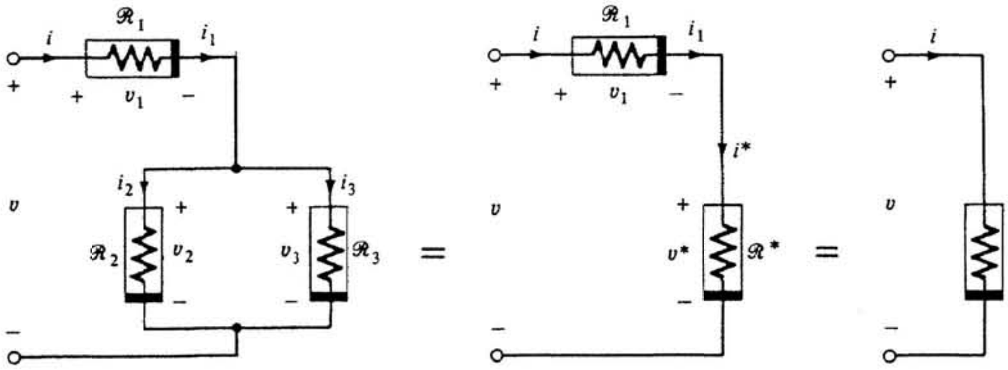

Figure 2.19 Reduction of a ladder circuit with nonlinear resistors.

We assume that the characteristics of $\mathscr{R}_2$ and $\mathscr{R}_3$ are voltage-controlled and specified by

$$i_2 = \hat{i}_2(v_2) \qquad \text{and} \qquad i_3 = \hat{i}_3(v_3)$$

The parallel connection has an equivalent resistor $\mathscr{R}^*$ which is voltage-controlled and specified by

$$i^* = g(v^*) \tag{2.23a}$$

where i^* and v^* are the branch current and branch voltage, respectively, of the resistor $\mathscr{R}^*$. The parallel connection requires $v^* = v_2 = v_3$ and $i^* = i_2 + i_3$. Therefore we have the characteristic of $\mathscr{R}^*$ which is related to that of $\mathscr{R}_2$ and $\mathscr{R}_3$ by

$$g(v^*) = \hat{i}_2(v^*) + \hat{i}_3(v^*) \qquad \text{for all } v^* \tag{2.23b}$$

The next step is to obtain the series connection of $\mathscr{R}_1$ and $\mathscr{R}^*$. Let us assume that the characteristic of $\mathscr{R}_1$ is current-controlled and specified by

$$v_1 = \hat{v}_1(i_1)$$

In order to proceed with the series connection of $\mathscr{R}_1$ and $\mathscr{R}^*$ we must also express $\mathscr{R}^*$ as a current-controlled resistor. This calls for finding just the inverse of the function $g(\cdot)$ in Eqs. (2.23*a*) and (2.23*b*), which is given by

$$v^* = g^{-1}(i^*) \qquad \text{for all } i^*$$

The series connection of $\mathscr{R}^*$ and $\mathscr{R}_1$ requires $i = i_1 = i^*$ and $v = v_1 + v^*$. Thus we obtain the characteristic of $\mathscr{R}$:

$$v = \hat{v}(i) \tag{2.24a}$$

where

$$\hat{v}(i) = \hat{v}_1(i) + g^{-1}(i) \qquad \text{for all } i \tag{2.24b}$$

In this problem, one key step is the determination of the inverse function $g^{-1}(\cdot)$. The question is therefore whether the inverse exists. If it does not, the characteristic of $\mathscr{R}$ cannot be written as in Eq. (2.24) because it is not current-controlled. One simple criterion which guarantees the existence of the inverse is that the v-i characteristic is *strictly monotonically increasing*, i.e., the slope, $g'(v^*)$ is positive for all v^*

REMARK The characteristic of the one-port shown in Fig. 2.19 can always be represented parametrically. Indeed, we have

$$i = i_1 = i^* \qquad \text{and} \qquad v = v_1 + v^*$$

Hence, using v^* as a parameter, we obtain

$$i = g(v^*)$$
$$v = \hat{v}_1[g(v^*)] + v^*$$

Example 3 (zener diode circuit) Consider the circuit shown in Fig. 2.20 where two zener diodes are connected back to back. The symbol of the zener diode and its characteristic are shown in Fig. 2.21, where E_z is called the *breakdown voltage*. The breakdown voltage depends on the doping impurity. For a heavily doped diode, E_z is in the neighborhood of 5 to 6 V.

For this circuit, we will use the approximate characteristic shown in Fig. 2.22*a*. The characteristic of the second diode with the terminals turned around is shown in Fig. 2.22*b*. We add the voltages to get the characteristic of the back-to-back connection of the diodes as shown in Fig. 2.22*c*. The next step is to consider the parallel connection of the linear resistor with negative conductance G, whose characteristic is shown in Fig. 2.22*d*. The resulting curve shown in Fig. 2.22*e* is the characteristic of the one-port. This characteristic is obtained by adding the currents i_a and i_b in Fig. 2.22*c* and *d*, respectively. The vertical portions of the characteristic need some explanation. When we consider the vertical portions of the characteristic in Fig. 2.22*c*, we may use the limiting process by first assuming that they each have a positive slope G_1 and let $G_1 \to \infty$. Thus adding the current $i_b = -Gv$, we note that $G_1 - G$ also approaches ∞. Therefore we have the two vertical portions of the characteristic in Fig. 2.22*e*.

Exercise Consider the one-port shown in Fig. 2.23*a* which consists of two tunnel diodes and two dc voltage sources. The v-i characteristic of the tunnel diode is given in Fig. 2.23*b*. Determine the driving-point characteristic of the one-port in Fig. 2.23*a*.

3 PIECEWISE-LINEAR TECHNIQUES

The method of piecewise-linear analysis is extremely useful in studying circuits with nonlinear resistors. We have seen in the last section that nonlinear v-i characteristics can be approximated by linear segments. This allows us to make simple calculations. In this section we will present some useful techniques which will be used later in piecewise-linear analysis. But, first we will introduce two ideal piecewise-linear models which are used as building blocks.

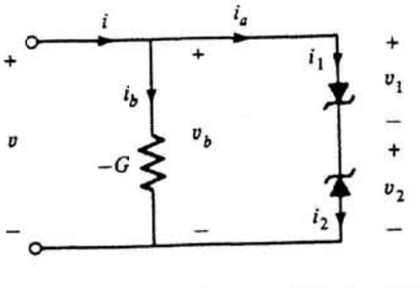

Figure 2.20 A series-parallel circuit with two back-to-back zener diodes.

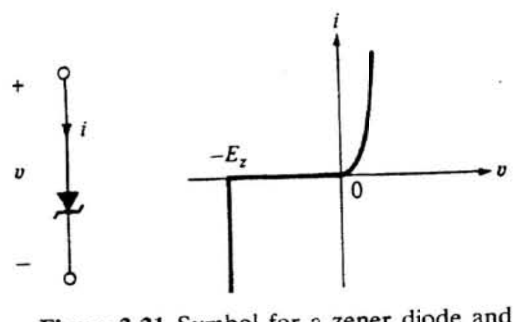

Figure 2.21 Symbol for a zener diode and its characteristic.